Nonlinear Super Integrable Couplings of Super Broer-Kaup-Kupershmidt Hierarchy and Its Super Hamiltonian Structures

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1. Introduction

With the development of soliton theory, super integrable systems associated with Lie super algebra have aroused growing attentions by many mathematicians and physicists. It was known that super integrable systems contained the odd variables, which would provide more prolific fields for mathematical researchers and physical ones. Several super integrable systems, including super AKNS hierarchy, super KdV hierarchy, super NLS-MKdV hierarchy, super Tu hierarchy, super Broer-Kaup-Kupershmidt hierarchy, have been studied in [1–8]. There are some interesting results on the super integrable systems, such as Darboux transformation in [9], super Hamiltonian structures in [10–12] binary nonlinearization in [13], and reciprocal transformation in [14].

The research of integrable couplings of the well-known integrable hierarchy has received considerable attention in [15–23]. A few approaches to construct linear integrable couplings of the classical soliton equation are presented by permutation, enlarging spectral problem, using matrix Lie algebra [24] constructing new loop Lie algebra and creating semidirect sums of Lie algebra. Recently, Ma [25] and Ma and Zhu [26] presented a scheme for constructing nonlinear continuous and discrete integrable couplings using the block type matrix algebra. However, there is one interesting question for us is how to generate nonlinear super integrable couplings for the super integrable hierarchy.

In this paper, we would like to construct nonlinear super integrable couplings of the super soliton equations through enlarging matrix Lie super algebra. We take the Lie algebra $\mathfrak{b}(0,1)$ as an example to illustrate the approach for extending Lie super algebras. Based on the enlarged Lie super algebra $\mathfrak{gl}(6,2)$, we work out nonlinear super integrable Hamiltonian couplings of the super Broer-Kaup-Kupershmidt hierarchy. Finally, we will reduce the nonlinear super Broer-Kaup-Kupershmidt integrable Hamiltonian couplings to some special cases.

2. Enlargement of Lie Super Algebra $\mathfrak{b}(0,1)$

Consider the Lie super algebra $\mathfrak{b}(0,1)$. Its basis is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
where $E_1$, $E_2$, and $E_3$ are even elements and $E_4$, and $E_5$ are odd elements. Their nonzero (anti)commutation relations are

\begin{align}
[E_1, E_2] &= 2E_2, \\
[E_1, E_3] &= -2E_3, \\
[E_1, E_4] &= E_4, \\
[E_2, E_3] &= E_1, \\
[E_2, E_4] &= -2E_2, \\
[E_3, E_4] &= E_5, \\
[E_4, E_5] &= 2E_3, \\
[E_5, E_3] &= 2E_5.
\end{align}

Let us enlarged the Lie super algebra $B(0, 1)$ to the Lie super algebra $gl(6, 2)$ with a basis

\begin{align}
e_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
e_8 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{align}

where $e_1, e_2, e_3, e_4, e_5,$ and $e_6$ are even, and $e_7$ and $e_8$ are odd.

The generator of Lie super algebra $gl(6, 2)$, $e_i$, $0 \leq i \leq 8$, satisfy the following (anti)commutation relations:

\begin{align}
[e_1, e_2] &= 2e_2, \\
[e_1, e_3] &= -2e_3, \\
[e_1, e_4] &= 2e_5, \\
[e_1, e_6] &= -2e_6, \\
[e_2, e_3] &= e_1, \\
[e_2, e_4] &= -2e_5, \\
[e_2, e_5] &= e_4, \\
[e_2, e_6] &= e_7, \\
[e_2, e_7] &= e_6, \\
[e_3, e_4] &= 2e_6, \\
[e_3, e_5] &= -e_4, \\
[e_3, e_7] &= e_8, \\
[e_4, e_5] &= 2e_5, \\
[e_4, e_6] &= e_4, \\
[e_4, e_8] &= 2e_3 - 2e_6, \\
[e_5, e_6] &= e_4, \\
[e_5, e_8] &= e_1 - e_4, \\
[e_6, e_8] &= 2e_3 - 2e_6, \\
[e_7, e_8] &= e_1 - e_4, \\
[e_7, e_4] &= 2e_5 - 2e_2, \\
\end{align}

