Research Article

Representations of 3-Dimensional Simple Multiplicative Hom-Lie Algebras

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The aim of this paper is to study representations of 3-dimensional simple multiplicative Hom-Lie algebras $(\mathfrak{g}; [\cdot, \cdot]; \alpha)$ (whose structure is of A_1 -type). In this paper we can see that a finite dimensional representation of $(\mathfrak{g}; [\cdot, \cdot]; \alpha)$ is not always completely reducible, and a representation of $(\mathfrak{g}; [\cdot, \cdot]; \alpha)$ is irreducible if and only if it is a regular Lie-type representation.

1. Introduction

In 2006, Hartwig, Larsson, and Silvestrov introduced the notion of a Hom-Lie algebra [1], which is a generalization of the notion of a Lie algebra. In particular, if $\alpha = \text{id}$, then a Hom-Lie algebra is exactly a Lie algebra.

Because the Hom-Lie algebras are closely related to discrete and deformed vector fields, differential calculus [2, 3], and mathematical physics [4, 5], the Hom-Lie algebras have attracted more and more attention and become an active topic in recent years [6–8].

The representation theory plays an important role in Lie theory [9–11]. By means of the representation theory, we would be more aware of the corresponding algebras. Thus it is meaningful to obtain more information about the representations of Hom-Lie algebras.

In [7] the author defined the representations of Hom-Lie algebras and the corresponding Hom-cochain complexes, and studied the cohomologies associated with the adjoint representation and the trivial representation. As is known, specific calculations about the representations of Hom-Lie algebras are still not solved. The diversity of the twist map of $\mathfrak{g}$ makes this topic interesting and complicated.

Thanks to the relationship between multiplicative Hom-Lie algebras with α invertible and Lie algebras (Lemma 3), the representation theory of Lie algebras can be a reference to what is considered. The representation of a 3-dimensional simple Lie algebra plays a crucial role in the representation

theory of semisimple Lie algebras over $\mathbb{C}$ [9]. By the same reason, in this paper, we study the representations of 3-dimensional simple multiplicative Hom-Lie algebras.

The paper is organized as follows. In Section 2 we study the structures of 3-dimensional simple multiplicative Hom-Lie algebra and show that 3-dimensional simple multiplicative Hom-Lie algebras are of A_1 -type. In Section 3, the representation (ρ, V, β) of a multiplicative Hom-Lie algebra with α invertible is investigated and shows that when β is invertible, (ρ, V, β) is of Lie-type, which makes it convenient to study representations of multiplicative Hom-Lie algebras. In Section 4, we study regular Lie-type representations of 3-dimensional simple multiplicative Hom-Lie algebras and reflect the existence and irreducibility of representations of this type. In Section 5, we work over finite dimensional representations of 3-dimensional simple multiplicative Hom-Lie algebras $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. In this section we can see that a finite dimensional representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is not always completely reducible and a representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is irreducible if and only if it is a regular Lie-type one.

Throughout this paper, unless otherwise stated, all algebras are finite dimensional and over the complex field $\mathbb{C}$.

2. The Structures of 3-Dimensional Simple Multiplicative Hom-Lie Algebras

First we give some important definitions on Hom-Lie algebras.

Definition 1 (see [1]). A Hom-Lie algebra is a triple $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ consisting of a vector space $\mathfrak{g}$ over $\mathbb{C}$, a linear self-map α , and a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$[x, y] = -[y, x], \quad \forall x, y \in \mathfrak{g}, \quad (1)$$

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad (2)$$

$$\forall x, y, z \in \mathfrak{g}.$$

A Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is said to be multiplicative if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ for any $x, y \in \mathfrak{g}$; see [7].

Definition 2. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra. If there exists a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]')$ such that $[x, y] = \alpha([x, y]')$ for all $x, y \in \mathfrak{g}$, then $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is said to be of Lie-type, and $(\mathfrak{g}, [\cdot, \cdot]')$ is called the compatible Lie algebra of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. Furthermore, if the compatible Lie algebra of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is an A (or B, C, D)-type Lie algebra, one calls $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ an A (or B, C, D)-type Hom-Lie algebra.

A subspace $\mathfrak{g}_1$ of $\mathfrak{g}$ is called an ideal of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ if $\alpha(\mathfrak{g}_1) \subseteq \mathfrak{g}_1$, $[\mathfrak{g}_1, \mathfrak{g}] \subseteq \mathfrak{g}_1$ are satisfied. A center of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is defined as

$$C(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0; [\alpha(x), \mathfrak{g}] = 0\}. \quad (3)$$

A Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is called simple if it has no nontrivial ideals and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Lemma 3. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra with α invertible. Then $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is Lie-type with the compatible Lie algebra $(\mathfrak{g}, [\cdot, \cdot]')$, where $[\cdot, \cdot]'$ is defined by $[x, y]' = \alpha^{-1}([x, y])$, for all $x, y \in \mathfrak{g}$.

