Research Article

Semigroup Method on a $M^X/G/1$ Queueing Model

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By using the Hille-Yosida theorem, Phillips theorem, and Fattorini theorem in functional analysis we prove that the $M^X/G/1$ queueing model with vacation times has a unique nonnegative time-dependent solution.

1. Introduction

The queueing system when the server become idle is not new. Miller [1] was the first to study such a model, where the server is unavailable during some random length of time for the $M/G/1$ queueing system. The $M/G/1$ queueing models of similar nature have also been reported by a number of authors, since Levy and Yechiali [2] included several types of generalizations of the classical $M/G/1$ queueing system. These generalizations are useful in model building in many real life situations such as digital communication, computer network, and production/inventory system [3–5].

At present, however, most studies are devoted to batch arrival queues with vacation because of its interdisciplinary character. Considerable efforts have been devoted to study these models by Baba [6], Lee and Srinivasan [7], Lee et al. [8, 9], Borthakur and Choudhury [10], and Choudhury [11, 12] among others. However, the recent progress of $M^X/G/1$ type queueing models of this nature has been served by Chae and Lee [13] and Medhi [14].

In 2002, Choudhury [15] studied the $M^X/G/1$ queueing model with vacation times. By using the supplementary variable technique [16] he established the corresponding queueing model and obtained the queue size distribution at a stationary (random) as well as a departure point of time under multiple vacation policy based on the following hypothesis.

“Time dependent solution of the model converges to a nonzero steady-state solution.” By reading the paper we find that the previous hypothesis, in fact, implies the following two hypothesis.

Hypothesis 1. The model has a nonnegative time-dependent solution.

Hypothesis 2. The time-dependent solution of the model converges to a nonzero steady-state solution.

In this paper we investigate Hypothesis 1. By using the Hille-Yosida theorem, Phillips theorem, and Fattorini theorem in functional analysis we prove that the $M^X/G/1$ queueing model with vacation times has a unique nonnegative time-dependent solution.

According to Choudhury [15], the $M^X/G/1$ queueing system with vacation times can be described by the following system of equations:

$$
dQ(t) = -\lambda Q(t) + \int_0^\infty v(x) P_{0,0}(x,t) dx + \int_0^\infty b(x) P_{1,1}(x,t) dx,$$

$$\frac{\partial P_{0,0}(x,t)}{\partial t} + \frac{\partial P_{0,0}(x,t)}{\partial x} = -[\lambda + v(x)] P_{0,0}(x,t),$$

$$\frac{\partial P_{0,n}(x,t)}{\partial t} + \frac{\partial P_{0,n}(x,t)}{\partial x} = -[\lambda + v(x)] P_{0,n}(x,t) + \lambda \sum_{k=1}^{n} c_k P_{0,n-k}(x,t), \quad n \geq 1,$$

$$\frac{\partial P_{1,n}(x,t)}{\partial t} + \frac{\partial P_{1,n}(x,t)}{\partial x} = -[\lambda + v(x)] P_{1,n}(x,t).$$
\[-[\lambda + b (x)] P_{1,n} (x, t)
\]
\[+ \lambda \sum_{k=1}^{n} c_k P_{1,n-k+1} (x, t), \quad n \geq 1,\]
\[P_{0,0} (t) = \lambda Q (t), \quad P_{0,n} (0, t) = 0, \quad n \geq 1,\]
\[P_{1,n} (0, t) = \int_{0}^{\infty} v (x) P_{0,n} (x, t) \, dx
\]
\[+ \int_{0}^{\infty} b (x) P_{1,n-1} (x, t) \, dx, \quad n \geq 1,\]
\[Q (0) = 1, \quad P_{0,n} (x, 0) = 0, \quad n \geq 0,\]
\[P_{1,n} (x, 0) = 0, \quad n \geq 1,\]

