Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2013, Article ID 520765, 8 pages
http://dx.doi.org/10.1155/2013/520765

Research Article

Nonlinear Super Integrable Couplings of Super Broer-Kaup-Kupershmidt Hierarchy and Its Super Hamiltonian Structures

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Received 25 February 2013; Revised 26 July 2013; Accepted 9 August 2013

Academic Editor: Xing Biao Hu

Nonlinear integrable couplings of super Broer-Kaup-Kupershmidt hierarchy based upon an enlarged matrix Lie super algebra were constructed. Then its super Hamiltonian structures were established by using super trace identity, and the conserved functionals were proved to be in involution in pairs under the defined Poisson bracket. As its reduction, nonlinear integrable couplings of the classical integrable hierarchy were obtained.

1. Introduction

With the development of soliton theory, super integrable systems associated with Lie super algebra have aroused growing attentions by many mathematicians and physicists. It was known that super integrable systems contained the odd variables, which would provide more prolific fields for mathematical researchers and physical ones. Several super integrable systems, including super AKNS hierarchy, super KdV hierarchy, super NLS-MKdV hierarchy, super Tu hierarchy, super Broer-Kaup-Kupershmidt hierarchy, have been studied in [1–8]. There are some interesting results on the super integrable systems, such as Darboux transformation in [9], super Hamiltonian structures in [10–12] binary nonlinearization in [13], and reciprocal transformation in [14].

The research of integrable couplings of the well-known integrable hierarchy has received considerable attention in [15–23] A few approaches to construct linear integrable couplings of the classical soliton equation are presented by permutation, enlarging spectral problem, using matrix Lie algebra [24] constructing new loop Lie algebra and creating semidirect sums of Lie algebra. Recently, Ma [25] and Ma and Zhu [26] presented a scheme for constructing nonlinear continuous and discrete integrable couplings using the block type matrix algebra. However, there is one interesting question for us is how to generate nonlinear super integrable couplings for the super integrable hierarchy.

In this paper, we would like to construct nonlinear super integrable couplings of the super soliton equations through enlarging matrix Lie super algebra. We take the Lie algebra $B(0,1)$ as an example to illustrate the approach for extending Lie super algebras. Based on the enlarged Lie super algebra $gl(6,2)$, we work out nonlinear super integrable Hamiltonian couplings of the super Broer-Kaup-Kupershmidt hierarchy. Finally, we will reduce the nonlinear super Broer-Kaup-Kupershmidt integrable Hamiltonian couplings to some special cases.

2. Enlargement of Lie Super Algebra $B(0,1)$

Consider the Lie super algebra $B(0,1)$. Its basis is

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$
$$E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1}$$

where E_1, E_2, and E_3 are even elements and E_4, and E_5 are odd elements. Their nonzero (anti)commutation relations are

$$\begin{aligned} &[E_1, E_2] = 2E_2, \quad [E_1, E_3] = -2E_3, \quad [E_1, E_4] = E_4, \\ &[E_1, E_5] = -E_5, \quad [E_2, E_3] = E_1, \quad [E_2, E_5] = E_4, \\ &[E_3, E_4] = E_5, \quad [E_4, E_4] = -2E_2, \quad [E_4, E_5] = E_1, \\ &[E_5, E_5] = 2E_3. \end{aligned} \tag{2}$$

Let us enlarged the Lie super algebra $B(0,1)$ to the Lie super algebra $gl(6,2)$ with a basis

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$e_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$e_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}, \qquad e_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}, \tag{3}$$

where e_1, e_2, e_3, e_4, e_5, and e_6 are even, and e_7, and e_8 are odd.

The generator of Lie super algebra $gl(6,2)$, e_i, $0 \le i \le 8$, satisfy the following (anti)commutation relations:

$$\begin{aligned} &[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_1, e_5] = 2e_5, \\ &[e_1, e_6] = -2e_6, \quad [e_1, e_7] = e_7, \quad [e_1, e_8] = -e_8, \\ &[e_2, e_3] = e_1, \quad [e_2, e_4] = -2e_5, \quad [e_2, e_6] = e_4, \\ &[e_2, e_8] = e_7, \quad [e_3, e_4] = 2e_6, \quad [e_3, e_5] = -e_4, \\ &[e_3, e_7] = e_8, \quad [e_4, e_5] = 2e_5, \quad [e_4, e_6] = -2e_6, \\ &[e_5, e_6] = e_4, \quad [e_7, e_8] = e_1 - e_4, \quad [e_7, e_7] = 2e_5 - 2e_2, \\ &[e_8, e_8] = 2e_3 - 2e_6, \\ &[e_1, e_4] = [e_2, e_5] = [e_2, e_7] = [e_3, e_6] = [e_3, e_8] \\ &\qquad = [e_4, e_7] = [e_4, e_8] = [e_5, e_7] = [e_5, e_8] \\ &\qquad = [e_6, e_7] = [e_6, e_8] = 0. \end{aligned} \tag{4}$$