Define a loop super algebra corresponding to the Lie superalgebra $gl(6, 2)$, denote by

\begin{align}
\mathfrak{gl}(6, 2) &= gl(6, 2) \otimes \mathbb{C}[\lambda, \lambda^{-1}] \\
&= \{ e, \lambda^m, e_i \in gl(6, 2), i = 1, 2, \ldots, 8, \ m = 0, \pm 1, \pm 2, \ldots \}.
(5)
\end{align}

The corresponding (anti)commutative relations are given as

\begin{align}
[e_i \lambda^m, e_j \lambda^n] = \{ e_i, e_j \} \lambda^{m+n}, \quad \forall e_i, e_j \in \mathfrak{gl}(6, 2). &
\end{align}

3. Nonlinear Super Integrable Couplings of Super Broer-Kaup-Kupershmidt Hierarchy

Let us start from an enlarged spectral problem associated with $gl(6, 2)$:

\begin{align}
\phi = U (u, \lambda) \phi, \\
U = e_1 (1 + re_1 (0) + se_2 (0) + te_3 (0)) \\
+ u_1 e_4 (0) + u_2 e_5 (0) + \alpha e_7 (0) + \beta e_8 (0) \\
\begin{pmatrix}
\lambda + r & s & u_1 & u_2 & \alpha \\
1 & -\lambda - r & 0 & -u_1 & \beta \\
0 & 0 & \lambda + r + u_1 & s + u_2 & 0 \\
0 & 0 & 1 & -\lambda - r - u_1 & 0 \\
\beta & -\alpha & -\beta & \alpha & 0
\end{pmatrix}.
\end{align}

where $r, s, u_1,$ and $u_2$ are even potentials but $\alpha$ and $\beta$ are odd ones.

In order to obtain super integrable couplings of super integrable hierarchy, we first solve the adjoint representation of (7),

\begin{align}
V = [U, V],
\end{align}

with

\begin{align}
V = A e_1 (0) + B e_2 (0) + C e_3 (0) + E e_4 (0) \\
+ F e_5 (0) + G e_6 (0) + P e_7 (0) + \delta e_8 (0) \\
\begin{pmatrix}
A & B & E & F & \rho \\
C & -A & G & -E & \delta \\
0 & 0 & A + E & B + F & 0 \\
0 & 0 & C + G & -A - E & 0 \\
\delta & -\rho & -\delta & \rho & 0
\end{pmatrix}.
\end{align}
where $A, B, C, E, F,$ and $G$ are commuting fields and $\rho,$ and $\delta$ are anticommuting fields. Then we obtain

$$A_x = -B + sC + \beta \rho + \alpha \delta,$$

$$B_x = -2sA + 2\lambda B + 2rB - 2\alpha \rho,$$

$$C_x = 2A - 2\lambda C - 2rC + 2\beta \delta,$$

$$E_x = u_2C - F + sG + u_2G - \alpha \delta - \beta \rho,$$

$$F_x = -2u_2A + 2u_1B - 2sE$$

$$- 2u_2E + 2\lambda F + 2rF + 2u_1E + 2\alpha \rho,$$

$$G_x = -2u_1C + 2E - 2\lambda G - 2rG - 2u_1G - 2\beta \delta.$$  

$$\rho_x = -\alpha A - \beta B + \lambda \rho + \rho_1 + s\delta,$$

$$\delta_x = \beta A - \alpha C + \rho - \lambda \delta - r\delta.$$  

Substituting

$$A = \sum_{m=0} A_m \lambda^{-m}, \quad B = \sum_{m=0} B_m \lambda^{-m},$$

$$C = \sum_{m=0} C_m \lambda^{-m}, \quad E = \sum_{m=0} E_m \lambda^{-m},$$

$$F = \sum_{m=0} F_m \lambda^{-m}, \quad G = \sum_{m=0} G_m \lambda^{-m},$$

$$\rho = \sum_{m=0} \rho_m \lambda^{-m}, \quad \delta = \sum_{m=0} \delta_m \lambda^{-m},$$