(1)

where \((x, t) \in [0, \infty) \times [0, \infty)\); \(Q(t)\) represents the probability that there is no customer in the system and the server is idle at time \(t\); \(P_{0,n}(x, t) \, dx (n \geq 0)\) represents the probability that at time \(t\) there are \(n\) customers in the system and the server is on a vacation with elapsed vacation time of the server lying in \([x, x + dx)\). \(P_{1,n}(x, t) \, dx (n \geq 1)\) represents the probability that at time \(t\) there are \(n\) customers in the system with elapsed service time of the customer undergoing service lying in \([x, x + dx)\). \(\lambda\) is batch arrival rate of customers. \(c_k (k \geq 1)\) represents the probability that at every arrival epoch a batch of \(k\) external customers arrives and satisfies \(\sum_{k=1}^{n} c_k = 1\). \(v(x)\) is the vacation rate of the server, which satisfies

\[v(x) \geq 0, \quad \int_{0}^{\infty} v (x) \, dx = \infty.\]

(2)

\(b(x)\) is the service rate of the server satisfying

\[b(x) \geq 0, \quad \int_{0}^{\infty} b (x) \, dx = \infty.\]

(3)

2. Problem Formulation

We first formulate the system (1) as an abstract Cauchy problem on a suitable state space. For convenience we take some notations as follows:

\[
\Gamma_1 = \begin{pmatrix}
e^{-x} & 0 & 0 & \cdots \\
\lambda e^{-x} & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix}
0 & 0 & v(x) & 0 & \cdots \\
0 & 0 & 0 & v(x) & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

\[
\Gamma_3 = \begin{pmatrix}
0 & b(x) & 0 & 0 & \cdots \\
0 & 0 & b(x) & \cdots \\
0 & 0 & 0 & b(x) & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(4)

If we take state space

\[
X = \left\{ (P_0, P_1) \in \mathbb{R} \times L^1 \left[ 0, \infty \right), \ P_0 (x) \in L^1 \left[ 0, \infty \right), \ P_1 (x) \in L^1 \left[ 0, \infty \right) \mid \| (P_0, P_1) \| = |Q| + \sum_{i=1}^{n} \left| \int_{0}^{\infty} \left( P_{i,0} \right) \, dx \right| \right\}
\]

(5)

then it is obvious that \(X\) is a Banach space. In the following we define operators and their domains;

\[
A (P_0, P_1) = \begin{pmatrix}
-\lambda & 0 & 0 & \cdots \\
0 & -d/dx & 0 & \cdots \\
0 & 0 & -d/dx & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \begin{pmatrix}
Q \\
P_{0,0} (x) \\
P_{0,1} (x) \\
\vdots
\end{pmatrix},
\]

\[
D (A) = \begin{pmatrix}
\frac{dP_{0,n} (x)}{dx} & \frac{dP_{1,k} (x)}{dx} \in L^1 \left[ 0, \infty \right), \\
P_{0,n} (x), P_{1,k} (x) (n \geq 0, \ k \geq 1) \end{pmatrix}
\]

are absolutely continuous functions and satisfy

\[
P_0 (0) = \int_{0}^{\infty} \Gamma_1 P_0 (x) \, dx,
\]

\[
P_1 (0) = \int_{0}^{\infty} \Gamma_2 P_1 (x) \, dx
\]

(6)

\[
U (P_0, P_1) = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & \lambda_1 & 0 & 0 & \cdots \\
0 & \lambda_2 & \lambda_1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \begin{pmatrix}
Q \\
P_{0,0} (x) \\
P_{0,1} (x) \\
P_{0,2} (x) \\
\vdots
\end{pmatrix},
\]

(7)
where \( \tilde{\nu} = -[\lambda + \nu(x)] \), \( \tilde{b} = -[\lambda + b(x)] + \lambda c_1 \),

\[
E(P_0, P_1) = \left( \begin{array}{c}
\int_0^\infty v(x) P_{0,0}(x) \, dx + \int_0^\infty b(x) P_{1,1}(x) \, dx \\
0 \\
\vdots \\
0 \\
0 \\
\end{array} \right),
\]

\[
D(U) = D(E) = X.
\]

Then the previous system of equations (1) can be rewritten as an abstract Cauchy problem in the Banach space \( X \):

\[\frac{d}{dt} (P_0, P_1)(t) = (A + U + E)(P_0, P_1)(t), \quad \forall t \in [0, \infty),\]

\[
(P_0, P_1)(0) = \left( \begin{array}{c}
1 \\
0 \\
\vdots \\
0 \\
0 \\
\end{array} \right).
\]

3. Well-Posedness of The System (8)

Theorem 1. If \( v = \sup_{x \in [0, \infty)} v(x) < \infty, \ b = \sup_{x \in [0, \infty)} b(x) < \infty, \) then \( A + U + E \) generates a positive contraction \( C_0 \)-semigroup \( T(t) \).