Define a loop super algebra corresponding to the Lie superalgebra $gl(6,2)$, denote by

$$\begin{aligned} \tilde{g}l(6,2) &= gl(6,2) \otimes \mathbb{C}\left[\lambda, \lambda^{-1}\right] \\ &= \{e_i \lambda^m,\ e_i \in gl(6,2),\ i = 1, 2, \ldots, 8, \\ &\qquad m = 0, \pm 1, \pm 2, \ldots\}. \end{aligned} \tag{5}$$

The corresponding (anti)commutative relations are given as

$$\left[e_i \lambda^m, e_j \lambda^n\right] = \left[e_i, e_j\right] \lambda^{m+n}, \quad \forall e_i, e_j \in gl(6,2). \tag{6}$$

3. Nonlinear Super Integrable Couplings of Super Broer-Kaup-Kupershmidt Hierarchy

Let us start from an enlarged spectral problem associated with $gl(6,2)$:

$$\begin{aligned} &\phi_x = U(u, \lambda)\phi, \\ &U = e_1(1) + re_1(0) + se_2(0) + e_3(0) \\ &\quad + u_1 e_4(0) + u_2 e_5(0) + \alpha e_7(0) + \beta e_8(0) \\ &= \begin{pmatrix} \lambda + r & s & u_1 & u_2 & \alpha \\ 1 & -\lambda - r & 0 & -u_1 & \beta \\ 0 & 0 & \lambda + r + u_1 & s + u_2 & 0 \\ 0 & 0 & 1 & -\lambda - r - u_1 & 0 \\ \beta & -\alpha & -\beta & \alpha & 0 \end{pmatrix}, \end{aligned} \tag{7}$$

where r, s, u_1, and u_2 are even potentials but α and β are odd ones.

In ordr to obtain super integrable couplings of super integrable hierarchy, we first solve the adjoint representation of (7),

$$V_x = [U, V], \tag{8}$$

with

$$\begin{aligned} V &= Ae_1(0) + Be_2(0) + Ce_3(0) + Ee_4(0) \\ &\quad + Fe_5(0) + Ge_6(0) + \rho e_7(0) + \delta e_8(0) \\ &= \begin{pmatrix} A & B & E & F & \rho \\ C & -A & G & -E & \delta \\ 0 & 0 & A + E & B + F & 0 \\ 0 & 0 & C + G & -A - E & 0 \\ \delta & -\rho & -\delta & \rho & 0 \end{pmatrix}, \end{aligned} \tag{9}$$

where A, B, C, E, F, and G are commuting fields and ρ, and δ are anticommuting fields. Then we obtain

$$\begin{aligned}
A_x &= -B + sC + \beta\rho + \alpha\delta,\\
B_x &= -2sA + 2\lambda B + 2rB - 2\alpha\rho,\\
C_x &= 2A - 2\lambda C - 2rC + 2\beta\delta,\\
E_x &= u_2C - F + sG + u_2G - \alpha\delta - \beta\rho,\\
F_x &= -2u_2A + 2u_1B - 2sE\\
&\quad - 2u_2E + 2\lambda F + 2rF + 2u_1F + 2\alpha\rho,\\
G_x &= -2u_1C + 2E - 2\lambda G - 2rG - 2u_1G - 2\beta\delta,\\
\rho_x &= -\alpha A - \beta B + \lambda\rho + r\rho + s\delta,\\
\delta_x &= \beta A - \alpha C + \rho - \lambda\delta - r\delta.
\end{aligned} \tag{10}$$