into the previous equation gives the following recursive formulas:

$$A_{m,x} = -B_m + sC_m + \beta \rho_m + \alpha \delta_m,$$

$$B_{m,x} = -2sA_m + 2B_{m+1} + 2rB_m - 2\alpha \rho_m,$$

$$C_{m,x} = 2A_m - 2C_{m+1} - 2rC_m + 2\beta \delta_m,$$

$$E_{m,x} = u_2 C_m - F_m + sG_m + u_2 G_m - \alpha \delta_m - \beta \rho_m,$$

$$F_{m,x} = -2u_2 A_m + 2u_1 B_m - 2sE_m$$

$$- 2u_2E_m + 2F_{m+1} + 2rF_m + 2u_1E_m + 2\alpha \rho_m,$$

$$G_{m,x} = -2u_1 C_m + 2E_m - 2G_{m+1} - 2rG_m - 2u_1 G_m - 2\beta \delta_m,$$

$$\rho_{m,x} = -\alpha A_m - \beta B_m + \rho_{m+1} + r \rho_m + s \delta_m,$$

$$\delta_{m,x} = \beta A_m - \alpha C_m + \rho_m - \delta_{m+1} - r \delta_m.$$  

$$\rho_1 = \alpha, \quad \delta_1 = \beta, \quad A_2 = -\frac{1}{2} s - \alpha \beta,$$

$$B_2 = \frac{1}{2} s_x - rs, \quad C_2 = -r,$$

$$E_2 = \frac{1}{2} u_2 - \frac{1}{2} s u_2 - \frac{1}{2} s^2 + \alpha \beta,$$

$$F_2 = \frac{1}{2} u_{2x} + \frac{1}{2} s u_{2x} + \frac{1}{2} s^2 - s u_1 - r u_2 - \delta u_2 - s \rho_2 - u_1 u_2 - s \rho_1,$$

$$G_2 = -u_1 - \delta u_1 - \epsilon,$$  

$$\rho_3 = \alpha_x - r \alpha,$$

$$\delta_2 = -\beta_x - r \beta, \quad A_3 = -\frac{1}{4} s + s + \alpha \beta_{x} - \alpha_x \beta + 2r \alpha \beta,$$

$$B_3 = \frac{1}{4} s_{xx} - \frac{1}{2} r s - s_{xx} - \frac{1}{2} s^2 - s \alpha \beta + r^2 s + \alpha \alpha_x,$$

$$C_3 = \frac{1}{2} r x - \frac{1}{2} s - \alpha \beta + r^2 - \beta \beta_x,$$

$$E_3 = (e + 1) \left( ru_2 + s u_1 + u_1 u_2 - \frac{1}{4} u_{2x} \right) + \epsilon s - \frac{1}{4} e s_x$$

$$- \alpha \beta x + \alpha_x \beta - 2r \alpha \beta,$$

$$F_3 = (e + 1)$$

$$\times \left( -\frac{1}{1} s u_{1x} - s_{x} u_1 - \frac{1}{2} s u_{2x} - r u_{2x} - \frac{1}{2} u_{1x} u_2$$

$$- u_1 u_{2x} + \frac{1}{4} u_{2xx} - s u_2 - \frac{1}{2} u_{1x}^2 + r^2 u_{2x}$$

