Research Article

A Matrix Lie Superalgebra and Its Applications

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A matrix Lie superalgebra is established. As its applications, multicomponent super Ablowitz-Kaup-Newell-Segur (AKNS) equations and multicomponent super Dirac equations are constructed. By making use of supertrace identity, their super-Hamiltonian structures are presented, respectively.

1. Introduction

It is well-known that by making use of the trace identity, many integrable equations can be written as the Hamiltonian forms, which was first proposed by Tu in [1]. The key ideas include the following three aspects. Firstly, from the finite-dimensional Lie algebra

$$G = \{E_1, \dots, E_d\}, \quad (1)$$

Tu constructs a loop Lie algebra

$$\tilde{G} = \{E_1(n), \dots, E_d(n)\}, \quad (2)$$

where $E_i(n) = E_i \otimes \lambda^n = E_i \lambda^n$. Secondly, an iso-spectral problem ($\lambda_{t_n} = 0$)

$$\psi_x = U(u, \lambda) \psi \quad (3)$$

is considered, where λ is a spectral parameter and u is a potential. By solving the adjoint representation equation

$$V_x = [U, V] \quad (4)$$

and zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad V^{(n)} = (\lambda^n V)_+, \quad (5)$$

where plus denotes the choice of the nonnegative power of λ , Tu obtains integrable equations. Lastly, by using the trace identity

$$\frac{\delta}{\delta u} \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \left\langle V, \frac{\partial U}{\partial u} \right\rangle, \quad (6)$$

where γ is a undetermined constant, the obtained integrable equations can be written as the Hamiltonian form. Some well-known integrable Hamiltonian equations have been obtained by this method, such as AKNS equations, Kaup-Newell (KN) equations, and Wadati-Konno-Ichikawa (WKI) equations.

This method for constructing Hamiltonian structures of integrable equations was extended to superintegrable equations by Hu in [2], where the supertrace identity

$$\frac{\delta}{\delta u} \int \text{str} \left(V \frac{\partial U}{\partial \lambda} \right) dx = \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \right) \text{str} \left(\frac{\partial U}{\partial u} V \right) \quad (7)$$

was first proposed by Hu in [3] and proved by Ma et al. in [4]. Similarly, many integrable super-Hamiltonian equations have also been constructed, such as super AKNS equations, super Dirac equations [4], super coupled Korteweg-de Vries (cKdV) equations [5, 6], and super KN equations [7, 8].

It is a valuable generalization from one component to multicomponent in soliton equations because multicomponent soliton equations possess more complex structure and become more extensive than one-component ones. In physics, multicomponent integrable system is widely applied.

For example, mixed N-coupled nonlinear Schrödinger equations (NLS), two-component Bose-Einstein condensates (BECs), and coupled Schrödinger equation are, respectively, discussed in [9–11], and they obtain many new results for these multicomponent equations. In mathematics, the inverse scattering method provides us a powerful tool to find multicomponent extensions of one-component soliton equations, like multicomponent AKNS equations [12, 13], multicomponent NLS equation [14], and so on [15–18]. Moreover, from the point of mathematics, we find that multicomponent generalizations mainly include the following aspects:

- (1) symmetric space [14, 19, 20],
- (2) matrix algebra [13, 21–23],
- (3) soliton hierarchy associated with a matrix pseudodifferential operator [24, 25].

Based on results from the above analysis, some questions are listed as follows.

- (1) Do multicomponent superintegrable equations exist?
- (2) If they exist, how can the multicomponent superintegrable equations be constructed?
- (3) By making use of the supertrace identity (7), can multicomponent superintegrable equations be written as the super-Hamiltonian form?

The purpose of this paper is to answer these questions. The paper is organized as follows. In the next section, we construct a matrix Lie superalgebra $A(2m-1, m-1)$ [26]. As its applications, multicomponent super AKNS equations and multicomponent super Dirac equations are, respectively, constructed in Sections 3 and 4. Their super bi-Hamiltonian forms are also, respectively, constructed in these sections. Some conclusions and discussions are listed in the last section.

2. A Matrix Lie Superalgebra

Let us start with the following linear space $G = \{e_1, e_2, e_3, e_4, e_5\}$:

$$e_1 = \begin{pmatrix} E & O & O \\ O & -E & O \\ O & O & O \end{pmatrix}, \quad e_2 = \begin{pmatrix} O & E & O \\ O & O & O \\ O & O & O \end{pmatrix},$$

$$e_3 = \begin{pmatrix} O & O & O \\ E & O & O \\ O & O & O \end{pmatrix}, \quad e_4 = \begin{pmatrix} O & O & E \\ O & O & O \\ O & -E & O \end{pmatrix},$$

$$e_5 = \begin{pmatrix} O & O & O \\ O & O & E \\ E & O & O \end{pmatrix},$$

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3,$$

$$[e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$

$$[e_1, e_5] = -e_5, \quad [e_2, e_4] = [e_3, e_5] = 0,$$

$$[e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,$$

$$[e_4, e_4] = -2e_2, \quad [e_4, e_5] = e_1, \quad [e_5, e_5] = 2e_3, \quad (8)$$

where E is a $m \times m$ unit matrix, O is a $m \times m$ zero matrix, $G_0 = \{e_1, e_2, e_3\}$ is even, $G_1 = \{e_4, e_5\}$ is odd, $[a, b] = ab - (-1)^{p(a)p(b)}ba$ is the super Lie bracket, and $p(f)$ denotes the parity of the arbitrary element f .

