Research Article

Optimal Control of a Viscous Shallow Water Equation

Wenbin Zhang¹ and Jiangbo Zhou²

¹ Taizhou Institute of Science and Technology (NUST) Taizhou, Jiangsu 225300, China

² Nonlinear Scientific Research Center, Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China

Correspondence should be addressed to Wenbin Zhang; zwbust@126.com

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This paper is concerned with a viscous shallow water equation, which includes both the viscous Camassa-Holm equation and the viscous Degasperis-Procesi equation as its special cases. The optimal control under boundary conditions is given, and the existence of optimal solution to the equation is proved.

1. Introduction

Holm and Staley [1] studied the following family of evolution-ary 1+1 PDEs:

$$m_t + \underbrace{u_x m}_{\text{convection}} + b \underbrace{u m_x}_{\text{stretching}} = \varepsilon \underbrace{m_{xx}}_{\text{viscosity}}, \quad (1)$$

which describes the balance between convection and stretching for small viscosity in the dynamics of one-dimensional nonlinear waves in fluids. Here $u = p * m = \int_{-\infty}^{\infty} p(x - y)m(y)dy$, $m = u - u_{xx}$, and p is chosen to be the Green's function for the Helmholtz operator $1 - \partial_x^2$ on the line. In a recent study of soliton equations, it is found that (1) for $\varepsilon = 0$ and any $b \neq -1$ is included in the family of shallow water equations at quadratic order accuracy that are asymptotically equivalent under Kodama transformations [2].

When $\varepsilon = 0$, (1) becomes the b -family of equations:

$$m_t + u_x m + b u m_x = 0, \quad (2)$$

which describes a one-dimensional version of active fluid transport. It was shown by Degasperis and Procesi [3] that (2) cannot satisfy the asymptotic integrability condition unless $b = 2$ or $b = 3$; compare [2, 4, 5].

For $b = 2$ in (2), it becomes the Camassa-Holm (CH) equation:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad (3)$$

which is a model describing the unidirectional propagation of shallow water waves over a flat bottom [4]. Equation (3) has a bi-Hamiltonian structure [6] and is completely integrable [7, 8]. It admits, in addition to smooth waves, a multitude of traveling wave solutions with singularities: peakons, cuspons, stumpons, and composite waves [4, 9]. Its solitary waves are stable solitons [10, 11], retaining their shape and form after interactions [10]. The Cauchy problem of (3) has been studied extensively. Constantin [12] and Rodríguez-Blanco [13] investigated the locally well-posed for initial data $u_0 \in H^s(R)$ with $s > 3/2$. More interestingly, it has strong solutions that are global in time [11, 14] as well as solutions that blow up in finite time [11, 15, 16]. On the other hand, Bressan and Constantin [17] and Xin and Zhang [18] showed that (3) has global weak solutions with initial data $u_0 \in H^1$.

For $b = 3$ in (2), it becomes the Degasperis-Procesi (DP) equation:

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad (4)$$

which can be used as a model for nonlinear shallow water dynamics, and its asymptotic accuracy is the same as (3). Degasperis et al. [5] presented that (4) has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to (3) peakons [4, 10, 19]. Dullin et al. [20] showed that (4) can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. The numerical stability of solitons and peakons, the multisoliton solutions

and their peakon limits, and an inverse scattering method to compute N-peakon solutions to (4) have been investigated, respectively, in [21–23]. After (4) appeared, it has attracted many researchers to discover its dynamics (see [24–31]). Yin [24, 25] proved the local well-posedness of (4) with initial data $u_0 \in H^s(R)$ ($s > 3/2$) on the line and on the circle and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to (4) were studied in [29, 30]. Similar to (3) [11, 15, 18, 22–24], (4) has not only global strong solutions [27, 28] but also blow-up solutions [26, 28–30]. Apart from these, Coclite and Karlsen [31] proved that it has global entropy weak solutions in $L^2(R) \cap BV(R)$ and $L^2(R) \cap L^4(R)$.

Recently, Lai and Wu [32] found that, in both models (1) and (2), the coefficient of uu_x is equal to the coefficient of $u_x u_{xx}$ plus the one of uu_{xxx} . That is,

$$3 = 2 + 1, \quad 4 = 3 + 1. \quad (5)$$

Then, they studied the global solutions and blow-up phenomena to the following generalized equation:

$$u_t - u_{xxt} + (a + b)uu_x = au_x u_{xx} + buu_{xxx}. \quad (6)$$

In this paper, we study the optimal control problem for the following equation:

$$u_t - u_{xxt} - \varepsilon(u_{xx} - u_{xxx}) + (a + b)uu_x = au_x u_{xx} + buu_{xxx}, \quad (7)$$