Proof. Let $[x, y]' = \alpha^{-1}([x, y])$ for any $x, y \in \mathfrak{g}$. Since $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is multiplicative, we have

$$\begin{aligned} \alpha([x, y]) &= [\alpha(x), \alpha(y)] \implies \alpha^2([x, y]') \\ &= \alpha([\alpha(x), \alpha(y)]'), \end{aligned} \quad (4)$$

and thus $\alpha([x, y]') = [\alpha(x), \alpha(y)]'$ follows.

In the following we will show that $(\mathfrak{g}, [\cdot, \cdot]')$ is a Lie algebra. First it is obvious that $[\cdot, \cdot]'$ is skew-symmetric. Next $\forall x, y, z \in \mathfrak{g}$,

$$\begin{aligned} \varpi_{x,y,z} [x, [y, z]]' &= \varpi_{x,y,z} \alpha^{-1} [x, \alpha^{-1}([y, z])] \\ &= \varpi_{x,y,z} \alpha^{-1} [\alpha^{-1} \alpha(x), \alpha^{-1}([y, z])] \\ &= \varpi_{x,y,z} \alpha^{-2} ([\alpha(x), [y, z]]) = 0, \end{aligned} \quad (5)$$

where $\varpi_{x,y,z}$ denotes a summation over the cyclic permutation on x, y, z . Now it follows that $(\mathfrak{g}, [\cdot, \cdot]')$ is a Lie algebra. $\square$

Theorem 4. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a 3-dimensional simple multiplicative Hom-Lie algebra; then $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is A_1 -type and

$$\begin{aligned} &(\mathfrak{g}, [\cdot, \cdot], \alpha) \\ &= \{h, e, f \mid [h, e] = 2ae, [h, f] = -2a^{-1}f, [e, f] = h\}, \\ &\alpha(h, e, f) = (h, e, f) \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}_{0 \neq a \in \mathbb{C}}. \end{aligned} \quad (6)$$

Proof. If $\text{Ker}(\alpha) \neq 0$, then $\alpha(\text{Ker}(\alpha)) = 0$, $\alpha([\mathfrak{g}, \text{Ker}(\alpha)]) = 0$; that is, $\text{Ker}(\alpha)$ is a nontrivial ideal of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, which is a contradiction to the simplicity of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. So α is invertible. Now by Lemma 3, we have that $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is Lie-type with the 3-dimensional Hom-Lie admissible algebra $(\mathfrak{g}, [\cdot, \cdot]') = (\mathfrak{g}, \alpha^{-1}([\cdot, \cdot]))$.

If $(\mathfrak{g}, [\cdot, \cdot]')$ is an abelian Lie algebra, then we can deduce that $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ is also abelian, which is absurd.

Suppose that $(\mathfrak{g}, [\cdot, \cdot]')$ has a 1-dimensional center $\mathbb{C}x$. Note that $[x, \mathfrak{g}] = \alpha([\alpha(x), \mathfrak{g}]) = 0$, and for all $y \in \mathfrak{g}$, $\exists y' \in \mathfrak{g}$ such that $y = \alpha(y')$, we have

$$[\alpha(x), y] = [\alpha(x), \alpha(y')] = \alpha([x, y']) = 0. \quad (7)$$

That is, $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ has a 1-dimensional center $\mathbb{C}x$, which is a contradiction to the simplicity of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$.

If $[\mathfrak{g}, \mathfrak{g}]' \neq \mathfrak{g}$, then

$$[\mathfrak{g}, \mathfrak{g}] = \alpha^{-1}([\mathfrak{g}, \mathfrak{g}]') = [\alpha^{-1}(\mathfrak{g}), \alpha^{-1}(\mathfrak{g})]' = [\mathfrak{g}, \mathfrak{g}]' \neq \mathfrak{g}, \quad (8)$$

which is impossible.

Now we can get that $[\mathfrak{g}, \mathfrak{g}]' = \mathfrak{g}$. By Lie theory, $(\mathfrak{g}, [\cdot, \cdot]')$ is an A_1 -type Lie algebra with a basis $\{h, e, f\}$ and a bracket $[h, e]' = 2e$, $[h, f]' = -2f$, $[e, f]' = h$. On one hand, by the proof of Lemma 3, we have that α is an automorphism of $(\mathfrak{g}, [\cdot, \cdot]')$. On the other hand, by Lie theory, the automorphism of A_1 has the form $\alpha = \exp^{adkh}$, $k \in \mathbb{C}$. Thus

$$\alpha(h, e, f) = (h, e, f) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp^{2k} & 0 \\ 0 & 0 & \exp^{-2k} \end{pmatrix}. \quad (9)$$

Now it follows that

$$\begin{aligned} [h, e] &= \alpha([h, e]') = 2\exp^{2k}e, \\ [h, f] &= \alpha([h, f]') = -2\exp^{-2k}f, \\ [e, f] &= \alpha([e, f]') = h. \end{aligned} \quad (10)$$

Let $a = \exp^{2k} \neq 0$. The result follows. $\square$

3. The Representations of Multiplicative Hom-Lie Algebras

First we give the definition of the representations of multiplicative Hom-Lie algebras.