Proof. We split the proof of the theorem into four steps. Firstly, we prove that \((yI-A)^{-1}\) exists and is bounded for some \( y \). Secondly, we show that \( D(A) \) is dense in \( X \). Thirdly, we verify that \( U \) and \( E \) are bounded linear operators. Thus by using the Hille- Yosida theorem and the perturbation theorem of \( C_0 \)-semigroup we deduce that \( A + U + E \) generates a \( C_0 \)-semigroup \( T(t) \). Finally, we check that \( A + U + E \) is dispersive, and therefore we obtain the desired result.

For any given \((y_0, y_1) \in X \), we consider the equation \((yI - A)(P_0, P_1) = (y_0, y_1)\); that is,

\[ (y + \lambda) Q = y_Q, \]

through solving (9)–(11), we have

\[ Q = \frac{1}{y + \lambda} y_Q, \]

\[ P_{0,n}(x) = a_{0,n} e^{-\gamma x} + \int_0^x y_{0,n}(\tau) e^{\gamma \tau} d\tau, \quad n \geq 0, \]

\[ P_{1,n}(x) = a_{1,n} e^{-\gamma x} + \int_0^x y_{1,n+1}(\tau) e^{\gamma \tau} d\tau, \quad n \geq 1. \]

Combining (16) with (12) and (13), we obtain

\[ a_{0,0} = P_{0,0}(0) = \lambda Q = \frac{\lambda}{y + \lambda} y_Q, \]

\[ a_{0,n} = P_{0,n}(0) = 0, \quad n \geq 1. \]

Substituting (19) into (16), it follows that

\[ P_{0,n}(x) = e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma \tau} d\tau, \quad n \geq 1. \]

By combining (15), (16), (17), and (20) with (14), we deduce

\[ a_{1,n} = P_{1,n}(0) = \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma \tau} d\tau \, dx \]

\[ + a_{1,n+1} \int_0^\infty b(x) e^{-\gamma x} \, dx \]

\[ + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,n+1}(\tau) e^{\gamma \tau} d\tau \, dx, \quad n \geq 1, \]

\[ \implies a_{1,n} - a_{1,n+1} \int_0^\infty b(x) e^{-\gamma x} \, dx \]

\[ = \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma \tau} d\tau \, dx \]

\[ + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,n+1}(\tau) e^{\gamma \tau} d\tau \, dx. \]
\[ C = \begin{pmatrix} 1 - \int_0^\infty b(x) e^{-\gamma x} dx & 0 & 0 & \cdots \\ 0 & 1 & -\int_0^\infty b(x) e^{-\gamma x} dx & 0 & \cdots \\ 0 & 0 & 1 & -\int_0^\infty b(x) e^{-\gamma x} dx & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \end{pmatrix}, \]

\[ \vec{a}_1 = \begin{pmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ \vdots \end{pmatrix}, \]

then (21) can be rewritten as follows:

\[ \mathcal{G} \vec{a}_1 = \begin{pmatrix} \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,1}(\tau) e^{\gamma \tau} d\tau dx + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{0,2}(\tau) e^{\gamma \tau} d\tau dx \\ \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,2}(\tau) e^{\gamma \tau} d\tau dx + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{0,3}(\tau) e^{\gamma \tau} d\tau dx \\ \vdots \end{pmatrix}. \]