where a, b are positive constants. The optimal control is an important component of modern control theories and has a wider application in modern engineering. Two methods are introduced to study the control problems in PDE: one is using a low model method and then changing to an ODE model [33]; the other is using a quasi-optimal control method [34]. No matter which one is chosen, it is necessary to prove the existence of optimal solution according to the basic theory [35]. The control problems of nonlinear PDE have been studied extensively. Kunisch and Volkwein solved open-loop and closed-loop optimal control problems for the Burgers equation [36] and discussed the instantaneous control of the equation [37]. Vedantham [38] developed a technique to utilize the Cole-Hopf transformation to solve an optimal control problem for the Burgers equation. Øksendal [39] proved a sufficient maximum principle for the optimal control systems described by a quasi-linear stochastic heat equation. In [40], Ghattas and Bark studied the optimal control of two- and three-dimensional incompressible Navier-Stokes flows. Lagnese and Leugering [41] considered the problem of boundary optimal control of a wave equation with boundary dissipation. In [42], Yong established a unified existence theory of optimal controls for general semilinear evolutionary distributed parameter systems under the framework of mild (or weak) solutions for evolution equations. Yong and Zheng [43] considered the Cahn-Hilliard equation in a bounded domain with smooth boundary. Based on the energy estimates and the compact method, Ryu and Yagi [44, 45] studied Keller-Segel equations and adsorbate-induced phase transition model. Tian et al. [46–48] studied

the optimal control problems for parabolic equations, such as viscous CH equation, viscous DP equation, and viscous DGH equation. Under boundary condition, Zhao and Liu [49] studied viscous Cahn-Hilliard equation.

With $m = u - u_{xx}$, the optimal control problem for (7) we intend to investigate is

$$\min J(m, \bar{\omega}) = \frac{1}{2} \|Cm - z\|_S^2 + \frac{\delta}{2} \|\bar{\omega}\|_{L^2(Q_0)}^2, \quad (8)$$

subject to

$$\begin{aligned} m_t - \varepsilon m_{xx} + au_x m + bum_x &= B^* \bar{\omega} \quad \text{in } (0, T) \times \Omega, \\ u &= u_x = u_{xx} = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ m(0, x) &= m_0(x) \quad \text{in } \Omega, \end{aligned} \quad (9)$$

where $\Omega = (0, 1)$. Clearly, our control target is to match the given desired state z by adjusting the body force $\bar{\omega}$ in a control volume $Q_0 \subseteq Q = (0, T) \times \Omega$ in the L^2 -sense.

Notation. In this paper, we set that $V = H_0^1(0, 1)$, $H = L^2(0, 1)$, $V^* = H^{-1}(0, 1)$, and $H^* = L^2(0, 1)$ are dual space, respectively. Then, $V \hookrightarrow H = H^* \hookrightarrow V^*$, with each embedding being dense. The extension operator $B^* \in \mathcal{L}(L^2(Q_0), L^2(V^*))$ is given by

$$B^* q = \begin{cases} q, & q \in Q_0, \\ 0, & q \in \frac{Q}{Q_0}. \end{cases} \quad (10)$$

We supply the Lebesgue space L^2 with the norm $\|\cdot\|$ and the Sobolev space H^s , $s \in \mathbb{R}$ with the norm $\|\cdot\|_{H^s}$. For a fixed $T > 0$, we also defined a space $W(0, T; V)$ as

$$W(0, T; V) = \{m : m \in L^2(0, T; V), m_t \in L^2(0, T; V^*)\}, \quad (11)$$

which is a Hilbert space endowed with common inner product.

2. Existence of Unique Weak Solution

In this section, we prove the existence of a weak solution for the following equation:

$$m_t - \varepsilon m_{xx} + au_x m + bum_x = B^* \bar{\omega} \quad \text{in } (0, T) \times \Omega, \quad (12)$$

with the boundary conditions

$$u = u_x = u_{xx} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (13)$$

and the initial value

$$m(x, 0) = m_0(x) \quad \text{in } \Omega, \quad (14)$$

where $m = u - u_{xx}$, $\Omega = (0, 1)$, $B^* \bar{\omega} \in L^2(V^*)$, and a control $\bar{\omega} \in L^2(Q_0)$. Now, we give the definition of the weak solution in the space $W(0, T; V)$.

Definition 1. A function $m(x, t) \in W(0, T; V)$ is called a weak solution to (12), if

$$\begin{aligned} \frac{d}{dt} (m, \phi) + \varepsilon (m_x, \phi_x) + a (u_x m, \phi) + b (u m_x, \phi) \\ = (B^* \bar{\omega}, \phi)_{V^*, V}, \end{aligned} \quad (15)$$

is valid for all $\phi \in V$, a.e. $t \in [0, T]$ and $m_0(x) \in H$.

By using the standard Galerkin method and some a priori estimates, one can obtain the following theorem, which ensures the existence of a unique weak solution to the viscous shallow water equation.