Substituting

$$\begin{aligned}
A &= \sum_{m\geq 0} A_m\lambda^{-m}, & B &= \sum_{m\geq 0} B_m\lambda^{-m},\\
C &= \sum_{m\geq 0} C_m\lambda^{-m}, & E &= \sum_{m\geq 0} E_m\lambda^{-m},\\
F &= \sum_{m\geq 0} F_m\lambda^{-m}, & G &= \sum_{m\geq 0} G_m\lambda^{-m},\\
\rho &= \sum_{m\geq 0} \rho_m\lambda^{-m}, & \delta &= \sum_{m\geq 0} \delta_m\lambda^{-m},
\end{aligned} \tag{11}$$

into the previous equation gives the following recursive formulas:

$$\begin{aligned}
A_{m,x} &= -B_m + sC_m + \beta\rho_m + \alpha\delta_m,\\
B_{m,x} &= -2sA_m + 2B_{m+1} + 2rB_m - 2\alpha\rho_m,\\
C_{m,x} &= 2A_m - 2C_{m+1} - 2rC_m + 2\beta\delta_m,\\
E_{m,x} &= u_2C_m - F_m + sG_m + u_2G_m - \alpha\delta_m - \beta\rho_m,\\
F_{m,x} &= -2u_2A_m + 2u_1B_m - 2sE_m\\
&\quad - 2u_2E_m + 2F_{m+1} + 2rF_m + 2u_1F_m + 2\alpha\rho_m,\\
G_{m,x} &= -2u_1C_m + 2E_m - 2G_{m+1} - 2rG_m - 2u_1G_m - 2\beta\delta_m,\\
\rho_{m,x} &= -\alpha A_m - \beta B_m + \rho_{m+1} + r\rho_m + s\delta_m,\\
\delta_{m,x} &= \beta A_m - \alpha C_m + \rho_m - \delta_{m+1} - r\delta_m.
\end{aligned} \tag{12}$$

From these equations, we can successively deduce

$$\begin{aligned}
&A_0 = 1, \quad B_0 = C_0 = F_0 = G_0 = \rho_0 = \delta_0 = 0,\\
&E_0 = \varepsilon = \text{constant}, \quad A_1 = 0, \quad B_1 = s, \quad C_1 = 1,\\
&E_1 = 0, \quad F_1 = u_2 + \varepsilon u_2 + \varepsilon s, \quad G_1 = \varepsilon,\\
&\rho_1 = \alpha, \quad \delta_1 = \beta, \quad A_2 = -\frac{1}{2}s - \alpha\beta,\\
&B_2 = \frac{1}{2}s_x - rs, \quad C_2 = -r,\\
&E_2 = -\frac{1}{2}u_2 - \frac{1}{2}\varepsilon u_2 - \frac{1}{2}\varepsilon s + \alpha\beta,\\
&F_2 = \frac{1}{2}u_{2x} + \frac{1}{2}\varepsilon u_{2x} + \frac{1}{2}\varepsilon s_x - su_1\\
&\quad - ru_2 - \varepsilon ru_2 - \varepsilon rs - u_1u_2 - \varepsilon u_1u_2 - \varepsilon su_1,\\
&G_2 = -u_1 - \varepsilon u_1 - \varepsilon r, \quad \rho_2 = \alpha_x - r\alpha,\\
&\delta_2 = -\beta_x - r\beta, \quad A_3 = -\frac{1}{4}s_x + rs + \alpha\beta_x - \alpha_x\beta + 2r\alpha\beta,\\
&B_3 = \frac{1}{4}s_{xx} - \frac{1}{2}r_xs - rs_x - \frac{1}{2}s^2 - s\alpha\beta + r^2s + \alpha\alpha_x,\\
&C_3 = \frac{1}{2}r_x - \frac{1}{2}s - \alpha\beta + r^2 - \beta\beta_x,\\
&E_3 = (\varepsilon + 1)\left(ru_2 + su_1 + u_1u_2 - \frac{1}{4}u_{2x}\right) + \varepsilon rs - \frac{1}{4}\varepsilon s_x\\
&\quad - \alpha\beta_x + \alpha_x\beta - 2r\alpha\beta,\\
&F_3 = (\varepsilon + 1)\\
&\quad \times\left(-\frac{1}{2}su_{1x} - s_xu_1 - \frac{1}{2}r_xu_2 - ru_{2x} - \frac{1}{2}u_{1x}u_2\right.\\
&\qquad - u_1u_{2x} + \frac{1}{4}u_{2xx} - su_2 - \frac{1}{2}u_2^2 + r^2u_2\\
&\qquad \left. + su_1^2 + u_1^2u_2 + 2ru_1u_2 + 2rsu_1\right)\\
&\quad + s\alpha\beta + \frac{1}{4}\varepsilon s_{xx} - \frac{1}{2}\varepsilon s^2\\
&\quad - \varepsilon rs_x - \frac{1}{2}\varepsilon r_xs + \varepsilon r^2s - \alpha\alpha_x,\\
&G_3 = \frac{1}{2}u_{1x} + \frac{1}{2}\varepsilon u_{1x} + \frac{1}{2}\varepsilon r_x + 2ru_1 - \frac{1}{2}u_2\\
&\quad - \frac{1}{2}\varepsilon u_2 - \frac{1}{2}\varepsilon s + \alpha\beta + 2\varepsilon ru_1 + \varepsilon r^2 + u_1^2 + \varepsilon u_1^2 + \beta\beta_x,\\
&\rho_3 = \alpha_{xx} - 2r\alpha_x - r_x\alpha - \frac{1}{2}s\alpha + \frac{1}{2}s_x\beta + s\beta_x + r^2\alpha,\\
&\delta_3 = \beta_{xx} + r_x\beta + 2r\beta_x - \frac{1}{2}s\beta + \alpha_x + r^2\beta.
\end{aligned} \tag{13}$$