It is easy to prove that the linear space G is a matrix Lie superalgebra $A(2m-1, m-1)$. The corresponding loop superalgebra $\bar{G}$ is presented as

$$\begin{aligned} e_i(n) &= e_i \lambda^n, & [e_1(m), e_2(n)] &= 2e_2(m+n), \\ [e_1(m), e_3(n)] &= -2e_3(m+n), \\ [e_2(m), e_3(n)] &= e_1(m+n), \\ [e_1(m), e_4(n)] &= e_4(m+n), \\ [e_1(m), e_5(n)] &= -e_5(m+n), \\ [e_2(m), e_4(n)] &= [e_3(m), e_5(n)] = 0, \\ [e_2(m), e_5(n)] &= e_4(m+n), \\ [e_3(m), e_4(n)] &= e_5(m+n), \\ [e_4(m), e_4(n)] &= -2e_2(m+n), \\ [e_4(m), e_5(n)] &= e_1(m+n), \\ [e_5(m), e_5(n)] &= 2e_3(m+n), \\ \deg e_i(n) &= n, \quad (i = 1, 2, 3, 4, 5). \end{aligned} \quad (9)$$

3. Multicomponent Super AKNS Hierarchy and Its Super-Hamiltonian Structure

In this section, we will derive multicomponent super AKNS hierarchy by the above matrix Lie superalgebra, and further, its super-Hamiltonian structure will be constructed.

Let us consider the following spectral problem:

$$\begin{aligned} \phi_x &= U(u, \lambda) \phi, \\ U &= -e_1(1) + v e_2(0) + w e_3(0) \\ &\quad + \alpha e_4(0) + \beta e_5(0), \end{aligned} \quad (10)$$

which can be written in the following matrix form:

$$\phi_x = U(u, \lambda) \phi = \begin{pmatrix} -\lambda E & v & \alpha \\ w & \lambda E & \beta \\ \beta & -\alpha & 0 \end{pmatrix} \phi, \quad (11)$$

where $v = \text{diag}(v_1, \dots, v_m)$, $w = \text{diag}(w_1, \dots, w_m)$, $\alpha = \text{diag}(\alpha_1, \dots, \alpha_m)$, $\beta = \text{diag}(\beta_1, \dots, \beta_m)$, $u = (v, w, \alpha, \beta)^T$, $p(\lambda) = p(v_j) = p(w_j) = 0$, and $p(\alpha_j) = p(\beta_j) = 1$ ($1 \leq j \leq m$). Note that $U(u, \lambda) \in A(2m-1, m-1)$. Taking

$$V = \begin{pmatrix} a & b & \rho \\ c & -a & \delta \\ \delta & -\rho & 0 \end{pmatrix}, \quad (12)$$

where $a = \text{diag}(a_1, \dots, a_m)$, $b = \text{diag}(b_1, \dots, b_m)$, $c = \text{diag}(c_1, \dots, c_m)$, $\rho = \text{diag}(\rho_1, \dots, \rho_m)$, and $\delta = \text{diag}(\delta_1, \dots, \delta_m)$, the adjoint representation equation (4) gives

$$\begin{aligned} a_{k,x} &= v_k c_k - w_k b_k + \alpha_k \delta_k + \beta_k \rho_k, \\ b_{k,x} &= -2\lambda b_k - 2v_k a_k - 2\alpha_k \rho_k, \\ c_{k,x} &= 2\lambda c_k + 2w_k a_k + 2\beta_k \delta_k, \\ \rho_{k,x} &= -\lambda \rho_k + v_k \delta_k - \alpha_k a_k - \beta_k b_k, \\ \delta_{k,x} &= \lambda \delta_k + w_k \rho_k - \alpha_k c_k + \beta_k a_k, \end{aligned} \quad (13)$$

where $1 \leq k \leq m$. Let $a_k = \sum_{j \geq 0} a_k^{(j)} \lambda^{-j}$, $b_k = \sum_{j \geq 0} b_k^{(j)} \lambda^{-j}$, $c_k = \sum_{j \geq 0} c_k^{(j)} \lambda^{-j}$, $\rho_k = \sum_{j \geq 0} \rho_k^{(j)} \lambda^{-j}$, and $\delta_k = \sum_{j \geq 0} \delta_k^{(j)} \lambda^{-j}$, and then (13) becomes

$$\begin{aligned} b_k^{(0)} &= c_k^{(0)} = \rho_k^{(0)} = \delta_k^{(0)} = 0, \\ a_{k,x}^{(j)} &= v_k c_k^{(j)} - w_k b_k^{(j)} + \alpha_k \delta_k^{(j)} + \beta_k \rho_k^{(j)}, \quad j \geq 0, \\ b_{k,x}^{(j)} &= -2b_k^{(j+1)} - 2v_k a_k^{(j)} - 2\alpha_k \rho_k^{(j)}, \quad j \geq 0, \\ c_{k,x}^{(j)} &= 2c_k^{(j+1)} + 2w_k a_k^{(j)} + 2\beta_k \delta_k^{(j)}, \quad j \geq 0, \\ \rho_{k,x}^{(j)} &= -\rho_k^{(j+1)} + v_k \delta_k^{(j)} - \alpha_k a_k^{(j)} - \beta_k b_k^{(j)}, \quad j \geq 0, \\ \delta_{k,x}^{(j)} &= \delta_k^{(j+1)} + w_k \rho_k^{(j)} - \alpha_k c_k^{(j)} + \beta_k a_k^{(j)}, \quad j \geq 0, \end{aligned} \quad (14)$$

where $1 \leq k \leq m$. Moreover, we find that (14) can be written as the following recursive form:

$$\begin{pmatrix} c_1^{(j+1)} & & & \\ & \ddots & & \\ & & c_m^{(j+1)} & \\ b_1^{(j+1)} & & & \\ & \ddots & & \\ & & b_m^{(j+1)} & \\ 2\delta_1^{(j+1)} & & & \\ & \ddots & & \\ & & 2\delta_m^{(j+1)} & \\ -2\rho_1^{(j+1)} & & & \\ & \ddots & & \\ & & -2\rho_m^{(j+1)} & \end{pmatrix} = \mathcal{L} \begin{pmatrix} c_1^{(j)} & & & \\ & \ddots & & \\ & & c_m^{(j)} & \\ b_1^{(j)} & & & \\ & \ddots & & \\ & & b_m^{(j)} & \\ 2\delta_1^{(j)} & & & \\ & \ddots & & \\ & & 2\delta_m^{(j)} & \\ -2\rho_1^{(j)} & & & \\ & \ddots & & \\ & & -2\rho_m^{(j)} & \end{pmatrix}, \quad (15)$$