Theorem 2. Let $m_0(x) \in H$ and $B^* \bar{\omega} \in L^2(V^*)$ then the problem (12)–(14) admits a local weak solution $m(x, t)$ which satisfies Definition 1.

Proof. Let $\{\varphi_i\}_{i=1}^\infty$ be an orthonormal basis in the space H consisting of eigenfunctions of the operator $A = -\partial_x^2$. For $k \in \mathcal{N}$, we define the discrete ansatz space by $V_k = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_k\} \subset V$. Let $m_k(t) = m_k(x, t) = \sum_{i=1}^k m_i^k(t) \varphi_i(x)$ with $m_k(\cdot, 0) \rightarrow m_0$ in H .

By analyzing the limiting behavior of the sequences of smooth functions $\{u_k\}$ and $\{m_k\}$, we can prove the existence of a weak solution to (12).

Performing the Galerkin procedure for the problem (12)–(14), we obtain

$$\begin{aligned} m_{k,t} - \varepsilon m_{k,xx} + a u_{k,x} m_k + b u_k m_{k,x} \\ = B^* \bar{\omega} \quad \text{in } (0, T) \times \Omega, \end{aligned} \quad (16)$$

with the boundary conditions

$$u_k = u_{k,x} = u_{k,xx} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (17)$$

and the initial value

$$m_k(x, 0) = m_{k,0}(x) \in H \quad \text{in } \Omega. \quad (18)$$

Clearly, (16) is an ordinary differential equation, and, according to standard ODE's theory, there is a unique solution to (16) in the interval $[0, t_k]$. What we should do is to show that the solution is uniformly bounded as $t_k \rightarrow T$. We will prove the existence of weak solution in the following steps.

Step 1. Taking the inner product of (16) with u_k in Ω , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_k\|^2 + \|u_{k,x}\|^2) + \varepsilon (\|u_{k,x}\|^2 + \|u_{k,xx}\|^2) \\ = -\frac{a-2b}{2} (u_{k,x}, u_{k,x}^2) + (B^* \bar{\omega}, u_k)_{V^*, V}. \end{aligned} \quad (19)$$

Let us estimate the first term of the right hand side of (19) as follows:

$$\left| -\frac{a-2b}{2} (u_{k,x}, u_{k,x}^2) \right| \leq \frac{|a-2b|}{2} \|u_{k,x}\|_{L^\infty} \|u_{k,x}\|^2. \quad (20)$$

By the Sobolev embedding theorem and the Poincaré inequality, we obtain

$$\|u_{k,x}\|_{L^\infty} \leq c_1 \|u_{k,x}\|_{H^1} \leq c_1 \|u_k\|_{H^2} \leq c_2 \|u_{k,xx}\|, \quad (21)$$

where c_1, c_2 are constants.

It follows from (20) and (21) that

$$\left| -\frac{a-2b}{2} (u_{k,x}, u_{k,x}^2) \right| \leq \frac{|a-2b|}{2} c_2 \|u_{k,xx}\| \cdot \|u_{k,x}\|^2. \quad (22)$$

Next, we estimate the second term of the right hand side of (19). Noting that $B^* \bar{\omega} \in L^2(V^*)$ is a control item, we can assume that $\|B^* \bar{\omega}\|_{V^*} \leq M$, where M is positive constant. Then

$$|(B^* \bar{\omega}, u_k)_{V^*, V}| \leq \|B^* \bar{\omega}\|_{V^*} \|u_k\|_V \leq c_3 M \|u_{k,x}\|. \quad (23)$$

Combining inequalities (22) and (23) with (19) and using the Young inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_k\|^2 + \|u_{k,x}\|^2) + \varepsilon (\|u_{k,x}\|^2 + \|u_{k,xx}\|^2) \\ \leq \frac{|a-2b|}{2} c_2 \|u_{k,xx}\| \cdot \|u_{k,x}\|^2 + c_3 M \|u_{k,x}\| \\ \leq \varepsilon \|u_{k,xx}\|^2 + \frac{(a-2b)^2 c_2^2}{4\varepsilon} \|u_{k,x}\|^4 + \varepsilon \|u_k\|^2 + \frac{c_3^2 M^2}{\varepsilon} \\ \leq \varepsilon (\|u_{k,xx}\|^2 + \|u_k\|^2) + \frac{(a-2b)^2 c_2^2}{4\varepsilon} (\|u_k\|^2 + \|u_{k,x}\|^2)^2 \\ + \frac{c_3^2 M^2}{\varepsilon}. \end{aligned} \quad (24)$$

Therefore, we obtain

$$\begin{aligned} \frac{d}{dt} (\|u_k\|^2 + \|u_{k,x}\|^2) \\ \leq \frac{(a-2b)^2 c_2^2}{2\varepsilon} (\|u_k\|^2 + \|u_{k,x}\|^2)^2 + \frac{2c_3^2 M^2}{\varepsilon}. \end{aligned} \quad (25)$$

