Research Article

LMI-Based Stability Criteria for Discrete-Time Neural Networks with Multiple Delays

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Discrete neural models are of great importance in numerical simulations and practical implementations. In the current paper, a discrete model of continuous-time neural networks with variable and distributed delays is investigated. By Lyapunov stability theory and techniques such as linear matrix inequalities, sufficient conditions guaranteeing the existence and global exponential stability of the unique equilibrium point are obtained. Introduction of LMIs enables one to take into consideration the sign of connection weights. To show the effectiveness of the method, an illustrative example, along with numerical simulation, is presented.

1. Introduction

During the past decades, various types of neural networks have been proposed and investigated intensively, since they play important roles and have found successful applications in fields such as pattern recognition, signal and image processing, nonlinear optimization problems, and parallel computation. The dynamical behaviors in neural models, such as the existence and their asymptotic stability of equilibria, periodic solutions, bifurcations, and chaos, have been the most active areas of research and have been extensively explored over the past years [1–22].

Due to the finite transmission speed of signals among neurons, time delays in interactions between neurons frequently happen and will cause complex dynamics in neural networks [6]; so it is necessary to introduce time delays into the neural models. So far, discrete, time-varying, and distributed delays have been, respectively, introduced to describe the dynamics of neural networks, and various sufficient conditions ensuring the stability have been given.

Note that in numerical simulation and practical implementations, discretization of continuous-time models is necessary and of great importance. On the other hand, the dynamics of discrete-time neural networks could be quite different from those of continuous versions and will display much more complicated behaviors. So it is of great theoretical and practical significance to study the dynamics of discrete neural models. For discrete models, such as discrete Hopfield, bidirectional associate memory, and Cohen-Grossberg neural networks, several authors [1, 7–22] have studied the existence and exponential stability of equilibria and periodic solutions.

In this paper, a discrete model with both variable and distributed time delays is introduced. By Lyapunov stability theory and linear matrix inequality (LMI) technique, sufficient conditions ensuring the existence and globally exponential stability of a unique equilibrium point are obtained. To show the effectiveness of our results, an illustrative example along with numerical simulation is presented. To our best knowledge, such general models have been seldom touched upon in the existing literatures. As we see, the obtained conditions are easy to verify. Furthermore, introduction of LMIs enables us to take into consideration the sign of connection weights. In contrast, sufficient conditions, for instance, in [7–12], depend on the absolute values of connection weights. That will ignore the differences between neuronal excitatory and inhibitory effects.
2. Preliminaries

Set \( Z \) to be the set of integers and \( Z^* \) the set of nonnegative integers; let \( N(a,b) \) represent the set of integers between \( a \) and \( b \) with \( a \leq b \), \( a, b \in Z \), namely, \( N(a,b) = \{a,a+1,\ldots,b\} \).

Consider the discrete-time neural networks with both variable and distributed delays:

\[
x_i(n+1) = a_i x_i(n) + \sum_{j=1}^{m} b_{ij} f_j (x_j(n-k(n))) + \sum_{p=1}^{\infty} c_{ij} \mathcal{H}_j(p) g_j (x_j(n-p)) + I_i,
\]

with initial values

\[
x_i(l) = \phi_i(l), \quad l \in N(-\infty,0),
\]

where \( x(n) = \text{col}(x_1(n),\ldots,x_m(n)) \in R^m \), \( x_i(n) \) are the states of the \( i \)th neuron at time \( n \); \( a_i \in (0,1) \) represents the rate with which the \( i \)th neuron resets its potential when isolated from others; \( b_{ij} \) and \( c_{ij} \) weigh the strengths of the \( j \)th unit on the \( i \)th unit; \( f_j \) and \( g_j \) are the nonlinear activation functions of the neurons; \( k(n) \) denotes the transmission delay along the axon of the \( j \)th unit. \( \mathcal{H}_j(p) \geq 0 \) is the delay kernel; \( I_i \) is the external input on the \( i \)th neuron at time \( n \); the initial value functions \( \phi_i(l) \) are bounded \( N(-\infty,0) \), \( i = 1,\ldots,m \).