where the recursive operator is given by

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & \mathcal{L}_{14} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & \mathcal{L}_{24} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & \mathcal{L}_{34} \\ \mathcal{L}_{41} & \mathcal{L}_{42} & \mathcal{L}_{43} & \mathcal{L}_{44} \end{pmatrix}, \quad (16)$$

where

$$\begin{aligned} \mathcal{L}_{11} &= \text{diag} \left(\frac{1}{2} \partial - w_1 \partial^{-1} v_1, \dots, \frac{1}{2} \partial - w_m \partial^{-1} v_m \right), \\ \mathcal{L}_{12} &= \text{diag} (w_1 \partial^{-1} w_1, \dots, w_m \partial^{-1} w_m), \\ \mathcal{L}_{13} &= \text{diag} \left(-\frac{1}{2} \beta_1 - \frac{1}{2} w_1 \partial^{-1} \alpha_1, \dots, -\frac{1}{2} \beta_m - \frac{1}{2} w_m \partial^{-1} \alpha_m \right), \\ \mathcal{L}_{14} &= \text{diag} \left(\frac{1}{2} w_1 \partial^{-1} \beta_1, \dots, \frac{1}{2} w_m \partial^{-1} \beta_m \right), \\ \mathcal{L}_{21} &= \text{diag} (-v_1 \partial^{-1} v_1, \dots, -v_m \partial^{-1} v_m), \\ \mathcal{L}_{22} &= \text{diag} \left(-\frac{1}{2} \partial + v_1 \partial^{-1} w_1, \dots, -\frac{1}{2} \partial + v_m \partial^{-1} w_m \right), \\ \mathcal{L}_{23} &= \text{diag} \left(-\frac{1}{2} v_1 \partial^{-1} \alpha_1, \dots, -\frac{1}{2} v_m \partial^{-1} \alpha_m \right), \\ \mathcal{L}_{24} &= \text{diag} \left(\frac{1}{2} \alpha_1 + \frac{1}{2} v_1 \partial^{-1} \beta_1, \dots, \frac{1}{2} \alpha_m + \frac{1}{2} v_m \partial^{-1} \beta_m \right), \\ \mathcal{L}_{31} &= \text{diag} (2\alpha_1 - 2\beta_1 \partial^{-1} v_1, \dots, 2\alpha_m - 2\beta_m \partial^{-1} v_m), \\ \mathcal{L}_{32} &= \text{diag} (2\beta_1 \partial^{-1} w_1, \dots, 2\beta_m \partial^{-1} w_m), \\ \mathcal{L}_{33} &= \text{diag} (\partial - \beta_1 \partial^{-1} \alpha_1, \dots, \partial - \beta_m \partial^{-1} \alpha_m), \\ \mathcal{L}_{34} &= \text{diag} (w_1 + \beta_1 \partial^{-1} \beta_1, \dots, w_m + \beta_m \partial^{-1} \beta_m), \\ \mathcal{L}_{41} &= \text{diag} (2\alpha_1 \partial^{-1} v_1, \dots, 2\alpha_m \partial^{-1} v_m), \\ \mathcal{L}_{42} &= \text{diag} (2\beta_1 - 2\alpha_1 \partial^{-1} w_1, \dots, 2\beta_m - 2\alpha_m \partial^{-1} w_m), \\ \mathcal{L}_{43} &= \text{diag} (-v_1 + \alpha_1 \partial^{-1} \alpha_1, \dots, -v_m + \alpha_m \partial^{-1} \alpha_m), \\ \mathcal{L}_{44} &= \text{diag} (-\partial - \alpha_1 \partial^{-1} \beta_1, \dots, -\partial - \alpha_m \partial^{-1} \beta_m). \end{aligned} \quad (17)$$

After a direct calculation, we obtain that $a_{k,x}^{(0)} = 0$. Choosing $a_k^{(0)} = -1$ ($1 \leq k \leq m$) and constant of integration to be zero, the first few terms are listed as follows:

$$\begin{aligned} a_k^{(1)} &= 0, & b_k^{(1)} &= v_k, & c_k^{(1)} &= w_k, \\ \rho_k^{(1)} &= \alpha_k, & \delta_k^{(1)} &= \beta_k, \\ a_k^{(2)} &= \frac{1}{2} v_k w_k + \alpha_k \beta_k, & b_k^{(2)} &= -\frac{1}{2} v_{k,x}, \\ c_k^{(2)} &= \frac{1}{2} w_{k,x}, & \rho_k^{(2)} &= -\alpha_{k,x}, & \delta_k^{(2)} &= \beta_{k,x}, \end{aligned} \quad (18)$$

where $1 \leq k \leq m$.

Then, let us consider the spectral problem (11) with the following auxiliary spectral problem:

$$\phi_{t_n} = V^{(n)}\phi, \quad V^{(n)} = (\lambda^n V)_+. \quad (19)$$

The compatibility conditions of (11) and (19), that is, the zero curvature equations (5) give an infinite hierarchy of nonlinear partial differential equations:

$$\begin{aligned} v_{k,t_n} &= -2b_k^{(n+1)}, \quad 1 \leq k \leq m, \\ w_{k,t_n} &= 2c_k^{(n+1)}, \quad 1 \leq k \leq m, \\ \alpha_{k,t_n} &= -\rho_k^{(n+1)}, \quad 1 \leq k \leq m, \\ \beta_{k,t_n} &= \delta_k^{(n+1)}, \quad 1 \leq k \leq m, \end{aligned} \quad (20)$$

or

$$\begin{aligned} v_{t_n} &= -2b^{(n+1)}, & w_{t_n} &= 2c^{(n+1)}, \\ \alpha_{t_n} &= -\rho^{(n+1)}, & \beta_{t_n} &= \delta^{(n+1)}, \end{aligned} \quad (21)$$

where $a^{(j)} = \text{diag}(a_1^{(j)}, \dots, a_m^{(j)})$, $b^{(j)} = \text{diag}(b_1^{(j)}, \dots, b_m^{(j)})$, $c^{(j)} = \text{diag}(c_1^{(j)}, \dots, c_m^{(j)})$, $\rho^{(j)} = \text{diag}(\rho_1^{(j)}, \dots, \rho_m^{(j)})$, $\delta^{(j)} = \text{diag}(\delta_1^{(j)}, \dots, \delta_m^{(j)})$, with $j \geq 0$. When $m = 1$, (21) is equivalent to the super AKNS soliton hierarchy [27–30], and thus (21) is called multicomponent super AKNS hierarchy.