$$+ s u_1^2 + u_1 u_2 + 2r u_1 u_2 + 2r s u_1 \right)$$

$$+ s \alpha \beta + \frac{1}{4} e s_{xx} - \frac{1}{2} e s^2$$

$$- e s x - \frac{1}{2} e s_{x} s + e r^2 s - \alpha \alpha_x,$$

$$G_3 = \frac{1}{2} u_{1x} + \frac{1}{2} e u_{1x} + \frac{1}{2} e u_{x} + 2 r u_1 - \frac{1}{2} u_2$$

$$- \frac{1}{2} e u_2 - \frac{1}{2} s \alpha \beta + 2e u_1 + e r^2 + u_2 + u_1^2 + s \beta \beta_x,$$

$$\rho_5 = \alpha_{xx} - 2r \alpha \alpha - r \delta$$

$$\delta_3 = \beta_{xx} + r \beta_{x} + 2r \beta_{x} - \frac{1}{2} s \beta + \alpha_x + r^2 \beta.$$  

Equations (12) can be written as

$$\begin{pmatrix}
4A_{m+1} + 2E_{m+1} \\
2C_{m+1} + G_{m+1} \\
2A_{m+1} + 2E_{m+1} \\
C_{m+1} + G_{m+1}
\end{pmatrix}
= L
\begin{pmatrix}
4A_m + 2E_m \\
2C_m + G_m \\
2A_m + 2E_m \\
2C_m + G_m
\end{pmatrix},$$

where
with
\[
L_{33} = \frac{1}{2} \partial - \partial^{-1} r \partial - \partial^{-1} u_1 \partial,
\]
\[
L_{34} = -s - u_2 - \partial^{-1} s \partial - \partial^{-1} u_2 \partial.
\] (16)

Then, let us consider the spectral problem (7) with the following auxiliary spectral problem:
\[
\phi_{t_n} = V^{(n)} \phi,
\] (17)

where
\[
V^{(n)} = \sum_{j=0}^{n} \left( \begin{array}{cccc}
A_j & B_j & E_j & F_j \\
C_j & -A_j & G_j & -E_j \\
0 & 0 & A_j + E_j & B_j + F_j \\
0 & 0 & C_j + G_j & -A_j - E_j \\
\delta_j - \rho_j & -\delta_j & \rho_j & 0
\end{array} \right) \lambda^{n-j},
\]
\[
-(-C_{n+1} + 0 - G_{n+1} + 0) + \left( \begin{array}{cccc}
0 & 0 & -C_{n+1} + G_{n+1} + 0 \\
0 & 0 & 0 & C_{n+1} + G_{n+1} \\
0 & 0 & 0 & 0
\end{array} \right) \lambda^{n-j}.
\] (18)

From the compatible condition, \( \phi_{x_{t_n}} = \phi_{t_n x} \), according to (7) and (17), we get the zero curvature equation:
\[
U_{t_n} - V^{(n)} + [U, V^{(n)}] = 0,
\] (19)

which gives a nonlinear Lax super integrable hierarchy:
\[
\mathbf{u}_{t_n} = \left( \begin{array}{c}
\frac{r}{s} \\
u_1 \\
u_2 \\
\frac{\alpha}{\beta}
\end{array} \right) t_n
\]
\[
= \left( \begin{array}{c}
-C_{n+1,x} \\
2B_{n+1} - 2s C_{n+1} \\
-G_{n+1,x} \\
-2u_2 C_{n+1} + 2C_{n+1} + 2s C_{n+1} - 2u_2 G_{n+1} \\
-\rho_{n+1} + \alpha C_{n+1} \\
-\delta_{n+1} + \beta C_{n+1}
\end{array} \right).
\] (20)

The super integrable hierarchy (20) is a nonlinear super integrable couplings for the super BKK hierarchy:
\[
\mathbf{\bar{u}}_n = \left( \begin{array}{c}
\frac{r}{s} \\
u_1 \\
u_2 \\
\frac{\alpha}{\beta}
\end{array} \right) t_n
\]
\[
= \left( \begin{array}{c}
-\frac{C_{n+1,x}}{2} + 2B_{n+1} - 2s C_{n+1} \\
-\frac{C_{n+1}}{2} + 2B_{n+1} - 2s C_{n+1} - 2u_2 G_{n+1} \\
-\rho_{n+1} + \alpha C_{n+1} \\
-\delta_{n+1} + \beta C_{n+1}
\end{array} \right).
\] (21)