Equations (12) can be written as

$$\begin{pmatrix} 4A_{m+1} + 2E_{m+1}\\ 2C_{m+1} + G_{m+1}\\ 2A_{m+1} + 2E_{m+1}\\ C_{m+1} + G_{m+1}\\ 2\delta_{m+1}\\ -2\rho_{m+1} \end{pmatrix} = L \begin{pmatrix} 4A_m + 2E_m\\ 2C_m + G_m\\ 2A_m + 2E_m\\ C_m + G_m\\ 2\delta_m\\ -2\rho_m \end{pmatrix}, \tag{14}$$

where

$$L=\begin{pmatrix} \frac{1}{2}\partial-\partial^{-1}r\partial & -s-\partial^{-1}s\partial & -\partial^{-1}u_1\partial & -u_2-\partial^{-1}u_2\partial & -\frac{1}{2}\alpha-\partial^{-1}\alpha\partial & \frac{1}{2}\beta-\partial^{-1}\beta\partial \\ \frac{1}{2} & -\frac{1}{2}\partial-r & 0 & -2u_1 & \frac{1}{2}\beta & 0 \\ 0 & 0 & L_{33} & L_{34} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\partial-r-u_1 & 0 & 0 \\ \beta & -2\alpha & -\beta & \alpha & -\partial-r & -1 \\ -\alpha+\beta\partial & -2s\beta & \alpha-\beta\partial & 2s\beta & s+\alpha\beta & \partial-r \end{pmatrix}. \tag{15}$$

with

$$\begin{aligned} L_{33}&=\frac{1}{2}\partial-\partial^{-1}r\partial-\partial^{-1}u_1\partial, \\ L_{34}&=-s-u_2-\partial^{-1}s\partial-\partial^{-1}u_2\partial. \end{aligned} \tag{16}$$

Then, let us consider the spectral problem (7) with the following auxiliary spectral problem:

$$\phi_{t_n}=V^{(n)}\phi, \tag{17}$$

with

$$V^{(n)}=\sum_{j=0}^{n}\begin{pmatrix} A_j & B_j & E_j & F_j & \rho_j \\ C_j & -A_j & G_j & -E_j & \delta_j \\ 0 & 0 & A_j+E_j & B_j+F_j & 0 \\ 0 & 0 & C_j+G_j & -A_j-E_j & 0 \\ \delta_j & -\rho_j & -\delta_j & \rho_j & 0 \end{pmatrix}\lambda^{n-j} + \begin{pmatrix} -C_{n+1} & 0 & -G_{n+1} & 0 & 0 \\ 0 & C_{n+1} & 0 & G_{n+1} & 0 \\ 0 & 0 & -C_{n+1}-G_{n+1} & 0 & 0 \\ 0 & 0 & 0 & C_{n+1}+G_{n+1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