It is easy to calculate

\[ \mathcal{G}^{-1} = \begin{pmatrix} 1 - \int_0^\infty b(x) e^{-\gamma x} dx & \left( \int_0^\infty b(x) e^{-\gamma x} dx \right)^2 & \left( \int_0^\infty b(x) e^{-\gamma x} dx \right)^3 & \cdots \\ 0 & 1 & \int_0^\infty b(x) e^{-\gamma x} dx & \left( \int_0^\infty b(x) e^{-\gamma x} dx \right)^2 & \cdots \\ 0 & 0 & 1 & \int_0^\infty b(x) e^{-\gamma x} dx & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \]

From which together with (24), we derive

\[ a_{1,n} = \sum_{k=0}^\infty \left( \int_0^\infty b(x) e^{-\gamma x} dx \right)^k \times \left\{ \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,k+n}(\tau) e^{\gamma \tau} d\tau dx \\ + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,k+n+1}(\tau) e^{\gamma \tau} d\tau dx \right\}, \quad n \geq 1. \]

(26)

By using Fubini theorem we estimate (16) and (17) as follows (assume that \( \gamma > v + b \)):

\[ \|P_{0,n}\|_{L^1[0,\infty)} \leq \int_0^\infty |a_{0,n}| e^{-\gamma x} dx + \int_0^\infty e^{-\gamma x} \int_0^x |y_{0,n}(\tau)| e^{\gamma \tau} d\tau dx \\ + b \int_0^\infty |y_{1,k+n+1}(\tau)| e^{\gamma \tau} \int_\tau^\infty e^{-\gamma x} dx d\tau \]

(27)

From (26) and Fubini theorem, we deduce

\[ \sum_{n=1}^\infty |a_{1,n}| \leq \sum_{n=1}^\infty \sum_{k=0}^\infty \left( \frac{b}{\gamma} \right)^k \]

\[ \times \left\{ v \int_0^\infty |y_{0,k+n}(\tau)| e^{\gamma \tau} \int_\tau^\infty e^{-\gamma x} dx d\tau \\ + b \int_0^\infty |y_{1,k+n+1}(\tau)| e^{\gamma \tau} \int_\tau^\infty e^{-\gamma x} dx d\tau \right\}. \]
\[
\begin{align*}
&= \frac{v}{y} \sum_{k=0}^{\infty} \left( \frac{b}{y} \right)^k \sum_{n=0}^{\infty} \|y_{0,n}\|_L^1[0,\infty) \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{b}{y} \right)^k \sum_{n=0}^{\infty} \|y_{1,n}\|_L^1[0,\infty) \\
&\leq \frac{v}{y} \sum_{k=0}^{\infty} \left( \frac{b}{y} \right)^k \sum_{n=0}^{\infty} \|y_{0,n}\|_L^1[0,\infty) \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{b}{y} \right)^k \sum_{n=0}^{\infty} \|y_{1,n}\|_L^1[0,\infty) \\
&= \frac{v}{y} \sum_{k=0}^{\infty} \left( \frac{b}{y} \right)^k \sum_{n=0}^{\infty} \|y_{0,n}\|_L^1[0,\infty) \\
&\quad + \frac{b}{y} \sum_{n=0}^{\infty} \|y_{1,n}\|_L^1[0,\infty) \\
&= \frac{v}{y} \sum_{n=0}^{\infty} \|y_{0,n}\|_L^1[0,\infty) + \frac{b}{y} \sum_{n=0}^{\infty} \|y_{1,n}\|_L^1[0,\infty) \\
\end{align*}
\]

(28)

By inserting (18), (19), and (28) into (27) and using inequality \((y + v - b)/y(y - b) < 1/(y - v - b)\), we estimate

\[
\| (P_0, P_1) \| \leq \frac{|y_0|}{y + \lambda} + \frac{1}{y} \sum_{n=0}^{\infty} |a_{0,n}| + \frac{1}{y} \sum_{n=0}^{\infty} \|y_{0,n}\|_L^1[0,\infty) \\
+ \frac{1}{y} \sum_{n=0}^{\infty} |a_{1,n}| + \frac{1}{y} \sum_{n=0}^{\infty} \|y_{1,n}\|_L^1[0,\infty) \\
\leq \frac{1}{y} |y_0| + \frac{v + b}{y} \sum_{n=0}^{\infty} \|y_{0,n}\|_L^1[0,\infty) \\
+ \frac{1}{y} \sum_{n=0}^{\infty} \|y_{1,n}\|_L^1[0,\infty) \\
= \frac{1}{y} \|y_0, y_1\|. \\
\]