In what follows, a super-Hamiltonian structure of (21) is derived by means of the supertrace identity (7). To this end, the following quantities are needed:

$$\begin{aligned} \text{str} \left(V \frac{\partial U}{\partial \lambda} \right) &= -2tra, & \text{str} \left(\frac{\partial U}{\partial v} V \right) &= c, \\ \text{str} \left(\frac{\partial U}{\partial w} V \right) &= b, & \text{str} \left(\frac{\partial U}{\partial \alpha} V \right) &= 2\delta, \\ \text{str} \left(\frac{\partial U}{\partial \beta} V \right) &= -2\rho. \end{aligned} \quad (22)$$

Thus, the supertrace identity (7) gives the following equality:

$$\begin{pmatrix} \frac{\delta}{\delta v_1} & \dots & \frac{\delta}{\delta v_m} \\ \frac{\delta}{\delta w_1} & \dots & \frac{\delta}{\delta w_m} \\ \frac{\delta}{\delta \alpha_1} & \dots & \frac{\delta}{\delta \alpha_m} \\ \frac{\delta}{\delta \beta_1} & \dots & \frac{\delta}{\delta \beta_m} \end{pmatrix} \int -2 \sum_{k=1}^m \sum_{j \geq 0} a_k^{(j)} \lambda^{-j} dx$$

$$= \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \right) \begin{pmatrix} \sum_{j \geq 0} c_1^{(j)} \lambda^{-j} & \dots & \sum_{j \geq 0} c_m^{(j)} \lambda^{-j} \\ \sum_{j \geq 0} b_1^{(j)} \lambda^{-j} & \dots & \sum_{j \geq 0} b_m^{(j)} \lambda^{-j} \\ 2 \sum_{j \geq 0} \delta_1^{(j)} \lambda^{-j} & \dots & 2 \sum_{j \geq 0} \delta_m^{(j)} \lambda^{-j} \\ -2 \sum_{j \geq 0} \rho_1^{(j)} \lambda^{-j} & \dots & -2 \sum_{j \geq 0} \rho_m^{(j)} \lambda^{-j} \end{pmatrix}. \quad (23)$$

Equating the coefficients of λ^{-j-2} on two sides of the above equality, we have

$$\begin{pmatrix} \frac{\delta}{\delta v_1} & \dots & \frac{\delta}{\delta v_m} \\ \frac{\delta}{\delta w_1} & \dots & \frac{\delta}{\delta w_m} \\ \frac{\delta}{\delta \alpha_1} & \dots & \frac{\delta}{\delta \alpha_m} \\ \frac{\delta}{\delta \beta_1} & \dots & \frac{\delta}{\delta \beta_m} \end{pmatrix} \int -2 \sum_{k=1}^m a_k^{(j+2)} dx = (\gamma - j - 1) \begin{pmatrix} c_1^{(j+1)} & \dots & c_m^{(j+1)} \\ b_1^{(j+1)} & \dots & b_m^{(j+1)} \\ 2\delta_1^{(j+1)} & \dots & 2\delta_m^{(j+1)} \\ -2\rho_1^{(j+1)} & \dots & -2\rho_m^{(j+1)} \end{pmatrix}. \quad (24)$$

By taking $j = 0$, we obtain that the constant $\gamma = 0$. Thus, we have

$$\begin{pmatrix} c_1^{(j+1)} & & & \\ & \ddots & & \\ & & c_m^{(j+1)} & \\ b_1^{(j+1)} & & & \\ & \ddots & & \\ & & b_m^{(j+1)} & \\ 2\delta_1^{(j+1)} & & & \\ & \ddots & & \\ & & 2\delta_m^{(j+1)} & \\ -2\rho_1^{(j+1)} & & & \\ & \ddots & & \\ & & -2\rho_m^{(j+1)} & \end{pmatrix} = \begin{pmatrix} \frac{\delta}{\delta v_1} & & & \\ & \ddots & & \\ & & \frac{\delta}{\delta v_m} & \\ \frac{\delta}{\delta w_1} & & & \\ & \ddots & & \\ & & \frac{\delta}{\delta w_m} & \\ \frac{\delta}{\delta \alpha_1} & & & \\ & \ddots & & \\ & & \frac{\delta}{\delta \alpha_m} & \\ \frac{\delta}{\delta \beta_1} & & & \\ & \ddots & & \\ & & \frac{\delta}{\delta \beta_m} & \end{pmatrix} H_j, \quad (25)$$

where $H_j = \int (2/(j+1)) \sum_{k=1}^m a_k^{(j+2)} dx$.

Therefore, (21) can be written in the following super bi-Hamiltonian form:

$$\begin{pmatrix} v_1 & & & \\ & \ddots & & \\ & & v_m & \\ w_1 & & & \\ & \ddots & & \\ & & w_m & \\ \alpha_1 & & & \\ & \ddots & & \\ & & \alpha_m & \\ \beta_1 & & & \\ & \ddots & & \\ & & \beta_m & \end{pmatrix}_{t_n} = J \begin{pmatrix} c_1^{(n+1)} & & & \\ & \ddots & & \\ & & c_m^{(n+1)} & \\ b_1^{(n+1)} & & & \\ & \ddots & & \\ & & b_m^{(n+1)} & \\ 2\delta_1^{(n+1)} & & & \\ & \ddots & & \\ & & 2\delta_m^{(n+1)} & \\ -2\rho_1^{(n+1)} & & & \\ & \ddots & & \\ & & -2\rho_m^{(n+1)} & \end{pmatrix} \\ = J \frac{\delta}{\delta u} H_n = J \mathcal{L} \frac{\delta}{\delta u} H_{n-1}, \quad (26)$$

where $J = \begin{pmatrix} 0 & -2E & 0 & 0 \\ 2E & 0 & 0 & 0 \\ 0 & 0 & 0 & (1/2)E \\ 0 & 0 & (1/2)E & 0 \end{pmatrix}$ is the supersymplectic operator.