From the compatible condition, $\phi_{x,t_n}=\phi_{t_n,x}$, according to (7) and (17), we get the zero curvature equation:

$$U_{t_n}-V_x^{(n)}+\left[U,V^{(n)}\right]=0, \tag{19}$$

which gives a nonlinear Lax super integrable hierarchy:

$$u_{t_n}=\begin{pmatrix} r \\ s \\ u_1 \\ u_2 \\ \alpha \\ \beta \end{pmatrix}_{t_n} = \begin{pmatrix} -C_{n+1,x} \\ 2B_{n+1}-2sC_{n+1} \\ -G_{n+1,x} \\ -2u_2C_{n+1}+2F_{n+1}-2sG_{n+1}-2u_2G_{n+1} \\ \rho_{n+1}-\alpha C_{n+1} \\ -\delta_{n+1}+\beta C_{n+1} \end{pmatrix}. \tag{20}$$

The super integrable hierarchy (20) is a nonlinear super integrable couplings for the super BKK hierarchy:

$$\tilde{u}_{t_n}=\begin{pmatrix} r \\ s \\ \alpha \\ \beta \end{pmatrix}_{t_n} = \begin{pmatrix} -C_{n+1,x} \\ 2B_{n+1}-2sC_{n+1} \\ \rho_{n+1}-\alpha C_{n+1} \\ -\delta_{n+1}+\beta C_{n+1} \end{pmatrix}. \tag{21}$$

4. Super Hamiltonian Structures

A direct calculation reads

$$\begin{aligned} &\operatorname{Str}(U_\lambda,V)=4A+2E, && \operatorname{Str}(U_r,V)=4A+2E, \\ &\operatorname{Str}(U_s,V)=2C+G, && \operatorname{Str}\left(U_{u_1},V\right)=C+G, \\ &\operatorname{Str}\left(U_{u_2},V\right)=B+F, && \operatorname{Str}(U_\alpha,V)=2\delta, \\ &\operatorname{Str}\left(U_\beta,V\right)=-2\rho. \end{aligned} \tag{22}$$

Substituting the above results into the super trace identity in [11, 12],

$$\frac{\delta}{\delta u}\int \operatorname{Str}\left(\frac{\delta U}{\delta\lambda}V\right)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\operatorname{Str}\left(\frac{\delta U}{\delta u}V\right) \tag{23}$$

yields that

$$\frac{\delta}{\delta u}\int(4A+2E)\,\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} 4A+2E \\ 2C+G \\ 2A+2E \\ C+G \\ 2\delta \\ -2\rho \end{pmatrix}. \tag{24}$$

Comparing the coefficients of λ^{-n-1} on both side of (24),

$$\frac{\delta}{\delta u}\int\left(4A_{n+1}+2E_{n+1}\right)\mathrm{d}x=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} 4A_n+2E_n \\ 2C_n+G_n \\ 2A_n+2E_n \\ C_n+G_n \\ 2\delta_n \\ -2\rho_n \end{pmatrix}, \quad n\geq 0. \tag{25}$$

From the initial values in (12), we obtain $\gamma = 0$. Thus we have

$$\frac{\mathcal{H}_n}{\delta u} = \begin{pmatrix} 4A_n + 2E_n \\ 2C_n + G_n \\ 2A_n + 2E_n \\ C_n + G_n \\ 2\delta_n \\ -2\rho_n \end{pmatrix}, \qquad \mathcal{H}_n = -\int \frac{1}{n+1}\left(4A_{n+1} + 2E_{n+1}\right) \mathrm{d}x, \quad n \geq 0. \tag{26}$$

It then follows that the nonlinear super integrable couplings (20) possess the following super Hamiltonian form

$$u_{t_n} = K_n(u) = J\frac{\delta \mathcal{H}_n}{\delta u}, \tag{27}$$

where

$$J = \begin{pmatrix} 0 & -\partial & 0 & \partial & 0 & 0 \\ -\partial & 0 & \partial & 0 & \alpha & -\beta \\ 0 & \partial & 0 & -2\partial & 0 & 0 \\ \partial & 0 & -2\partial & 0 & -\alpha & \beta \\ 0 & -\alpha & 0 & \alpha & 0 & -\frac{1}{2} \\ 0 & \beta & 0 & -\beta & -\frac{1}{2} & 0 \end{pmatrix}. \tag{28}$$

is a super Hamiltonian operator and $\mathcal{H}_n$ $(n \geq 0)$ are Hamiltonian functions.