(29)

Equation (29) shows that \((yI - A)^{-1}\) exists for \(y > v + b\) and

\[
(yI - A)^{-1} : X \rightarrow D(A), \quad \|(yI - A)^{-1}\| \leq \frac{1}{y - v - b}. \\
\]

(30)

As far as the second step is concerned, from \(|Q| + \sum_{n=0}^{\infty} \|P_{0,n}\|_L^1[0,\infty) + \sum_{n=1}^{\infty} \|P_{1,n}\|_L^1[0,\infty) < \infty\) for \((P_0, P_1) \in X\)

it follows that, for any \(e > 0\), there exists a positive integer \(N\) such that \(\sum_{n=N}^{\infty} \|P_{0,n}\|_L^1[0,\infty) < e\), \(\sum_{n=N}^{\infty} \|P_{1,n}\|_L^1[0,\infty) < \infty\). Let

\[
L = \left\{ (P_0, P_1) \mid \begin{array}{c} P_0 (x) = (Q, P_{0,0} (x), P_{1,1} (x), \ldots), \\
P_1 (x) = (P_{1,1} (x), P_{1,2} (x), \ldots), \\
P_{0,j} (x), P_{1,j} (x) \in L^1[0,\infty), \ \ i = 0, 1, \ldots, N; \\
j = 1, 2, \ldots, N \\
N \text{ is a finite positive integer} \end{array} \right\},
\]

(31)

then \(L\) is dense in \(X\). If we set

\[
Z = \left\{ (P_0, P_1) \mid \begin{array}{c} P_0 (x) = (Q, P_{0,0} (x), P_{0,1} (x), \ldots), \\
P_{0,j} (x) \in C_0^{\infty} [0,\infty), \\
P_{0,j} (x), P_{1,j} (x) \in C_0^{\infty} [0,\infty), \ \ i = 0, 1, \ldots, N; \\
j = 1, 2, \ldots, l \\
t \text{ such that } P_{0,j} (x) = 0, \ x \in [0, c_{0,j}], \\
P_{1,j} (x) = 0, \ x \in [0, c_{1,j}], \\
i = 0, 1, 2, \ldots, k; \ j = 1, 2, \ldots, l \end{array} \right\}
\]

(32)

then by the relationship \(C_0^{\infty} [0,\infty) \text{ and } L^1[0,\infty) \) in Adams [17], we know that \(Z\) is dense in \(L\). Hence in order to prove denseness of \(D(A)\), it suffices to prove that \(D(A)\) is dense in \(Z\). Take any \((P_0, P_1) \in Z\), then there are a finite positive integer \(l\) and positive numbers \(c_{0,j} > 0, c_{1,j} > 0\) such that \(P_{0,j} (x) = 0, \ x \in [0, c_{0,j}], \ P_{1,j} (x) = 0, \ x \in [0, c_{1,j}], \)

\[
\begin{align*}
P_0 (x) &= (Q, P_{0,0} (x), P_{1,1} (x), \ldots), \\
P_{0,j} (x) &= 0, \ x \in [0, 2c_{0,j}], i = 0, 1, 2, \ldots, l, \\
P_{1,j} (x) &= 0, \ x \in [0, 2c_{1,j}], j = 1, 2, \ldots, l, \\
\end{align*}
\]

(33)

then

\[
\begin{align*}
P_0 (x) &= (Q, P_{0,0} (x), P_{0,1} (x), \ldots), \\
P_{0,j} (x) &= 0, \ x \in [0, 2c_{0,j}], i = 0, 1, 2, \ldots, l, \\
P_{1,j} (x) &= 0, \ x \in [0, 2c_{1,j}], j = 1, 2, \ldots, l, \\
\end{align*}
\]