To investigate stability of system (1), make further assumptions:

(H1) suppose that \( b_{ij}, c_{ij}, I_i \in R \), for \( i = 1,\ldots,m \), and \( 0 \leq k(n) \leq k \) for \( n \in N(0,\infty) \), with \( k \) being constant;

(H2) suppose that \( \mathcal{H}_j(p) \geq 0 \), \( \sum_{p=1}^{\infty} \mathcal{H}_j(p) = 1 \), and \( \sum_{p=1}^{\infty} \mathcal{H}_j(p)v^p < +\infty \), for some \( v > 1 \), all \( i, j = 1,\ldots,m \);

(H3) assume that functions \( f_j \) and \( g_j \) are bounded and satisfy

\[
f_j \leq f_j^+ (\xi - \eta) \leq f_j^-, \quad g_j \leq g_j^+ (\xi - \eta) \leq g_j^-,
\]

for any \( \xi, \eta \in R \), where \( f_j^+, f_j^-, g_j^+ \), and \( g_j^- \) are some constants and can be positive, negative, or zero, \( j = 1,\ldots,m \). So they are less restrictive than sigmoid activation functions and Lipschitz-type ones.

For any \( \phi = (\phi_1,\ldots,\phi_m) \), a solution of systems (1) and (2) is a vector-valued function \( x : Z^* \rightarrow R^m \) satisfying system (1) and initial conditions (2) for \( n \in Z^* \). In this paper, it is always assumed that neural model (1) admits a solution represented by \( x(n,\phi) \) or simply \( x(n) \). Since the activation functions \( f_j, g_j \) are bounded, it is not difficult to check that system (1) has at least one equilibrium point by Brouwer’s fixed point theorem. So with loss of generality, assume that \( f_j(0) = g_j(0) = 0 \); that is, \( x = 0 \) is an equilibrium point. Throughout the paper, denote

\[
\|x(n)\|^2 = \sum_{i=1}^{m} |x_i(n)|^2,
\]

\[
k_M = \sup_n \{k(n)\}, \quad k_m = \inf_n \{k(n)\},
\]

\[
F_1 = \text{diag} \{f_1^+, \ldots, f_m^+\}, \quad F_2 = \text{diag} \{f_1^-, \ldots, f_m^-\}, \quad G_1 = \text{diag} \{g_1^+, \ldots, g_m^+\}, \quad G_2 = \text{diag} \{g_1^-, \ldots, g_m^-\},
\]

\[
A = \text{diag} \{a_1, \ldots, a_m\}, \quad B = (b_{ij})_{m \times m}, \quad C = (c_{ij})_{m \times m},
\]

\[
f = (f_1, \ldots, f_m)^T, \quad g = (g_1, \ldots, g_m)^T,
\]

\[
\mathcal{H} = \text{diag} \{\mathcal{H}_1, \ldots, \mathcal{H}_m\}.
\]

System (1) can be rewritten into the form

\[
x(n+1) = Ax(n) + Bf(x(n-k(n))) + C \sum_{p=1}^{\infty} \mathcal{H}_j(p) g(x(n-p)).
\]

Definition 1. The equilibrium point \( x = 0 \) of system (1) is globally exponentially stable if there exist constants \( \eta > 1 \) and \( C^* > 0 \) such that for any solution \( x(n,\phi) \) of system (1) with initial conditions \( \phi \), it holds that

\[
\|x(n,\phi)\|^2 \leq C^* \eta^{-n} \sup_{l \in N(-\infty,0)} \|\phi(l)\|^2, \quad \forall n \in Z^*.
\]

3. Exponential Stability of Equilibrium Points

By Lyapunov stability theory and LMI technique, the global exponential stability of the equilibrium point is established. Clearly, if \( x = 0 \) is exponentially stable, the equilibrium point is unique. Now we will investigate the exponential stability of the origin.

Theorem 2. Suppose that (H1)–(H3) hold and further there exist a number \( v \geq \eta > 1 \), positive definite matrix \( P \), \( \Sigma = \text{diag} \{e_1, \ldots, e_m\} \), and semipositive diagonal matrices
\[ U = \text{diag} \{ u_1, \ldots, u_m \}, \ V = \text{diag} \{ v_1, \ldots, v_m \}, \ \text{and} \ S = \text{diag} \{ s_1, \ldots, s_m \}, \ \text{such that} \]

\[ W \]

\[
\Phi_{11} = \eta A P A - P - 2U F_1 - 2V G_1,
\Phi_{33} = (k_M - k_m + 1) \Sigma - 2U,
\Phi_{44} = \eta B^T P B - \eta^{-k_m} \Sigma - 2S,
\]

\[ \alpha = \text{diag} \{ \alpha_1, \ldots, \alpha_m \}, \ \alpha_j = \sum_{p=1}^{\infty} \mathcal{H}_j(p) \eta^p. \]