It can be verified that

$$J^* = \begin{pmatrix} 0 & \partial & 0 & -\partial & 0 & 0 \\ \partial & 0 & -\partial & 0 & -\alpha & \beta \\ 0 & -\partial & 0 & 2\partial & 0 & 0 \\ -\partial & 0 & 2\partial & 0 & \alpha & -\beta \\ 0 & \alpha & 0 & -\alpha & 0 & -\frac{1}{2} \\ 0 & -\beta & 0 & \beta & -\frac{1}{2} & 0 \end{pmatrix}, \qquad J^* L = L^* J^*, \tag{29}$$

where J^* and L^* are conjugate operators of J and L, respectively.

If we define the following Poisson bracket

$$\begin{aligned} [\mathcal{H}_n, \mathcal{H}_m] &= \left(\frac{\delta \mathcal{H}_n}{\delta u}, J\frac{\delta \mathcal{H}_m}{\delta u}\right) \\ &= \int \left(\frac{\delta \mathcal{H}_n}{\delta r}, \frac{\delta \mathcal{H}_n}{\delta s}, \frac{\delta \mathcal{H}_n}{\delta u_1}, \frac{\delta \mathcal{H}_n}{\delta u_2}, \frac{\delta \mathcal{H}_n}{\delta \alpha}, \frac{\delta \mathcal{H}_n}{\delta \beta}\right) \\ &\quad \times J \begin{pmatrix} \frac{\delta \mathcal{H}_m}{\delta r} \\ \frac{\delta \mathcal{H}_m}{\delta s} \\ \frac{\delta \mathcal{H}_m}{\delta u_1} \\ \frac{\delta \mathcal{H}_m}{\delta u_2} \\ \frac{\delta \mathcal{H}_m}{\delta \alpha} \\ \frac{\delta \mathcal{H}_m}{\delta \beta} \end{pmatrix} \mathrm{d}x, \end{aligned} \tag{30}$$

where $(\cdot,\cdot)$ denotes the inner product, as $(\delta\mathcal{H}_n/\delta\alpha)(\delta\mathcal{H}_m/\delta\beta) = -(\delta\mathcal{H}_m/\delta\beta)(\delta\mathcal{H}_n/\delta\alpha)$, we can obtain

$$\begin{aligned} [\mathcal{H}_n, \mathcal{H}_m] &= \int \left(\frac{\delta \mathcal{H}_m}{\delta r}, \frac{\delta \mathcal{H}_m}{\delta s}, \frac{\delta \mathcal{H}_m}{\delta u_1}, \frac{\delta \mathcal{H}_m}{\delta u_2}, \frac{\delta \mathcal{H}_m}{\delta \alpha}, \frac{\delta \mathcal{H}_m}{\delta \beta}\right) \\ &\quad \times J^* \begin{pmatrix} \frac{\delta \mathcal{H}_n}{\delta r} \\ \frac{\delta \mathcal{H}_n}{\delta s} \\ \frac{\delta \mathcal{H}_n}{\delta u_1} \\ \frac{\delta \mathcal{H}_n}{\delta u_2} \\ \frac{\delta \mathcal{H}_n}{\delta \alpha} \\ \frac{\delta \mathcal{H}_n}{\delta \beta} \end{pmatrix} \mathrm{d}x = -\left[\mathcal{H}_m, \mathcal{H}_n\right]. \end{aligned} \tag{31}$$

From (14) and (31), we have

$$\frac{\delta \mathcal{H}_{m+1}}{\delta u} = L\frac{\delta \mathcal{H}_m}{\delta u}, \qquad \frac{\delta \mathcal{H}_m}{\delta u} = L^m \frac{\delta \mathcal{H}_0}{\delta u}. \tag{32}$$

Suppose k, l are arbitrary nonnegative integers consider

$$[\mathcal{H}_k, \mathcal{H}_{k+l}] = \left(L^k \frac{\delta \mathcal{H}_0}{\delta u}, JL^{k+l}\frac{\delta \mathcal{H}_0}{\delta u}\right). \tag{33}$$