It, thus, transpires that

$$\begin{aligned} \|u_k\|^2 + \|u_{k,x}\|^2 \\ \leq \frac{2c_2 M}{|a-2b| c_2} \tan \left\{ \frac{c_2 c_3 M |a-2b|}{\varepsilon} t + c_4 \right\} \triangleq \rho_1^2, \end{aligned} \quad (26)$$

where $t \in [0, T]$, $T < \pi\varepsilon/2c_2c_3M|a-2b|$ and $c_4 = \arctan(|a-2b|c_2(\|u_{k,0}\|^2 + \|u_{k,x}(0)\|^2)/2c_3M)$.

From the above discussion, we know that $\|u_k\| \leq \rho_1$ and $\|u_{k,x}\| \leq \rho_1$. By the Sobolev embedding theorem, we obtain

$$\|u_k\|_{L^\infty} \leq \|u_k\|_{H^1} = \|u_k\|_V \leq (\|u_k\| + \|u_{k,x}\|) \leq c_5. \quad (27)$$

Step 2. We prove a uniform $L^2(0, T; V)$ bound on a sequence $\{m_k\}$. Taking the inner product of (16) with m_k in Ω , we have

$$\begin{aligned} \frac{1}{2} \|m_k\|^2 + \varepsilon \|m_{k,x}\|^2 = (2a-b) (u_k, m_k m_{k,x}) \\ + (B^* \bar{\omega}, m_k)_{V^*, V}. \end{aligned} \quad (28)$$

Let us estimate the first term of the right hand side of (28). By the Poincaré inequality and the Hölder inequality, we have

$$\begin{aligned}
& |(2a - b)(u_k, m_k m_{k,x})| \\
& \leq |2a - b| \|u_k\|_{L^\infty} \|m_k\| \cdot \|m_{k,x}\| \\
& \leq \frac{c_5 |2a - b|}{2} (\|m_k\|^2 + \|m_{k,x}\|^2) \\
& \leq \frac{c_5 |2a - b|}{2} (c_6 \|m_{k,x}\|^2 + \|m_{k,x}\|^2) \\
& = c_7 \|m_{k,x}\|^2,
\end{aligned} \tag{29}$$

where $c_7 = (c_5 |2a - b|/2)(c_6 + 1)$.

Furthermore, we estimate the second term of the right hand side of (28) in the following way:

$$|(B^* \omega, m_k)_{V^*, V}| \leq M \|m_k\|_V \leq c_8 M \|m_{k,x}\|. \tag{30}$$

Combining inequalities (29) and (30) with (28), we have

$$\frac{1}{2} \frac{d}{dt} \|m_k\|^2 + \varepsilon \|m_{k,x}\|^2 \leq c_7 \|m_{k,x}\|^2 + c_8 M \|m_{k,x}\|, \tag{31}$$

where $\varepsilon > c_7$. It follows from (31) with the Young inequality that

$$\frac{d}{dt} \|m_k\|^2 + (\varepsilon - c_7) \|m_{k,x}\|^2 \leq \frac{c_8^2 M^2}{\varepsilon - c_7}. \tag{32}$$

Integrating the pervious inequality with respect to t on $[0, T]$, we obtain

$$\int_0^T \|m_{k,x}\|^2 dt \leq \frac{1}{\varepsilon - c_7} \left[\frac{c_8^2 M^2 T}{\varepsilon - c_7} + \|m_{k,0}\|^2 \right] \triangleq \rho_2^2. \tag{33}$$

Furthermore, we have

$$\|m_k\|_{L^2(0,T;V)}^2 = \int_0^T \|m_k\|_{H^1}^2 dt \leq c_8 \int_0^T \|m_{k,x}\|^2 dt \triangleq \rho_3^2. \tag{34}$$

From (32), we also have

$$\frac{d}{dt} \|m_k\|^2 \leq \frac{c_8^2 M^2}{\varepsilon - c_7}. \tag{35}$$

Integrating the pervious inequality with respect to $t < T$ on $[0, t]$, we obtain

$$\|m_k\|^2 \leq \frac{c_8^2 M^2 t}{\varepsilon - c_7} + \|m_{k,0}\|^2 \triangleq \rho_4^2, \quad \forall t \in [0, T]. \tag{36}$$

Step 3. We prove a uniform $L^2(0, T; V^*)$ bound on a sequence $\{m_{k,t}\}$. By (16) and the Sobolev embedding theorem, we have

$$\begin{aligned}
\|m_{k,t}\|_{V^*} & \leq \varepsilon \|m_k\|_V + a \|u_k\| \|m_k\|_V \\
& \quad + b \|u_k\|_V \|m_k\| + \|B^* \omega\|_{V^*} \\
& \leq \varepsilon \|m_k\|_V + a \rho_1 \|m_k\|_V + b c_5 \|m_k\| + M.
\end{aligned} \tag{37}$$