4. Super Hamiltonian Structures

A direct calculation reads
\[
\text{Str}(U_{\lambda}, V) = 4A + 2E,
\]
\[
\text{Str}(U_{\nu}, V) = 2C + G,
\]
\[
\text{Str}(U_{\nu}, V) = 2B + F,
\]\n\[
\text{Str}(U_{\alpha}, V) = 2\delta,
\]
\[
\text{Str}(U_{\beta}, V) = -2\rho.
\] (22)

Substituting the above results into the super trace identity in [11, 12],
\[
\delta \frac{\delta}{\delta u} \int \text{Str} \left( \frac{\delta U}{\delta \lambda} \frac{\delta V}{\delta \lambda} \right) dx = \lambda^{-\gamma} \frac{\delta}{\delta \lambda} \lambda^\gamma \text{Str} \left( \frac{\delta U}{\delta u} \frac{\delta V}{\delta u} \right),
\] (23)

yields that
\[
\delta \frac{\delta}{\delta u} \int (4A + 2E) dx = \lambda^{-\gamma} \frac{\delta}{\delta \lambda} \lambda^\gamma \left( \begin{array}{c}
4A + 2E \\
2C + G \\
2B + 2F \\
2A + 2E \\
2\delta \\
-2\rho
\end{array} \right),
\] (24)

Comparing the coefficients of \( \lambda^{-m} \) on both side of (24),
\[
\delta \frac{\delta}{\delta u} \int (4A_{n+1} + 2E_{n+1}) dx = \lambda^{-\gamma} \frac{\delta}{\delta \lambda} \lambda^\gamma \left( \begin{array}{c}
4A_{n+1} + 2E_{n+1} \\
2C_{n+1} + G_{n+1} \\
2A_{n+1} + 2E_{n+1} \\
2\delta_{n+1} \\
-2\rho_{n+1}
\end{array} \right) + \left( \begin{array}{c}
4A + 2E \\
2C + G \\
2B + 2F \\
2A + 2E \\
2\delta \\
-2\rho
\end{array} \right),
\] (25)
From the initial values in (12), we obtain $\gamma = 0$. Thus we have

$$H_n = \begin{pmatrix}
4A_n + 2E_n \\
2C_n + G_n \\
2\delta_n - 2\rho_n
\end{pmatrix}, \quad n = 0, 1, 2, 3, \ldots$$

\[(26)\]

$$H_n = - \int_{n+1}^{n+2} (4A_n + 2E_n) \, dx, \quad n \geq 0.$$  

\[(26)\]

It then follows that the nonlinear super integrable couplings (20) possess the following super Hamiltonian form

$$u_t = K_n(u) = J \delta H_n \delta u,$$  

\[(27)\]

where

$$J = \begin{pmatrix}
0 & -\partial & 0 & \partial & 0 & 0 \\
-\partial & 0 & \partial & 0 & \alpha & -\beta \\
0 & \partial & 0 & -2\partial & 0 & 0 \\
\partial & 0 & -2\partial & 0 & -\alpha & \beta \\
0 & -\alpha & 0 & \alpha & 0 & -\frac{1}{2} \\
0 & \beta & 0 & -\beta & 0 & -\frac{1}{2}
\end{pmatrix}.$$  

\[(28)\]

is a super Hamiltonian operator and $H_n (n \geq 0)$ are Hamiltonian functions. It can be verified that

$$J^* L = L^* J^*,$$

\[(29)\]

where $J^*$ and $L^*$ are conjugate operators of $J$ and $L$, respectively. If we define the following Poisson bracket