By using the properties of J and L in (29), we can obtain

$$\begin{aligned}
&[\mathscr{H}_k, \mathscr{H}_{k+l}] \\
&= \left(L^k \frac{\delta \mathscr{H}_0}{\delta u}, JL^{k+l} \frac{\delta \mathscr{H}_0}{\delta u}\right) = \left(L^k \frac{\delta \mathscr{H}_0}{\delta u}, (JL)\, L^{k+l-1} \frac{\delta \mathscr{H}_0}{\delta u}\right) \\
&= \left((JL)^* L^k \frac{\delta \mathscr{H}_0}{\delta u}, L^{k+l-1} \frac{\delta \mathscr{H}_0}{\delta u}\right) \\
&= \left(L^* J^* L^k \frac{\delta \mathscr{H}_0}{\delta u}, L^{k+l-1} \frac{\delta \mathscr{H}_0}{\delta u}\right) \\
&= \left(J^* L^{k+1} \frac{\delta \mathscr{H}_0}{\delta u}, L^{k+l-1} \frac{\delta \mathscr{H}_0}{\delta u}\right) \\
&= \left(L^{k+1} \frac{\delta \mathscr{H}_0}{\delta u}, JL^{k+l-1} \frac{\delta \mathscr{H}_0}{\delta u}\right) \\
&= \left(\frac{\delta \mathscr{H}_{k+1}}{\delta u}, J \frac{\delta \mathscr{H}_{k+l-1}}{\delta u}\right) = [\mathscr{H}_{k+1}, \mathscr{H}_{k+l-1}].
\end{aligned} \tag{34}$$

Ascending the subscript of left factor $\mathscr{H}_k$, then we have

$$[\mathscr{H}_k, \mathscr{H}_{k+l}] = [\mathscr{H}_{k+l}, \mathscr{H}_k]. \tag{35}$$

By using the antisymmetric law of Poisson bracket (31), we have

$$[\mathscr{H}_k, \mathscr{H}_{k+l}] = 0. \tag{36}$$

So the conserved functionals $\{\mathscr{H}_n\}_1^\infty$ are in involution in pairs under the Poisson bracket (30).

5. Reductions

Taking $\alpha = \beta = 0$, the hierarchy (27) reduces to a nonlinear integrable couplings of the Broer-Kaup-Kupershmidt hierarchy.

When $n = 2$ in (27), we obtain the nonlinear super couplings of the second super Broer-Kaup-Kupershmidt equations:

$$\begin{aligned}
r_{t_2} &= -\frac{1}{2} r_{xx} - 2rr_x + \frac{1}{2} s_x + \alpha_x \beta + \alpha \beta_x + \beta \beta_{xx}, \\
s_{t_2} &= \frac{1}{2} s_{xx} - 2r_x s - 2rs_x + 2\alpha\alpha_x + 2s\beta\beta_x, \\
u_{1,t_2} &= (\varepsilon + 1)\left(-\frac{1}{2} u_{1xx} - 2r_x u_1 - 2ru_{1x} + \frac{1}{2} u_{2x} \right. \\
&\qquad \left. -2u_1 u_{1x} - \frac{1}{2} r_{xx} + \frac{1}{2} s_x - 2rr_x\right) \\
&\quad + \frac{1}{2} r_{xx} - \frac{1}{2} s_x - \alpha_x \beta - \alpha \beta_x + 2rr_x - \beta \beta_{xx}, \\
u_{2,t_2} &= (\varepsilon + 1)\left(\frac{1}{2} u_{2xx} - 2s_x u_1 - 2su_{1x} - 2r_x u_2 \right. \\
&\qquad - 2ru_{2x} - 2u_{1x} u_2 - 2u_1 u_{2x} \\
&\qquad \left. + \frac{1}{2} s_{xx} - 2r_x s - 2rs_x\right) \\
&\quad - \frac{1}{2} s_{xx} + 2r_x s + 2rs_x - 2\alpha\alpha_x - 2s\beta\beta_x, \\
\alpha_{t_2} &= \alpha_{xx} - 2r\alpha_x - \frac{3}{2} r_x \alpha + \frac{1}{2} s_x \beta + s\beta_x + \alpha\beta\beta_x, \\
\beta_{t_2} &= -\beta_{xx} - \frac{1}{2} r_x \beta - 2r\beta_x - \alpha_x.
\end{aligned} \tag{37}$$