Furthermore, we have

$$\|m_{k,t}\|_{V^*}^2 \leq 3(\varepsilon + a \rho_1)^2 \|m_k\|_V^2 + 3b^2 c_5^2 \|m_k\|^2 + 3M^2. \tag{38}$$

Integrating (38) with respect to t on $[0, T]$, we deduce that

$$\|m_{k,t}\|_{L^2(0,T;V^*)}^2 \leq 3(\varepsilon + a \rho_1)^2 \rho_3^2 + 3(b^2 c_5^2 \rho_4^2 + M^2) T \triangleq \rho_5^2. \tag{39}$$

From the pervious discussion one has the following.

- (a) For every $T < \pi \varepsilon / 2 c_2 c_3 M |a - 2b|$, ρ_3 and ρ_4 are two constants and the sequence $\{m_k\}$ is bounded in $L^2(0, T; H)$ as well as in $L^2(0, T; V)$, which is independent of the dimension of ansatz space k .
- (b) For every $T < \pi \varepsilon / 2 c_2 c_3 M |a - 2b|$ and ρ_5 is constant, the sequence $\{m_{k,t}\}$ is bounded in $L^2(0, T; V^*)$, which is independent of the dimension of ansatz space k .

Note that, (a) and (b) the above mentioned are equivalent to $\{m_k\} \subset W(0, T; V)$ bounded, and $W(0, T; V)$ is compactly embedded into $C(0, T; H)$. Then, we conclude the convergence of a subsequence, again denoted by $\{m_k\}$ weak in $W(0, T; V)$, weakstar in $L^\infty(0, T; H)$, and strong in $L^2(0, T; H)$ to a function $m(x, t) \in W(0, T; V)$.

The proof of the uniqueness of the solution is similar to Theorem 1 in [37], so we omit it here. This completes the proof of the theorem. $\square$

Our next result describes that the norm of weak solution can be controlled by initial value and control item.

Theorem 3. Let $m_0 \in H$ and $B^* \omega \in L^2(V^*)$; then there exists a positive constant C , such that

$$\|m\|_{W(0,T;V)}^2 \leq C \left(\|m_0\|^2 + \|\omega\|_{L^2(Q_0)}^2 \right). \tag{40}$$

Proof. Taking the inner product of (12) with u in Ω , by using the same argument as in the proof of Theorem 2, we have

$$\|u\| \leq \rho_1, \quad \|u_x\| \leq \rho_1, \quad \|u\|_{L^\infty} \leq c_5, \tag{41}$$

where ρ_1 and c_5 are positive constants as in Theorem 2. Taking the inner product of (12) with m in Ω , we have

$$\frac{1}{2} \|m\|^2 + \varepsilon \|m_x\|^2 = (2a - b)(u, mm_x) + (B^* \omega, m)_{V^*, V}. \tag{42}$$

Let us estimate the first term of the right hand side of (42) as follows:

$$\begin{aligned}
|(2a - b)(u, mm_x)| & \leq |2a - b| \|u\|_{L^\infty} \|m\| \cdot \|m_x\| \\
& \leq \frac{c_5 |2a - b|}{2} (\|m\|^2 + \|m_x\|^2) \\
& \leq \frac{c_5 |2a - b|}{2} (c_6 \|m_x\|^2 + \|m_x\|^2) \\
& = c_7 \|m_x\|^2.
\end{aligned} \tag{43}$$

Furthermore, we estimate the second term of the right hand side of (42) in the following way:

$$(B^* \omega, m)_{V^*, V} \leq \|B^* \omega\|_{V^*} \|m\|_V \leq c_8 \|B^* \omega\|_{V^*} \|m_x\|. \quad (44)$$

Combining inequalities (43) and (44) with (42), we have

$$\frac{1}{2} \frac{d}{dt} \|m\|^2 + (\varepsilon - c_7) \|m_x\|^2 \leq c_8 \|B^* \omega\|_{V^*} \|m_x\|, \quad (45)$$

where $\varepsilon > c_7$.