(34)

where \(0 < 2s < \min(c_{0,0}, c_{0,1}, \ldots, c_{0,l}, c_{1,1}, \ldots, c_{1,l})\). Define

\[
\begin{align*}
f_0^i (0) &= (Q, f_{0,0}^i (0), f_{0,1}^i (0), \ldots, f_{0,l}^i (0), 0, \ldots), \\
&= (Q, \lambda Q, 0, \ldots, 0, 0, \ldots), \\
f_1^j (0) &= (f_{1,1}^j (0), f_{1,2}^j (0), \ldots, f_{1,l}^j (0), 0, \ldots), \\
\end{align*}
\]

(35)
where
\[
f'_{1,j}(0) = \int_{2s}^{\infty} v(x) P_{0,j}(x) \, dx + \int_{2s}^{\infty} b(x) P_{l+j}(x) \, dx,
\]
\[j = 1, 2, \ldots, l-1,
\]
\[
f''_1(0) = \int_{2s}^{\infty} v(x) P_{0,j}(x) \, dx,
\]
\[f'_0(x) = (Q, f'_{0,0}(x), f'_{0,1}(x), \ldots, f'_{0,l}(x), 0, \ldots),
\]
\[f'_1(x) = (f'_1, f'_2, \ldots, f'_{l}(x), 0, \ldots),
\]
(36)

Moreover, we obtain that the operators in (37) generate a \(C_0\)-semigroup.

The previous two formulas show that \((f''_0, f'_1) \in D(A)\).

Then it is not difficult to verify that \((Q, f'_{0,0}) \in D(A)\).

Moreover,
\[
\| (P_0, P_1) - (f''_0, f'_1) \| = \sum_{i=0}^{l} \int_{2s}^{\infty} \left| P_{0,j}(x) - f''_{0,j}(x) \right| \, dx
\]
\[+ \sum_{j=1}^{l} \int_{2s}^{\infty} \left| P_{l+j}(x) - f'_{l+j}(x) \right| \, dx
\]
\[= \sum_{i=0}^{l} \int_{2s}^{\infty} \left| f''_{0,i}(0) \left(1 - \frac{x}{s} \right)^2 \right| \, dx
\]
\[+ \sum_{i=0}^{l} \left| \mu_{0,i} \right| \int_{2s}^{\infty} (x-s)^2(x-2s)^2 \, dx
\]
\[+ \sum_{j=1}^{l} \int_{2s}^{\infty} \left| f'_{l,j}(0) \left(1 - \frac{x}{s} \right)^2 \right| \, dx
\]
\[+ \sum_{j=1}^{l} \left| \mu_{l,j} \right| \int_{2s}^{\infty} (x-s)^2(x-2s)^2 \, dx
\]
(37)

The previous two formulas show that \(U\) and \(E\) are bounded operators. It is easy to check that \(U\) and \(E\) are linear operators. Hence from the perturbation theorem of \(C_0\)-semigroup [18], we obtain that \(A + U + E\) generates a \(C_0\)-semigroup \(T(t)\).

Lastly, we will prove that \(A + U + E\) is a dispersive operator. For \((P_0, P_1) \in D(A)\) we take \((\phi_0, \phi_1)\) as
\[
\phi_0(x) = \begin{pmatrix} [Q] & [P_{0,0}(x)] & [P_{0,1}(x)] & \ldots \end{pmatrix},
\]
\[\phi_1(x) = \begin{pmatrix} [P_{1,1}(x)] & [P_{1,2}(x)] & \ldots \end{pmatrix}.
\]
(40)
where
\[
[Q]^+ = \begin{cases} 
Q, & \text{if } Q > 0, \\
0, & \text{if } Q \leq 0,
\end{cases}
\]
\[
[P_{0,n}(x)]^+ = \begin{cases} 
P_{0,n}(x), & \text{if } P_{0,n}(x) > 0, \\
0, & \text{if } P_{0,n}(x) \leq 0, 
\end{cases} \quad n \geq 0,
\]
\[
[P_{1,n}(x)]^+ = \begin{cases} 
P_{1,n}(x), & \text{if } P_{1,n}(x) > 0, \\
0, & \text{if } P_{1,n}(x) \leq 0, 
\end{cases} \quad n \geq 1.
\]