Example 1. Let $m = 2, n = 2$ in (21), and we have

$$\begin{aligned} v_{1,t_2} &= -\frac{1}{2} v_{1,xx} + v_1^2 w_1 + 2v_1 \alpha_1 \beta_1 - 2\alpha_1 \alpha_{1,x}, \\ v_{2,t_2} &= -\frac{1}{2} v_{2,xx} + v_2^2 w_2 + 2v_2 \alpha_2 \beta_2 - 2\alpha_2 \alpha_{2,x}, \end{aligned}$$

$$w_{1,t_2} = \frac{1}{2} w_{1,xx} - v_1 w_1^2 - 2w_1 \alpha_1 \beta_1 - 2\beta_1 \beta_{1,x},$$

$$w_{2,t_2} = \frac{1}{2} w_{2,xx} - v_2 w_2^2 - 2w_2 \alpha_2 \beta_2 - 2\beta_2 \beta_{2,x},$$

$$\alpha_{1,t_2} = -\alpha_{1,xx} - v_1 \beta_{1,x} + \frac{1}{2} v_1 w_1 \alpha_1 - \frac{1}{2} v_{1,x} \beta_1,$$

$$\alpha_{2,t_2} = -\alpha_{2,xx} - v_2 \beta_{2,x} + \frac{1}{2} v_2 w_2 \alpha_2 - \frac{1}{2} v_{2,x} \beta_2,$$

$$\beta_{1,t_2} = \beta_{1,xx} + w_1 \alpha_{1,x} + \frac{1}{2} w_{1,x} \alpha_1 - \frac{1}{2} v_1 w_1 \beta_1,$$

$$\beta_{2,t_2} = \beta_{2,xx} + w_2 \alpha_{2,x} + \frac{1}{2} w_{2,x} \alpha_2 - \frac{1}{2} v_2 w_2 \beta_2,$$

(27)

which is the first nonlinear two-component super AKNS equations.

4. MultiComponent Super Dirac Hierarchy and Its Super-Hamiltonian Structure

As another application of the matrix Lie superalgebra, multicomponent super Dirac hierarchy will be constructed, and further, its super-Hamiltonian structure will also be obtained.

Let us consider the following spectral problem:

$$\phi_x = U(\tilde{u}, \lambda) \phi,$$

$$\begin{aligned} U &= e_2(1) - e_3(1) + r e_1(0) + s e_2(0) \\ &\quad + s e_3(0) + \alpha e_4(0) + \beta e_5(0), \end{aligned} \quad (28)$$

which can be written in the following matrix form:

$$\phi_x = U(\tilde{u}, \lambda) \phi = \begin{pmatrix} r & \lambda E + s & \alpha \\ -\lambda E + s & -r & \beta \\ \beta & -\alpha & 0 \end{pmatrix} \phi, \quad (29)$$

where $r = \text{diag}(r_1, \dots, r_m)$, $s = \text{diag}(s_1, \dots, s_m)$, $\alpha = \text{diag}(\alpha_1, \dots, \alpha_m)$, $\beta = \text{diag}(\beta_1, \dots, \beta_m)$, $\tilde{u} = (r, s, \alpha, \beta)^T$, $p(\lambda) = p(r_i) = p(s_i) = 0$, and $p(\alpha_i) = p(\beta_i) = 1$ ($1 \leq i \leq m$). $U(\tilde{u}, \lambda) \in A(2m-1, m-1)$. Solving the adjoint representation equation (4), where

$$V = \begin{pmatrix} C & A+B & \rho \\ A-B & -C & \delta \\ \delta & -\rho & 0 \end{pmatrix}, \quad (30)$$

with $A = \text{diag}(A_1, \dots, A_m)$, $B = \text{diag}(B_1, \dots, B_m)$, $C = \text{diag}(C_1, \dots, C_m)$, $\rho = \text{diag}(\rho_1, \dots, \rho_m)$, and $\delta = \text{diag}(\delta_1, \dots, \delta_m)$, we have

$$\begin{aligned} A_{k,x} &= 2r_k B_k - 2\lambda C_k - \alpha_k \rho_k + \beta_k \delta_k, \\ B_{k,x} &= 2r_k A_k - 2s_k C_k - \alpha_k \rho_k - \beta_k \delta_k, \\ C_{k,x} &= 2\lambda A_k - 2s_k B_k + \alpha_k \delta_k + \beta_k \rho_k, \\ \rho_{k,x} &= \lambda \delta_k + r_k \rho_k + s_k \delta_k \\ &\quad - C_k \alpha_k - \beta_k A_k - \beta_k B_k, \\ \delta_{k,x} &= -\lambda \rho_k + s_k \rho_k - r_k \delta_k \\ &\quad - \alpha_k A_k + \alpha_k B_k + \beta_k C_k, \end{aligned} \quad (31)$$

where $1 \leq k \leq m$. Let $A_k = \sum_{j \geq 0} A_k^{(j)} \lambda^{-j}$, $B_k = \sum_{j \geq 0} B_k^{(j)} \lambda^{-j}$, $C_k = \sum_{j \geq 0} C_k^{(j)} \lambda^{-j}$, $\rho_k = \sum_{j \geq 0} \rho_k^{(j)} \lambda^{-j}$, and $\delta_k = \sum_{j \geq 0} \delta_k^{(j)} \lambda^{-j}$, and then we have