Integrating (45) with respect to t on $[0, T]$, we derive that

$$\begin{aligned} \frac{1}{2} \|m(T)\|^2 - \frac{1}{2} \|m_0\|^2 + (\varepsilon - c_7) \|m_x\|_{L^2(0,T;H)}^2 \\ \leq c_8 \|B^* \omega\|_{L^2(0,T;V^*)} \|m_x\|_{L^2(0,T;H)}. \end{aligned} \quad (46)$$

According to the Young inequality, we have

$$\begin{aligned} c_8 \|B^* \omega\|_{L^2(0,T;V^*)} \|m_x\|_{L^2(0,T;H)} \\ \leq \frac{(\varepsilon - c_7)}{2} \|m_x\|_{L^2(0,T;H)}^2 + \frac{c_8^2}{2(\varepsilon - c_7)} \|B^* \omega\|_{L^2(0,T;V^*)}^2. \end{aligned} \quad (47)$$

Substituting (47) into (46) yields

$$(\varepsilon - c_7) \|m_x\|_{L^2(0,T;H)}^2 \leq \|m_0\|^2 + \frac{c_8^2}{(\varepsilon - c_7)} \|B^* \omega\|_{L^2(0,T;V^*)}^2. \quad (48)$$

It follows from (48) that

$$\begin{aligned} \|m\|_{L^2(0,T;V)}^2 &\leq c_8 \int_0^T \|m_x\|^2 dt \\ &\leq \frac{c_8}{(\varepsilon - c_7)} \|m_0\|^2 + \frac{c_8^3}{(\varepsilon - c_7)^2} \|B^* \omega\|_{L^2(0,T;V^*)}^2 \\ &\leq c_9 (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)})^2, \end{aligned} \quad (49)$$

where $c_9 = \max\{c_8/(\varepsilon - c_7), c_8^3/(\varepsilon - c_7)^2\}$.

Combining inequality (45) with $\varepsilon > c_7$, we also have

$$\frac{1}{2} \frac{d}{dt} \|m\|^2 \leq c_8 \|B^* \omega\|_{V^*} \|m\|_V. \quad (50)$$

Integrating the pervious inequality with respect to t yields

$$\begin{aligned} \|m\|^2 &\leq \|m_0\|^2 + 2c_8 \|B^* \omega\|_{L^2(0,T;V^*)} \|m\|_{L^2(0,T;V)} \\ &\leq \|m_0\|^2 + 2c_8 \|B^* \omega\|_{L^2(0,T;V^*)} \sqrt{c_9} \\ &\quad \times (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)}) \\ &\leq c_{10} (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)})^2, \end{aligned} \quad (51)$$

where $c_{10} = 2 \max\{1, c_8 \sqrt{c_9}\}$.

Combining (12) with the Sobolev embedding theorem, we obtain

$$\begin{aligned} \|m_t\|_{V^*} &\leq \varepsilon \|m\|_V + a \|u\| \|m\|_V + b \|u\|_V \|m\| + \|B^* \omega\|_{V^*} \\ &\leq (\varepsilon + a\rho_1) \|m\|_V + bc_5 \|m\| + \|B^* \omega\|_{V^*}. \end{aligned} \quad (52)$$

Furthermore, we have

$$\|m_t\|_{V^*}^2 \leq 3(\varepsilon + a\rho_1)^2 \|m\|_V^2 + 3b^2 c_5^2 \|m\|^2 + 3\|B^* \omega\|_{V^*}^2. \quad (53)$$

Integrating (53) with respect to t on $[0, T]$, we deduce that

$$\begin{aligned} \|m_t\|_{L^2(0,T;V^*)}^2 &\leq 3(\varepsilon + a\rho_1)^2 \|m\|_{L^2(0,T;V)}^2 \\ &\quad + 3b^2 c_5^2 \int_0^T \|m\|^2 dt + 3\|B^* \omega\|_{L^2(0,T;V^*)}^2 \\ &\leq 3(\varepsilon + a\rho_1)^2 c_9 (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)})^2 \\ &\quad + 3b^2 c_5^2 c_{10} T (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)})^2 \\ &\quad + 3\|B^* \omega\|_{L^2(0,T;V^*)}^2 \\ &\leq c_{11} (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)})^2, \end{aligned} \quad (54)$$

where $c_{11} = 3[(\varepsilon + a\rho_1)^2 c_9 + b^2 c_5^2 c_{10} T + 1]$.

Taking into account (49) and (54), we arrive at

$$\begin{aligned} \|m\|_{W(0,T;V)}^2 &= \|m\|_{L^2(0,T;V)}^2 + \|m_t\|_{L^2(0,T;V^*)}^2 \\ &\leq (c_9 + c_{11}) (\|m_0\| + \|B^* \omega\|_{L^2(0,T;V^*)})^2 \\ &\leq C (\|m_0\| + \|\omega\|_{L^2(Q_0)})^2, \end{aligned} \quad (55)$$

where $C = c_9 + c_{11}$. This completes the proof of the theorem. $\square$

3. Existence of Optimal Solution to the Problem of (7)

In this section, we consider the optimal control problem of (7) and prove the existence of optimal solution based on Lions' theory (see [50]).