If we define \( V_{0,i} = \{ x \in [0,\infty) \mid P_{0,i}(x) > 0 \} \) and \( W_{0,i} = \{ x \in [0,\infty) \mid P_{0,i}(x) \leq 0 \} \) for \( i = 0, 1, 2, \ldots \), then by a short argument we calculate
\[
\int_0^\infty \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx
= \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx
+ \int_{W_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \quad (42)
= \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx
= \int_0^\infty \frac{d[P_{0,i}(x)]^+}{dx} dx = -[P_{0,i}(0)^+]^+, \quad i \geq 0.
\]

Similar to (42), we get
\[
\int_0^\infty \frac{dP_{1,i}(x)}{dx} \frac{[P_{1,i}(x)]^+}{P_{1,i}(x)} dx = -[P_{1,i}(0)^+]^+, \quad j \geq 1. \quad (43)
\]

By using boundary conditions on \((P_0, P_i) \in D(A), (42), (43),\) and \(\sum_{k=1}^\infty c_k = 1\) for such \((\phi_0, \phi_i)\), we derive
\[
\langle (A + U + E)(P_0, P_i), (\phi_0, \phi_i) \rangle
= -\lambda Q + \int_0^\infty v(x) P_{0,0}(x) dx
+ \int_0^\infty b(x) P_{1,1}(x) dx \frac{[Q]^+}{Q}
+ \int_0^\infty \left\{ - \frac{dP_{0,0}(x)}{dx} - [\lambda + v(x)] P_{0,0}(x) \right\} \frac{[P_{0,0}(x)]^+}{P_{0,0}(x)} dx
+ \sum_{n=1}^\infty \int_0^\infty \left\{ - \frac{dP_{0,n}(x)}{dx} - [\lambda + v(x)] P_{0,n}(x) \right\} \frac{[P_{0,n}(x)]^+}{P_{0,n}(x)} dx
+ \lambda \sum_{k=1}^n \int_0^\infty \left\{ - \frac{dP_{0,n-k}(x)}{dx} - [\lambda + v(x)] P_{0,n-k}(x) \right\} \frac{[P_{0,n-k}(x)]^+}{P_{0,n-k}(x)} dx
+ \lambda \sum_{k=1}^\infty \int_0^\infty \left\{ - \frac{dP_{1,1}(x)}{dx} - [\lambda + b(x)] P_{1,1}(x) \right\} dx
\leq \left( \frac{[Q]^+}{Q} - 1 \right)
\times \left( \int_0^\infty v(x) [P_{0,0}(x)]^+ dx + \int_0^\infty b(x) [P_{1,1}(x)]^+ dx \right)
- \sum_{n=1}^\infty \int_0^\infty \lambda [P_{0,n}(x)]^+ dx + \lambda \sum_{k=1}^\infty \int_0^\infty [P_{0,n-k}(x)]^+ dx
- \sum_{n=1}^\infty \int_0^\infty \lambda [P_{1,n}(x)]^+ dx + \lambda \sum_{k=1}^\infty \int_0^\infty [P_{1,n-k}(x)]^+ dx
\leq 0.
\]
Equation (44) shows that \( A + U + E \) is a dispersive operator. From which together with the first step, the second step, and the Phillips theorem, we know that \( A + U + E \) generates a positive contraction \( C_0 \)-semigroup [18]. By the uniqueness of a \( C_0 \)-semigroup we conclude that this semigroup is just \( T(t) \).

It is not difficult to see that \( X^* \), the dual space of \( X \), is
\[
X^* = \left\{ \left[ q_0^*, q_1^* \right] \in \mathbb{R} \times \mathbb{R} \mid \begin{array}{c}
q_0^* = (q_1^*, q_0^*, q_1^*, \ldots) \\
q_1^* = (q_1^*, q_0^*, q_1^*, \ldots)
\end{array} \in L^\infty(0, \infty) \times L^\infty(0, \infty),
\right\}
\]
(45)