To investigate the exponential stability of the origin, it is necessary to calculate the difference \( \Delta V(n) = V(n+1) - V(n) \) along the trajectory of (6). From (6), we have

\[
\Delta V_1(n) = \eta^{n+1} x^T(n+1) P x(n+1) - \eta^n x^T(n) P x(n)
\]

\[
= \eta^{n+1} \left[ x^T(n) A P x(n) + 2 x^T(n) A P f x(n-k(n)) + f^T(n) B^T P B f x(n-k(n)) + 2 f^T(n) B^T P C f x(n) + \phi(x(n)) C^T P C f x(n) \right]
\]

\[
- \eta^n x^T(n) P x(n),
\]

where \( \phi(x(n)) = \sum_{p=1}^{\infty} \mathcal{H}(p) g(x(n-p)). \) Since \( k_m \leq k(n) \leq k_M, \) one obtains

\[
\Delta V_2(n) = \sum_{s=n+1-k(n+1)}^{n} \eta^s f^T(x(s)) \Sigma f(x(s)) - \sum_{s=n-k(n)}^{n-1} \eta^s f^T(x(s)) \Sigma f(x(s))
\]

\[
\leq \eta^n f^T(x(n)) \Sigma f(x(n))
\]

Then the origin of system (1) is exponentially stable.

Proof. Define a Lyapunov functional \( V(n) = V(x_1, \ldots, x_m)(n) \) as follows:

\[ V(n) = V_1(n) + V_2(n) + V_3(n) + V_4(n), \]

where

\[
V_1(n) = \eta^n x^T(n) P x(n),
V_2(n) = \sum_{s=n-k(n)}^{n-1} \eta^s f^T(x(s)) \Sigma f(x(s)),
V_3(n) = \sum_{r=-k+1}^{-k_m+1} \sum_{s=r+2}^{n-1} \eta^s f^T(x(s)) \Sigma f(x(s)),
V_4(n) = \sum_{j=1}^{m} \sum_{p=1}^{\infty} \mathcal{H}_j(p) \sum_{s=r-p}^{n-1} \eta^s g^2_j(x_j(s)) e_j.
\]
\[
\Delta V_n (n) = \sum_{j=1}^{m} \sum_{p=1}^{\infty} \mathcal{H}_j (p) [\eta^{p} g_j^2 (x(n)) - \eta^n g_j^2 (x (n-p)) ] e_j \\
= \eta^n \sum_{j=1}^{m} \sum_{p=1}^{\infty} \mathcal{H}_j (p) g_j^2 (x (n)) e_j \\
- \eta^n \sum_{j=1}^{m} \sum_{p=1}^{\infty} \mathcal{H}_j (p) g_j^2 (x (n-p)) e_j \\
= \eta^n [g^T (x(n)) \alpha \Sigma g (x(n)) \\
- \phi^T (x(n)) \Sigma \phi (x(n))] .
\]

Therefore, we have
\[
\Delta V (n) = \Delta V_1 (n) + \Delta V_2 (n) + \Delta V_3 (n) \\
\leq \eta^n \left[ x^T (n) (\eta \text{APA} - P) x (n) \\
+ 2x^T (n) A P B f (x (n-k (n))) \\
+ 2x^T (n) A P C \phi (x (n)) \\
+ (k M - k_m + 1) f^T \\
\times (x (n)) \Sigma f (x (n)) \\
+ f^T (x (n-k (n))) (\eta B^T P B - \eta^{+ M} \Sigma) \\
\times f (x (n-k (n))) \\
+ 2 f^T (x (n-k (n))) B^T P C \phi (x (n)) \\
+ g^T (x(n)) \alpha \Sigma g (x(n)) \\
+ \phi (x(n))^T (\eta C^T P C - \Sigma) \phi (x(n)) \right] .
\]
\]

From (H3), one has
\[
(f_j (x_j (n)) - f_j x_j (n)) (f_j (x_j (n)) - f_j x_j (n)) \leq 0 , \\
(g_j (x_j (n)) - g_j x_j (n)) (g_j (x_j (n)) - g_j x_j (n)) \leq 0 , \\
j = 1, 2, \ldots, m.
\]