Allowing a control $\omega \in L^2(Q_0)$, we study the following control system:

$$\begin{aligned} m_t - \varepsilon m_{xx} + au_x m + bum_x &= B^* \omega \quad \text{in } (0, T) \times \Omega, \\ u &= u_x = u_{xx} = 0 \quad \text{on } (0, T) \times \partial\Omega, \\ m(x, 0) &= m_0(x) \in H \quad \text{in } \Omega, \end{aligned} \quad (56)$$

where $m = u - u_{xx}$ and $\Omega = (0, 1)$. Let $C \in L(W(0, T; V), S)$ be an observation operator, and let S be a real Hilbert space of

observations. Then, we choose performance index of tracking type

$$J(m, \omega) = \frac{1}{2} \|Cm - z\|_S^2 + \frac{\delta}{2} \|\omega\|_{L^2(Q_0)}^2, \quad (57)$$

where $z \in S$ is a desired state and $\delta > 0$ is fixed. Optimal control problem of (7) is

$$\min J(m, \omega), \quad (58)$$

where (m, ω) satisfies (56).

Let $X = W(0, T; V) \times L^2(Q_0)$, $Y = L^2(0, T; V) \times H$; we define an operator $e = e(e_1, e_2) : X \rightarrow Y$ by

$$\begin{aligned} e_1 &= (-\Delta)^{-1} (m_t - \varepsilon m_{xx} + au_x m + bum_x - B^* \omega), \\ e_2 &= m(x, 0) - m_0, \end{aligned} \quad (59)$$

where Δ is an operator from V to V^* . Then, we rewrite (58) in the following form:

$$\begin{aligned} \min & J(m, \omega) \\ \text{s.t.} & e(m, \omega) = 0. \end{aligned} \quad (60)$$

Now, we give the main result.

Theorem 4. *There exists an optimal control solution (m^*, ω^*) to the problem (56).*

Proof. Let $(m, \omega) \in X$ satisfying the equation $e(m, \omega) = 0$. According to (57) and Theorem 3, we have

$$J(m, \omega) \geq \frac{\delta}{2} \|\omega\|_{L^2(Q_0)}^2, \quad (61)$$

$$\|\omega\|_{L^2(Q_0)} \longrightarrow \infty, \quad \text{as } \|m\|_{W(0, T; V)} \longrightarrow \infty.$$

Then,

$$J(m, \omega) \longrightarrow \infty, \quad \text{as } \|(m, \omega)\|_X \longrightarrow \infty. \quad (62)$$

Note that the norm is weakly lower semicontinuous [51]; we have that J is weakly lower semicontinuous. Since $J(m, \omega) \geq 0$, for all $(m, \omega) \in X$, holds, there exists $\kappa \geq 0$ defined by

$$\kappa = \inf \{J(m, \omega) \mid (m, \omega) \in X, e(m, \omega) = 0\}, \quad (63)$$

which means the existence of a minimizing sequence $\{(m^n, \omega^n)\}_{n \in \mathcal{N}}$ in X such that

$$\kappa = \lim_{n \rightarrow \infty} J(m^n, \omega^n), \quad e(m^n, \omega^n) = 0, \quad \forall n \in \mathcal{N}. \quad (64)$$

According to (62), there exists an element $(m^*, \omega^*) \in X$ such that, when $n \rightarrow \infty$,

$$m^n \longrightarrow m^*, \quad \text{weakly}, m \in W(0, T; V), \quad (65)$$

$$\omega^n \longrightarrow \omega^*, \quad \text{weakly}, \omega \in L^2(Q_0). \quad (66)$$

From (65), it is easy to check that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T (m_t^n(t) - m_t^*(t), \phi(t))_{V^*, V} dt &= 0, \\ \forall \phi &\in L^2(0, T; V). \end{aligned} \quad (67)$$

Thanks to the fact that $W(0, T; V)$ is compactly embedded into $L^2(0, T; L^\infty)$ [52] and $W(0, T; V)$ is continuously embedded into $C(0, T; H)$ [53], we obtain that $m^n \rightarrow m^*$ strongly in $L^2(0, T; L^\infty)$ and $m^n \rightarrow m^*$ strongly in $C(0, T; H)$. Furthermore, we have that $u^n \rightarrow u^*$ strongly in $C(0, T; H)$.

Since the sequence $\{m^n\}_{n \in \mathcal{N}}$ converges weakly and $\|m^n\|_{W(0, T; V)}$ is bounded [54], based on embedding theorem, we obtain that $\|m^n\|_{L^2(0, T; L^\infty)}$ is bounded. In fact, $\|m^*\|_{L^2(0, T; L^\infty)}$ is also bounded, because $m^n \rightarrow m^*$ strongly in $L^2(0, T; L^\infty)$.