It is easy to check that \( X^* \) is a Banach space. If we take a set \( S \) in \( X \) as
\[
S = \{ (P_0, P_1) \in X \mid Q \geq 0, P_{0,n}(x) \geq 0, n \geq 0 \},
\]
then \( S \) is a cone in \( X \). For \( (P_0, P_1) \in D(A) \cap S \), we take
\[
(q_0^*, q_1^*) = \| (P_0, P_1) \| \left( \begin{array}{c}
1 \\
0 \\
\vdots
\end{array} \right), \quad (q_0^*, q_1^*) \in X^*,
\]
(47)

then we have
\[
\langle (P_0, P_1), (q_0^*, q_1^*) \rangle
= \| (P_0, P_1) \| \int_0^\infty \frac{dP_0}{dx} + \int_0^\infty \frac{dP_1}{dx} d(P_0, P_1)
\]
(48)

that is
\[
(q_0^*, q_1^*) \in \theta((P_0, P_1))
= \left\{ \left[ q_0^*, q_1^* \right] \in X^* \mid \langle (P_0, P_1), (q_0^*, q_1^*) \rangle \right\}
\]
(49)

For such \( (q_0^*, q_1^*) \), by using boundary conditions on \( (P_0, P_1) \in D(A) \cap S \) and \( \sum_{k=1}^\infty c_k = 1 \), we have
\[
\langle (A + U + E)(P_0, P_1), (q_0^*, q_1^*) \rangle
= \| (P_0, P_1) \|
\times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x) dx + \int_0^\infty b(x) P_{1,1}(x) dx \right\}
+ \int_0^\infty \left\{ -\frac{dP_{0,0}}{dx} - [\lambda + v(\lambda)] P_{0,0}(x) \right\} \| (P_0, P_1) \| dx
\]
(50)

Which shows that \( A + U + E \) is a conservative operator. So we can use the Fattorini theorem [19] and state it as follows.

**Theorem 2.** \( T(t) \) is isometric for the initial value of the system (8); that is,
\[
\| T(t)(P_0, P_1)(0) \| = \| (P_0, P_1)(0) \|, \quad \forall t \in [0, \infty).
\]
(51)
Proof. Since $A + U + E$ is conservative with respect to $\theta$ and $(P_0, P_1)(0) \in D(A^2) \cap S$, from the Taylor expansion of $T(t + h)$ for $t, h \geq 0$ we have

\[
T(t + h)(P_0, P_1)(0) = T(t)(P_0, P_1)(0) + h(A + U + E)T(t)(P_0, P_1)(0) + \int_{t}^{t+h} (t + h - s) A(s)U(s)E(s) \, ds,
\]


Since $\epsilon$ is arbitrary, it follows that $\|T(t_0 + \eta)(P_0, P_1)(0)\| = \|T(t_0)(P_0, P_1)(0)\|$, which contradicts the fact that $t_0$ is the right endpoint of $\Omega$. Hence $\Omega = [0, \infty)$. That is, $\|T(t)(P_0, P_1)(0)\| = \|(P_0, P_1)(0)\|$ for $t \in [0, \infty)$. The proof of the theorem is complete. \qed

From Theorems 1 and 2 we obtain the main result in this paper.

**Theorem 3.** If $v = \sup_{x \in [0, \infty)} v(x) < \infty$, $b = \sup_{x \in [0, \infty)} b(x) < \infty$, then the system (8) has a unique nonnegative time-dependent solution $(P_0, P_1)(x,t)$, which can be expressed as

\[
(P_0, P_1)(x,t) = T(t)(P_0, P_1)(0), \quad \forall t \in [0, \infty).
\]

From which together with Theorem 2 (i.e., (51)) we have

\[
\|(P_0, P_1)(\cdot,t)\| = \|T(t)(P_0, P_1)(0)\| = \|(P_0, P_1)(0)\| = 1, \quad \forall t \in [0, \infty),
\]

this just reflects the physical background of the problem. \qed

**4. Concluding Remarks**

If we know the spectrum of $A + U + E$ on the imaginary axis, then by Theorem 1 and Theorem 14 in Gupur et al. [18], we obtain the asymptotic behavior of the time-dependent solution of the system (8), which describes Hypothesis 2. It is our next research work.

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**References**