$$\begin{aligned} A_k^{(0)} &= C_k^{(0)} = \rho_k^{(0)} = \delta_k^{(0)} = 0, \\ A_{k,x}^{(j)} &= -2C_k^{(j+1)} + 2r_k B_k^{(j)} - \alpha_k \rho_k^{(j)} + \beta_k \delta_k^{(j)}, \\ B_{k,x}^{(j)} &= 2r_k A_k^{(j)} - 2s_k C_k^{(j)} - \alpha_k \rho_k^{(j)} - \beta_k \delta_k^{(j)}, \\ C_{k,x}^{(j)} &= 2A_k^{(j+1)} - 2s_k B_k^{(j)} + \alpha_k \delta_k^{(j)} + \beta_k \rho_k^{(j)}, \\ \rho_{k,x}^{(j)} &= \delta_k^{(j+1)} + r_k \rho_k^{(j)} + s_k \delta_k^{(j)} \\ &\quad - \alpha_k C_k^{(j)} - \beta_k A_k^{(j)} - \beta_k B_k^{(j)}, \\ \delta_{k,x}^{(j)} &= -\rho_k^{(j+1)} + s_k \rho_k^{(j)} - r_k \delta_k^{(j)} \\ &\quad - \alpha_k A_k^{(j)} + \alpha_k B_k^{(j)} + \beta_k C_k^{(j)}, \end{aligned} \quad (32)$$

where $j \geq 0$, $1 \leq k \leq m$, which can be written as a recursive form:

$$\begin{pmatrix} C_1^{(j+1)} & & & \\ & \ddots & & \\ & & C_m^{(j+1)} & \\ A_1^{(j+1)} & & & \\ & \ddots & & \\ & & A_m^{(j+1)} & \\ \delta_1^{(j+1)} & & & \\ & \ddots & & \\ & & \delta_m^{(j+1)} & \\ -\rho_1^{(j+1)} & & & \\ & \ddots & & \\ & & -\rho_m^{(j+1)} & \end{pmatrix} = L \begin{pmatrix} C_1^{(j)} & & & \\ & \ddots & & \\ & & C_m^{(j)} & \\ A_1^{(j)} & & & \\ & \ddots & & \\ & & A_m^{(j)} & \\ \delta_1^{(j)} & & & \\ & \ddots & & \\ & & \delta_m^{(j)} & \\ -\rho_1^{(j)} & & & \\ & \ddots & & \\ & & -\rho_m^{(j)} & \end{pmatrix}, \quad (33)$$

where the recursive operator is given by

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}, \quad (34)$$

with

$$\begin{aligned} L_{11} &= \text{diag}(-2r_1 \partial^{-1} s_1, \dots, -2r_m \partial^{-1} s_m), \\ L_{12} &= \text{diag}\left(-\frac{1}{2} \partial + 2r_1 \partial^{-1} r_1, \dots, -\frac{1}{2} \partial + 2r_m \partial^{-1} r_m\right), \\ L_{13} &= \text{diag}\left(\frac{1}{2} \beta_1 - r_1 \partial^{-1} \beta_1, \dots, \frac{1}{2} \beta_m - r_m \partial^{-1} \beta_m\right), \\ L_{14} &= \text{diag}\left(\frac{1}{2} \alpha_1 + r_1 \partial^{-1} \alpha_1, \dots, \frac{1}{2} \alpha_m + r_m \partial^{-1} \alpha_m\right), \\ L_{21} &= \text{diag}\left(\frac{1}{2} \partial - 2s_1 \partial^{-1} s_1, \dots, \frac{1}{2} \partial - 2s_m \partial^{-1} s_m\right), \\ L_{22} &= \text{diag}(2s_1 \partial^{-1} r_1, \dots, 2s_m \partial^{-1} r_m), \\ L_{23} &= \text{diag}\left(-\frac{1}{2} \alpha_1 - s_1 \partial^{-1} \beta_1, \dots, -\frac{1}{2} \alpha_m - s_m \partial^{-1} \beta_m\right), \\ L_{24} &= \text{diag}\left(\frac{1}{2} \beta_1 + s_1 \partial^{-1} \alpha_1, \dots, \frac{1}{2} \beta_m + s_m \partial^{-1} \alpha_m\right), \\ L_{31} &= \text{diag}(\alpha_1 - 2\beta_1 \partial^{-1} s_1, \dots, \alpha_m - 2\beta_m \partial^{-1} s_m), \\ L_{32} &= \text{diag}(\beta_1 + 2\beta_1 \partial^{-1} r_1, \dots, \beta_m + 2\beta_m \partial^{-1} r_m), \\ L_{33} &= \text{diag}(-s_1 - \beta_1 \partial^{-1} \beta_1, \dots, -s_m - \beta_m \partial^{-1} \beta_m), \\ L_{34} &= \text{diag}(-\partial + r_1 + \beta_1 \partial^{-1} \alpha_1, \dots, -\partial + r_m + \beta_m \partial^{-1} \alpha_m), \\ L_{41} &= \text{diag}(-\beta_1 + 2\alpha_1 \partial^{-1} s_1, \dots, -\beta_m + 2\alpha_m \partial^{-1} s_m), \\ L_{42} &= \text{diag}(\alpha_1 - 2\alpha_1 \partial^{-1} r_1, \dots, \alpha_m - 2\alpha_m \partial^{-1} r_m), \\ L_{43} &= \text{diag}(\partial + r_1 + \alpha_1 \partial^{-1} \beta_1, \dots, \partial + r_m + \alpha_m \partial^{-1} \beta_m), \\ L_{44} &= \text{diag}(s_1 - \alpha_1 \partial^{-1} \alpha_1, \dots, s_m - \alpha_m \partial^{-1} \alpha_m). \end{aligned} \quad (35)$$