$$[H_n, H_m] = \left( \frac{\delta H_n}{\delta u}, \frac{\delta H_m}{\delta u} \right),$$

\[(30)\]

where $(\cdot, \cdot)$ denotes the inner product, as $(\delta H_n/\delta \alpha)(\delta H_m/\delta \beta) = -(\delta H_m/\delta \beta)(\delta H_n/\delta \alpha)$, we can obtain

$$[H_n, H_m] = \left( \frac{\delta H_n}{\delta r}, \frac{\delta H_n}{\delta s}, \frac{\delta H_n}{\delta u_1}, \frac{\delta H_n}{\delta u_2}, \frac{\delta H_n}{\delta \alpha}, \frac{\delta H_n}{\delta \beta} \right) \times J^* \left( \frac{\delta H_m}{\delta u_1}, \frac{\delta H_m}{\delta u_2}, \frac{\delta H_m}{\delta \alpha}, \frac{\delta H_m}{\delta \beta} \right) dx = - [H_m, H_n].$$

\[(31)\]

From (14) and (31), we have

$$\frac{\delta H_{n+1}}{\delta u} = L \frac{\delta H_n}{\delta u}, \quad \frac{\delta H_m}{\delta u} = L^m \frac{\delta H_0}{\delta u}.\]  

\[(32)\]

Suppose $k, l$ are arbitrary nonnegative integers consider

$$[H_k, H_{k+l}] = \left( L^k \frac{\delta H_0}{\delta u}, L^{k+l} \frac{\delta H_0}{\delta u} \right).$$

\[(33)\]
By using the properties of $J$ and $L$ in (29), we can obtain

$$\left[ H_k, H_{k+l} \right] = \left( L^k \frac{\delta H_0}{\delta u}, (JL)^{k+l-1} \frac{\delta H_0}{\delta u} \right)$$

$$= \left( (JL)^* L^k \frac{\delta H_0}{\delta u}, L^{k+l-1} \frac{\delta H_0}{\delta u} \right)$$

$$= \left( J^* L^{k+l} \frac{\delta H_0}{\delta u}, L^{k+l-1} \frac{\delta H_0}{\delta u} \right)$$

$$= \left( \frac{\delta H_{k+l}}{\delta u}, \frac{\delta H_{k+l-1}}{\delta u} \right) = \left[ H_{k+l}, H_{k+l-1} \right].$$

(34)

Ascending the subscript of left factor $H_k$, then we have

$$\left[ H_k, H_{k+l} \right] = \left[ H_{k+l}, H_k \right].$$

(35)

By using the antisymmetric law of Poisson bracket (31), we have

$$\left[ H_k, H_{k+l} \right] = 0.$$

(36)

So the conserved functionals $[H_n]_{n=1}^\infty$ are in conservation in pairs under the Poisson bracket (30).

5. Reductions

Taking $\alpha = \beta = 0$, the hierarchy (27) reduces to a nonlinear integrable couplings of the Broer-Kaup-Kupershmidt hierarchy.

When $n = 2$ in (27), we obtain the nonlinear super couplings of the second super Broer-Kaup-Kupershmidt equations:

$$r_{12} = -\frac{1}{2} r_{xx} - 2 r_{x} + \frac{1}{2} s_{x} + \alpha_{x} \beta + \alpha \beta_{x},$$

$$s_{12} = \frac{1}{2} s_{xx} - 2 r_{s} - 2 r s + 2 \alpha \alpha_{x} + 2 s \beta_{x},$$

$$u_{12} = (\epsilon + 1) \left( -\frac{1}{2} u_{1xx} - 2 r_{su} u_{1} - 2 ru_{1s} + \frac{1}{2} u_{2x} - 2 u_{1xx} - 2 r u_{2s} + \frac{1}{2} r_{xx} - 2 r_{s} \right)$$