Particularly, taking $\alpha = \beta = 0$ in (37), we can get the nonlinear integrable couplings of the second order Broer-Kaup-Kupershmidt equations:

$$\begin{aligned}
r_{t_2} &= -\frac{1}{2} r_{xx} - 2rr_x + \frac{1}{2} s_x, \\
s_{t_2} &= \frac{1}{2} s_{xx} - 2r_x s - 2rs_x, \\
u_{1,t_2} &= (\varepsilon + 1)\left(-\frac{1}{2} u_{1xx} - 2r_x u_1 - 2ru_{1x} + \frac{1}{2} u_{2x} \right. \\
&\qquad \left. -2u_1 u_{1x} - \frac{1}{2} r_{xx} + \frac{1}{2} s_x - 2rr_x\right) \\
&\quad + \frac{1}{2} r_{xx} - \frac{1}{2} s_x + 2rr_x, \\
u_{2,t_2} &= (\varepsilon + 1) \\
&\quad \times \left(\frac{1}{2} u_{2xx} - 2s_x u_1 - 2su_{1x} \right. \\
&\qquad - 2r_x u_2 - 2ru_{2x} - 2u_{1x} u_2 - 2u_1 u_{2x} \\
&\qquad \left. + \frac{1}{2} s_{xx} - 2r_x s - 2rs_x\right) \\
&\qquad - \frac{1}{2} s_{xx} + 2r_x s + 2rs_x.
\end{aligned} \tag{38}$$

Let $\varepsilon = 0$ in (37); we have

$$\begin{aligned}
r_{t_2} &= -\frac{1}{2} r_{xx} - 2rr_x + \frac{1}{2} s_x + \alpha_x \beta + \alpha \beta_x + \beta \beta_{xx}, \\
s_{t_2} &= \frac{1}{2} s_{xx} - 2r_x s - 2rs_x + 2\alpha\alpha_x + 2s\beta\beta_x, \\
u_{1,t_2} &= -\frac{1}{2} u_{1xx} - 2r_x u_1 - 2ru_{1x} + \frac{1}{2} u_{2x} - 2u_1 u_{1x} \\
&\quad - \alpha_x \beta - \alpha \beta_x - \beta \beta_{xx}, \\
u_{2,t_2} &= \frac{1}{2} u_{2xx} - 2s_x u_1 - 2su_{1x} - 2r_x u_2 - 2ru_{2x} \\
&\quad - 2u_{1x} u_2 - 2u_1 u_{2x} - 2\alpha\alpha_x - 2s\beta\beta_x,
\end{aligned}$$

$$\alpha_{t_2} = \alpha_{xx} - 2r\alpha_x - \frac{3}{2}r_x\alpha + \frac{1}{2}s_x\beta + s\beta_x + \alpha\beta\beta_x,$$
$$\beta_{t_2} = -\beta_{xx} - \frac{1}{2}r_x\beta - 2r\beta_x - \alpha_x. \tag{39}$$

If setting $\varepsilon = -1$, $u_1 = -r$, and $u_2 = -s$ in (37), we obtain the second order super Broer-Kaup-Kupershmidt equations:

$$\begin{aligned} r_{t_2} &= -\frac{1}{2}r_{xx} - 2rr_x + \frac{1}{2}s_x + \alpha_x\beta + \alpha\beta_x + \beta\beta_{xx}, \\ s_{t_2} &= \frac{1}{2}s_{xx} - 2r_xs - 2rs_x + 2\alpha\alpha_x + 2s\beta\beta_x, \\ \alpha_{t_2} &= \alpha_{xx} - 2r\alpha_x - \frac{3}{2}r_x\alpha + \frac{1}{2}s_x\beta + s\beta_x + \alpha\beta\beta_x, \\ \beta_{t_2} &= -\beta_{xx} - \frac{1}{2}r_x\beta - 2r\beta_x - \alpha_x. \end{aligned} \tag{40}$$

6. Remarks

In this paper, we introduced an approach for constructing nonlinear integrable couplings of super integrable hierarchy. Zhang [27] once employed two kinds of explicit Lie algebra F and G to obtain the nonlinear integrable couplings of the GJ hierarchy and Yang hierarchy, respectively. It is easy to see that Lie algebra F given in [27] is isomorphic to the Lie algebra span $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ in $gl(6, 2)$. So we can obtain nonlinear integrable couplings of super GJ and Yang hierarchy easily. The method in this paper can be applied to other super integrable systems for constructing their integrable couplings.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 61072147), the Natural Science Foundation of Henan Province (no. 132300410202), the Science and Technology Key Research Foundation of the Education Department of Henan Province (no. 12A110017) and the Youth Research Foundation of Shangqiu Normal University (no. 2011QN12).

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