Thus, it follows from the Hölder inequality that

$$\begin{aligned} & \left| \int_0^T \int_0^1 (u_x^n m^n - u_x^* m^*) \phi dx dt \right| \\ &= \left| \int_0^T \int_0^1 (u_x^n m^n - u_x^n m^* + u_x^n m^* - u_x^* m^*) \phi dx dt \right| \\ &\leq \left| \int_0^T \int_0^1 u_x^n (m^n - m^*) \phi dx dt \right| \\ &\quad + \left| \int_0^T \int_0^1 (u_x^n - u_x^*) m^* \phi dx dt \right| \\ &\leq \left| \int_0^T \|m^n - m^*\|_{L^\infty} \|u^n\|_H \|\phi\|_V dt \right| \\ &\quad + \left| \int_0^T \|m^*\|_{L^\infty} \|u^n - u^*\|_H \|\phi\|_V dt \right| \\ &\leq \|m^n - m^*\|_{L^2(0, T; L^\infty)} \|u^n\|_{C(0, T; H)} \|\phi\|_{L^2(0, T; V)} \\ &\quad + \|m^*\|_{L^2(0, T; L^\infty)} \|u^n - u^*\|_{C(0, T; H)} \|\phi\|_{L^2(0, T; V)} \\ &\longrightarrow 0, \quad \forall \phi \in L^2(0, T; V). \end{aligned} \quad (68)$$

Similarly,

$$\begin{aligned} & \left| \int_0^T \int_0^1 (u^* m^* - u^n m^n) \phi_x dx dt \right| \\ &= \left| \int_0^T \int_0^1 (u^* m^* - u^n m^* + u^n m^* - u^n m^n) \phi_x dx dt \right| \\ &\leq \left| \int_0^T \int_0^1 (u^* m^* - u^n m^*) \phi_x dx dt \right| \\ &\quad + \left| \int_0^T \int_0^1 (u^n m^* - u^n m^n) \phi_x dx dt \right| \\ &\leq \left| \int_0^T \|m^*\|_{L^\infty} \|u^* - u^n\|_H \|\phi\|_V dt \right| \\ &\quad + \left| \int_0^T \|u^n\|_H \|m^* - m^n\|_{L^\infty} \|\phi\|_V dt \right| \end{aligned}$$

$$\begin{aligned}
&\leq \|m^*\|_{L^2(0,T;L^\infty)} \|u^* - u^n\|_{C(0,T;H)} \|\phi\|_{L^2(0,T;V)} \\
&\quad + \|m^* - u^n\|_{L^2(0,T;L^\infty)} \|u^n\|_{C(0,T;H)} \|\phi\|_{L^2(0,T;V)} \\
&\longrightarrow 0, \quad \forall \phi \in L^2(0,T;V).
\end{aligned} \tag{69}$$

Then, we have

$$\begin{aligned}
&\left| \int_0^T \int_0^1 (u_x^n m_x^n - u^* m_x^*) \phi \, dx \, dt \right| \\
&= \left| \int_0^T \int_0^1 (-u_x^n m^n \phi - u^n m^n \phi_x \right. \\
&\quad \left. + u_x^* m^* \phi + u^* m^* \phi_x) \, dx \, dt \right| \\
&\leq \left| \int_0^T \int_0^1 (u_x^* m^* - u_x^n m^n) \phi \, dx \, dt \right| \\
&\quad + \left| \int_0^T \int_0^1 (u^* m^* - u^n m^n) \phi_x \, dx \, dt \right| \\
&\longrightarrow 0, \quad \forall \phi \in L^2(0,T;V).
\end{aligned} \tag{70}$$

By view of (66), we can obtain that

$$\left| \int_0^T \int_0^1 (B^* \bar{\omega}^n - B^* \bar{\omega}^*) \phi \, dx \, dt \right| \longrightarrow 0, \quad \forall \phi \in L^2(0,T;V). \tag{71}$$

According to the above discussion, we have that

$$e_1(m^*, \bar{\omega}^*) = 0, \quad \forall n \in \mathcal{N}. \tag{72}$$

Thanks to the fact that $m^* \in W(0,T;V)$ and $m^n \rightarrow m^*$ weakly in $W(0,T;V)$, we have $m^*(0) \in H$ and $m^n(0) \rightarrow m^*(0)$ weakly as $n \rightarrow \infty$. Furthermore, we obtain

$$(m^n(0) - m^*(0), \phi) \longrightarrow 0, \quad n \longrightarrow \infty, \quad \forall \phi \in H, \tag{73}$$

which means that $e_2(m^*, \bar{\omega}^*) = 0$.

Therefore, we have that

$$e(m^*, \bar{\omega}^*) = 0, \quad \text{in } Y. \tag{74}$$

Thus, there exists an optimal solution $(m^*, \bar{\omega}^*)$ to the problem (56). In the meantime, we can infer that there exists an optimal solution $(u^*, \bar{\omega}^*)$ to the viscous shallow water equation due to the relation $u = (1 - \partial_x^2)^{-1} m$. The proof of this theorem is completed. $\square$

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