$$+ \frac{1}{2} r_{xx} - 2 s_{x} - \alpha \beta - \alpha \beta_{x} + 2 r s - \beta \beta_{x},$$

$$u_{2J_2} = (\epsilon + 1) \left( -\frac{1}{2} u_{2xx} - 2 s_{u} u_{1} - 2 su_{1s} - 2 r_{u} u_{2s} - 2 u_{1xx} - 2 r_{u} u_{2s} + \frac{1}{2} r_{xx} - 2 r_{s} \right)$$

$$+ \frac{1}{2} r_{xx} - 2 s_{x} - \alpha \beta - \alpha \beta_{x} + 2 r s - \beta \beta_{x}.$$

(37)

Particularly, taking $\alpha = \beta = 0$ in (37), we can get the nonlinear integrable couplings of the second order Broer-Kaup-Kupershmidt equations:

$$r_{2} = -\frac{1}{2} r_{xx} - 2 r_{x} + \frac{1}{2} s_{x},$$

$$s_{2} = \frac{1}{2} s_{xx} - 2 r_{s} - 2 r s + 2 \alpha \alpha_{x} + 2 s \beta_{x},$$

$$u_{12} = (\epsilon + 1) \left( -\frac{1}{2} u_{1xx} - 2 r_{su} u_{1} - 2 ru_{1s} + \frac{1}{2} u_{2x} - 2 u_{1xx} - 2 r u_{2s} + \frac{1}{2} r_{xx} - 2 r_{s} \right)$$

$$+ \frac{1}{2} r_{xx} - 2 s_{x} - \alpha \beta - \alpha \beta_{x} + 2 r s - \beta \beta_{x},$$

$$u_{2J_2} = (\epsilon + 1) \left( -\frac{1}{2} u_{2xx} - 2 s_{u} u_{1} - 2 su_{1s} - 2 r_{u} u_{2s} - 2 u_{1xx} - 2 r_{u} u_{2s} + \frac{1}{2} r_{xx} - 2 r_{s} \right)$$

$$+ \frac{1}{2} r_{xx} - 2 s_{x} - \alpha \beta - \alpha \beta_{x} + 2 r s - \beta \beta_{x}.$$

(38)
\[\begin{align*}
\alpha_t &= \alpha_{xx} - 2r\alpha_x - \frac{3}{2}r_x\alpha + \frac{1}{2}s_x\beta + s\beta_x + \alpha\beta_x, \\
\beta_t &= -\beta_{xx} - \frac{1}{2}r_x\beta - 2r\beta_x - \alpha_x.
\end{align*}\]

If setting \( \varepsilon = -1, u_1 = -r, \) and \( u_2 = -s \) in (37), we obtain the second order super Broer-Kaup-Kupershmidt equations:

\[\begin{align*}
r_t &= -r_{xx} - 2rr_x + \frac{1}{2}s_x\beta + \alpha\beta_x + \beta_{xx}, \\
\alpha_t &= -2r\alpha_x - \frac{3}{2}r_x\alpha + \frac{1}{2}s_x\beta + s\beta_x + \alpha\beta_x, \\
\beta_t &= -\beta_{xx} - \frac{1}{2}r_x\beta - 2r\beta_x - \alpha_x.
\end{align*}\]

6. Remarks

In this paper, we introduced an approach for constructing nonlinear integrable couplings of super integrable hierarchy. Zhang [27] once employed two kinds of explicit Lie algebra \( F \) and \( G \) to obtain the nonlinear integrable couplings of the \( GJ \) hierarchy and Yang hierarchy, respectively. It is easy to see that Lie algebra \( F \) given in [27] is isomorphic to the Lie algebra span \( \{e_1, e_2, e_3, e_4, e_5 \} \) in \( gl(6|2) \). So we can obtain nonlinear integrable couplings of super \( G \) and Yang hierarchy easily. The method in this paper can be applied to other super integrable systems for constructing their integrable couplings.

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References


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