It is easy to verify that $B_{k,x}^{(0)} = 0$ ($1 \leq k \leq m$). Choosing $B_k^{(0)} = 1$ and constant of integration to be zero, the first few terms can be worked out as follows:

$$\begin{aligned} A_k^{(1)} &= s_k, & B_k^{(1)} &= 0, & C_k^{(1)} &= r_k, \\ \rho_k^{(1)} &= \alpha_k, & \delta_k^{(1)} &= \beta_k, \\ A_k^{(2)} &= \frac{1}{2} r_{k,x}, & B_k^{(2)} &= \frac{1}{2} (r_k^2 + s_k^2) + \alpha_k \beta_k, \\ C_k^{(2)} &= -\frac{1}{2} s_{k,x}, & \rho_k^{(2)} &= -\beta_{k,x}, & \delta_k^{(2)} &= \alpha_{k,x}. \end{aligned} \quad (36)$$

In what follows, the spectral problem (29) is associated with the auxiliary spectral problem:

$$\phi_{t_n} = V^{(n)} \phi, \quad (37)$$

where

$$V^{(n)} = \sum_{j=1}^n \begin{pmatrix} C_1^{(j)} & & & A_1^{(j)} + B_1^{(j)} & & \rho_1^{(j)} \\ & \ddots & & & \ddots & \\ & & C_m^{(j)} & & A_m^{(j)} + B_m^{(j)} & \rho_m^{(j)} \\ A_1^{(j)} - B_1^{(j)} & & & -C_1^{(j)} & & \delta_1^{(j)} \\ & \ddots & & & \ddots & \\ \delta_1^{(j)} & & A_m^{(j)} - B_m^{(j)} & -\rho_1^{(j)} & & -C_m^{(j)} \\ & & & & \ddots & 0 \\ & & \delta_m^{(j)} & & & \delta_m^{(j)} \\ & & & & -\rho_m^{(j)} & 0 \end{pmatrix} \lambda^{n-j}. \quad (38)$$

The compatibility condition between (29) and (37) gives the following super nonlinear soliton hierarchy:

$$\begin{aligned} r_{k,t_n} &= 2A_k^{(n+1)}, \\ s_{k,t_n} &= -2C_k^{(n+1)}, \\ \alpha_{k,t_n} &= \delta_k^{(n+1)}, \\ \beta_{k,t_n} &= -\rho_k^{(n+1)}, \end{aligned} \quad (39)$$

where $1 \leq k \leq m$. By denoting $A^{(j)} = \text{diag}(A_1^{(j)}, \dots, A_m^{(j)})$, $B^{(j)} = \text{diag}(B_1^{(j)}, \dots, B_m^{(j)})$, $C^{(j)} = \text{diag}(C_1^{(j)}, \dots, C_m^{(j)})$, $\rho^{(j)} = \text{diag}(\rho_1^{(j)}, \dots, \rho_m^{(j)})$, and $\delta^{(j)} = \text{diag}(\delta_1^{(j)}, \dots, \delta_m^{(j)})$, with $j \geq 0$, and (39) can be written as follows:

$$\begin{aligned} r_{t_n} &= 2A^{(n+1)}, \\ s_{t_n} &= -2C^{(n+1)}, \\ \alpha_{t_n} &= \delta^{(n+1)}, \\ \beta_{t_n} &= -\rho^{(n+1)}. \end{aligned} \quad (40)$$

Similarly, when $m = 1$, (40) is equivalent to the super Dirac soliton hierarchy [4, 30], and thus (40) is called multicomponent super Dirac hierarchy.

To derive super bi-Hamiltonian structure of (40), we need to use the supertrace identity (7). To this end, we firstly obtain the following equalities:

$$\begin{aligned} \text{str} \left(V \frac{\partial U}{\partial \lambda} \right) &= -2trB, & \text{str} \left(\frac{\partial U}{\partial s} V \right) &= 2A, \\ \text{str} \left(\frac{\partial U}{\partial r} V \right) &= 2C, & \text{str} \left(\frac{\partial U}{\partial \alpha} V \right) &= 2\delta, \\ \text{str} \left(\frac{\partial U}{\partial \beta} V \right) &= -2\rho. \end{aligned} \quad (41)$$

Thus, the supertrace identity (7) becomes the following equality:

$$\begin{pmatrix} \frac{\delta}{\delta r_1} & & & & & \\ & \ddots & & & & \\ & & \frac{\delta}{\delta s_1} & & & \\ & & & \ddots & & \\ & & & & \frac{\delta}{\delta s_m} & \\ \frac{\delta}{\delta \alpha_1} & & & & & \\ & \ddots & & & & \\ & & \frac{\delta}{\delta \alpha_m} & & & \\ \frac{\delta}{\delta \beta_1} & & & & & \\ & \ddots & & & & \\ & & \frac{\delta}{\delta \beta_m} & & & \end{pmatrix} \int -\sum_{k=1}^m \sum_{j \geq 0} B_k^{(j)} \lambda^{-j} dx$$

$$= \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \right) \begin{pmatrix} \sum_{j \geq 0} C_1^{(j)} \lambda^{-j} & & & & & \\ & \ddots & & & & \\ & & \sum_{j \geq 0} C_m^{(j)} \lambda^{-j} & & & \\ \sum_{j \geq 0} A_1^{(j)} \lambda^{-j} & & & \ddots & & \\ & & & & \sum_{j \geq 0} A_m^{(j)} \lambda^{-j} & \\ \sum_{j \geq 0} \delta_1^{(j)} \lambda^{-j} & & & & & \ddots \\ & & & & & & \sum_{j \geq 0} \delta_m^{(j)} \lambda^{-j} \\ -\sum_{j \geq 0} \rho_1^{(j)} \lambda^{-j} & & & & & & & \ddots \\ & & & & & & & & -\sum_{j \geq 0} \rho_m^{(j)} \lambda^{-j} \end{pmatrix}. \quad (42)$$

Equating the coefficients of λ^{-j-2} in the above equality, we have

$$\begin{pmatrix} \frac{\delta}{\delta r_1} & \cdots & \frac{\delta}{\delta r_m} \\ \frac{\delta}{\delta s_1} & \cdots & \frac{\delta}{\delta s_m} \\ \frac{\delta}{\delta \alpha_1} & \cdots & \frac{\delta}{\delta \alpha_m} \\ \frac{\delta}{\delta \beta_1} & \cdots & \frac{\delta}{\delta \beta_m} \end{pmatrix} \int -\sum_{k=1}^m B_k^{(j+2)} dx$$

$$= (\gamma - j - 1) \begin{pmatrix} C_1^{(j+1)} & \cdots & C_m^{(j+1)} \\ A_1^{(j+1)} & \cdots & A_m^{(j+1)} \\ \delta_1^{(j+1)} & \cdots & \delta_m^{(j+1)} \\ -\rho_1^{(j+1)} & \cdots & -\rho_m^{(j+1)} \end{pmatrix}.$$