Then for \( U = \text{diag} (u_1, \ldots, u_m) \geq 0, V = \text{diag} (\nu_1, \ldots, \nu_m) \geq 0, \) and \( S = \text{diag} (s_1, \ldots, s_m) \geq 0, \) one has
\[
\Delta V (n) \leq \Delta V (n) - 2 \sum_{j=1}^{m} v_j (f_j (x_j (n)) - f_j x_j (n)) \\
\times (f_j (x_j (n)) - f_j x_j (n)) \\
- 2 \sum_{j=1}^{m} v_j (g_j (x_j (n)) - g_j x_j (n)) \\
\times (g_j (x_j (n)) - g_j x_j (n)) \\
- 2 \sum_{j=1}^{m} s_j (f_j (x_j (n-k (n))) - f_j x_j (n-k (n))) \\
\times (f_j (x_j (n-k (n))) - f_j x_j (n-k (n))) \\
\leq \xi^T W \xi ,
\]
where \( \xi = (x^T (n), x^T (n-k (n)), f^T (x(n)), f^T (x(n-k (n))), g^T (x(n)), \phi (x(n)))^T . \) From \( W < 0, \) it follows that \( \Delta V (n) < 0. \) Note that
\[
V (n) \geq \lambda_m (P) \eta^n \| x (n) \|^2 , \\
V (0) \leq x^T (0) P x (0) + (k M - k_m + 1) \\
\times \sum_{s=-k_m}^{1} \eta^n f^T (x (s)) \Sigma f (x (s)) \\
+ \sum_{j=1}^{m} \sum_{p=1}^{\infty} \mathcal{H}_j (p) \sum_{s=-p}^{1} \eta^p g_j^2 (x_j (s)) \\
\leq [\lambda_m (P) + \delta_1 L (k M - k_m + 1 + \delta_2)] \sup_{s \in N(-\infty,0)} \| x (s) \|^2 ,
\]
where \( \lambda_m (P), \lambda_M (P) \) are the minimum and maximum eigenvalues of matrix \( P, \) respectively, \( \delta_1 = 1/(\eta-1), \delta_2 = \sum_{j=1}^{m} e_j \alpha_j, \) and \( L = \max (|f_j |, |f_j|, |g_j|^2, |g^T |), \) so one has
\[
\| x (n, \phi) \|^2 \leq C^* \eta^{-n} \sup_{l \in N(-\infty,0)} \| \phi (l) \| , \quad n \in Z^+ ,
\]
where \( C^* = \lambda_m^{-1} (P)[\lambda_m (P) + \delta_1 L (k M - k_m + 1 + \delta_2)]. \) This implies that the equilibrium solution \( x = 0 \) of system (1) is globally exponentially stable. The proof is completed. \( \Box \)

Remark 3. By employing LMI (8), the signs of \( b_{ij}, c_{ij} , \) that is, the differences between neural excitatory and inhibitory interaction, are taken into consideration.

Remark 4. If \( \eta = 1, \) the equilibrium point 0 of system (1) is said to be globally stable.
4. Numerical Example

Next, an illustrative example is given to show the effectiveness of the obtained results. Consider the discrete-time neural model (6) with parameters:

\[
A = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.6 \end{pmatrix}, \quad B = \begin{pmatrix} 0.05 & -0.1 \\ -0.1 & 0.15 \end{pmatrix}, \quad C = \begin{pmatrix} 0.1 & -0.1 \\ 0.05 & 0.15 \end{pmatrix},
\]

\[
f_1(u) = \tanh(2u), \quad f_2(u) = \tanh(-2u),
\]

\[
g_1(u) = g_2(u) = \arctan(u), \quad \mathcal{X}_1(p) = \mathcal{X}_2(p) = \frac{3}{4p}, \quad k(n) = 4 + 2(-1)^n,
\]

then it is not difficult to see that \(F_1 = G_1 = 0, F_2 = \text{diag}[2, -2], G_2 = \text{diag}[1, 1], k_M = 6, \text{ and } k_m = 2\). Take \(\eta = 2\), then by solving LMI (8), it has feasible solutions that are \(\Sigma = S = V = I\) and

\[
P = \begin{pmatrix} 19.8208 & -1.8309 \\ -1.8309 & 17.0882 \end{pmatrix}, \quad U = \begin{pmatrix} 2.5213 & 0 \\ 0 & 2.5710 \end{pmatrix},
\]

so from Theorems 2, this system admits a unique equilibrium 0, with all other solutions converging to it exponentially as \(n \to \infty\) see Figures 1 and 2.

5. Conclusions

In the current paper, a class of discrete-time neural networks with both variable and distributed delays has been studied. Using Lyapunov stability and LMI technique, the existence and global exponential stability of the unique equilibrium point have been established. The obtained results are easy to verify, so they will be of practical use for applying discrete neural models.

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References


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