Let $j = 0$, and we have $\gamma = 0$. Thus, we obtain

$$\begin{pmatrix} C_1^{(j+1)} & \cdots & C_m^{(j+1)} \\ A_1^{(j+1)} & \cdots & A_m^{(j+1)} \\ \delta_1^{(j+1)} & \cdots & \delta_m^{(j+1)} \\ -\rho_1^{(j+1)} & \cdots & -\rho_m^{(j+1)} \end{pmatrix} = \begin{pmatrix} \frac{\delta}{\delta r_1} & \cdots & \frac{\delta}{\delta r_m} \\ \frac{\delta}{\delta s_1} & \cdots & \frac{\delta}{\delta s_m} \\ \frac{\delta}{\delta \alpha_1} & \cdots & \frac{\delta}{\delta \alpha_m} \\ \frac{\delta}{\delta \beta_1} & \cdots & \frac{\delta}{\delta \beta_m} \end{pmatrix} \tilde{H}_j,$$

$$\tilde{H}_j = \int \frac{1}{j+1} \sum_{k=1}^m B_k^{(j+2)} dx.$$

Therefore, (39) or (40) can be written in the following super bi-Hamiltonian structure:

$$\begin{pmatrix} r_1 & \cdots & r_m \\ s_1 & \cdots & s_m \\ \alpha_1 & \cdots & \alpha_m \\ \beta_1 & \cdots & \beta_m \end{pmatrix}_{t_n} = \tilde{J} \begin{pmatrix} C_1^{(n+1)} & \cdots & C_m^{(n+1)} \\ A_1^{(n+1)} & \cdots & A_m^{(n+1)} \\ \delta_1^{(n+1)} & \cdots & \delta_m^{(n+1)} \\ -\rho_1^{(n+1)} & \cdots & -\rho_m^{(n+1)} \end{pmatrix} \quad (45)$$

$$= \tilde{J} \frac{\delta}{\delta \bar{u}} \tilde{H}_n = \tilde{J} L \frac{\delta}{\delta \bar{u}} \tilde{H}_{n-1},$$

where $\tilde{J} = \begin{pmatrix} 0 & 2E & 0 & 0 \\ -2E & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}$ is the supersymplectic operator.

Example 2. Let $m = 2$, $n = 2$ in (45), and we obtain two-component super Dirac equations:

$$\begin{aligned} r_{1,t_2} &= -\frac{1}{2} s_{1,xx} + r_1^2 s_1 + s_1^3 \\ &\quad + 2s_1 \alpha_1 \beta_1 - \alpha_1 \alpha_{1,x} + \beta_1 \beta_{1,x}, \\ r_{2,t_2} &= -\frac{1}{2} s_{2,xx} + r_2^2 s_2 + s_2^3 \\ &\quad + 2s_2 \alpha_2 \beta_2 - \alpha_2 \alpha_{2,x} + \beta_2 \beta_{2,x}, \\ s_{1,t_2} &= \frac{1}{2} r_{1,xx} - r_1^3 - r_1 s_1^2 - 2r_1 \alpha_1 \beta_1 \\ &\quad - \alpha_1 \beta_{1,x} + \alpha_{1,x} \beta_1, \\ s_{2,t_2} &= \frac{1}{2} r_{2,xx} - r_2^3 - r_2 s_2^2 - 2r_2 \alpha_2 \beta_2 \\ &\quad - \alpha_2 \beta_{2,x} + \alpha_{2,x} \beta_2, \\ \alpha_{1,t_2} &= -\beta_{1,xx} + r_1 \beta_{1,x} - s_1 \alpha_{1,x} - \frac{1}{2} \alpha_1 s_{1,x} \\ &\quad + \frac{1}{2} \beta_1 r_{1,x} + \frac{1}{2} r_1^2 \beta_1 + \frac{1}{2} s_1^2 \beta_1, \\ \alpha_{2,t_2} &= -\beta_{2,xx} + r_2 \beta_{2,x} - s_2 \alpha_{2,x} - \frac{1}{2} \alpha_2 s_{2,x} \\ &\quad + \frac{1}{2} \beta_2 r_{2,x} + \frac{1}{2} r_2^2 \beta_2 + \frac{1}{2} s_2^2 \beta_2, \\ \beta_{1,t_2} &= \alpha_{1,xx} + s_1 \beta_{1,x} + r_1 \alpha_{1,x} + \frac{1}{2} \alpha_1 r_{1,x} \\ &\quad - \frac{1}{2} r_1^2 \alpha_1 - \frac{1}{2} s_1^2 \alpha_1 + \frac{1}{2} \beta_1 s_{1,x}, \end{aligned}$$

$$\begin{aligned}\beta_{2,t_2} = & \alpha_{2,xx} + s_2\beta_{2,x} + r_2\alpha_{2,x} + \frac{1}{2}\alpha_2r_{2,x} \\ & - \frac{1}{2}r_2^2\alpha_2 - \frac{1}{2}s_2^2\alpha_2 + \frac{1}{2}\beta_2s_{2,x}.\end{aligned}\quad (46)$$

5. Conclusions and Discussions

Starting from the matrix Lie superalgebra $A(2m-1, m-1)$, we have, respectively, constructed multicomponent super AKNS equations (21) and multicomponent super Dirac equations (40). By making use of the supertrace identity (7), (21), and (40) have been rewritten as integrable super-Hamiltonian forms (26) and (45), respectively. Moreover, we believe that many multicomponent superintegrable equations can also be constructed, which may be helpful to many physical and mathematical researchers in their future work.

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