Definition 5 (see [7]). Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra, V a finite dimensional vector space, and $\beta \in \text{gl}(V)$. If a linear map $\rho : \mathfrak{g} \rightarrow \text{gl}(V)$ satisfies

$$\rho([x, y])\beta(v) = \rho(\alpha(x))\rho(y)v - \rho(\alpha(y))\rho(x)v, \quad (11)$$

$$\forall x, y \in \mathfrak{g}, v \in V,$$

$$3\beta(\rho(x)v) = \rho(\alpha(x))\beta(v), \quad \forall x \in \mathfrak{g}, v \in V, \quad (12)$$

then (ρ, V, β) is called a representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, and (V, β) is called a Hom- $\mathfrak{g}$ -module via the action $xv = \rho(x)v$, for all $x \in \mathfrak{g}, v \in V$.

For a Hom- $\mathfrak{g}$ -module (V, β) , if a subspace $V_1 \subseteq V$ is invariant under β , then (V_1, β) is called a Hom- $\mathfrak{g}$ -submodule of (V, β) . A Hom- $\mathfrak{g}$ -module (V, β) is called irreducible, if it has precisely two Hom- $\mathfrak{g}$ -submodules (itself and 0). A Hom- $\mathfrak{g}$ -module (V, β) is called completely reducible if $V = V_1 \oplus \dots \oplus V_s$, where $\beta(V_i) \subseteq V_i$ ($i = 1, \dots, s$) and (V_i, β) ($i = 1, \dots, s$) are irreducible Hom- $\mathfrak{g}$ -submodules.

Proposition 6. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie algebra with α invertible, (ρ, V, β) its representation with β invertible, and $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}')$ the compatible Lie algebra. Let $\rho' = \beta^{-1}\rho$; then (ρ', V) is a representation of $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}')$.

Proof. Equation (12) is equivalent to

$$\rho(\alpha(x)) = \beta\rho(x)\beta^{-1}, \quad \forall x \in \mathfrak{g}. \quad (13)$$

Equation (11) is equivalent to

$$\rho[x, y] = \beta\rho(x)\beta^{-1}\rho(y)\beta^{-1} - \beta\rho(y)\beta^{-1}\rho(x)\beta^{-1},$$

$$\forall x \in \mathfrak{g}, v \in V. \quad (14)$$

Denote that by $\alpha(x) = x'$, (13) can be rewritten as $\rho(\alpha^{-1}(x')) = \beta^{-1}\rho(x')\beta$; by the arbitrary of x and the invertibility of α we have

$$\rho(\alpha^{-1}(x)) = \beta^{-1}\rho(x)\beta, \quad \forall x \in \mathfrak{g}. \quad (15)$$

On vector space $\beta^{-1}\rho(\mathfrak{g})$, for all $x, y \in \mathfrak{g}$, define a commutator bracket $[\cdot, \cdot]'$ as

$$[\beta^{-1}\rho(x), \beta^{-1}\rho(y)]'$$

$$= \beta^{-1}\rho(x)\beta^{-1}\rho(y) - \beta^{-1}\rho(y)\beta^{-1}\rho(x) \quad (16)$$

$$= [\rho'(x), \rho'(y)]'.$$

Clearly, $(\beta^{-1}\rho(\mathfrak{g}), [\cdot, \cdot]')$ is a Lie algebra.

On the other hand, for all $x, y \in \mathfrak{g}$,

$$\rho'([x, y]_{\mathfrak{g}}') = \rho'(\alpha^{-1}([x, y]))$$

$$= \beta^{-1}\rho([\alpha^{-1}(x), \alpha^{-1}(y)])$$

$$= \beta^{-1}(\beta\rho(\alpha^{-1}(x))\beta^{-1}\rho(\alpha^{-1}(y))\beta^{-1}$$

$$- \beta\rho(\alpha^{-1}(y))\beta^{-1}\rho(\alpha^{-1}(x))\beta^{-1}) \quad (17)$$

$$= \beta^{-1}\rho(x)\beta\beta^{-1}\beta^{-1}\rho(y)\beta\beta^{-1}$$

$$- \beta^{-1}\rho(y)\beta\beta^{-1}\beta^{-1}\rho(x)\beta\beta^{-1}$$

$$= \beta^{-1}\rho(x)\beta^{-1}\rho(y) - \beta^{-1}\rho(y)\beta^{-1}\rho(x)$$

$$= [\rho'(x), \rho'(y)]'.$$

Then the result follows easily. $\square$

From Proposition 6, we can get a method of computing representations of a multiplicative Hom-Lie algebra with α, β invertible.

Let (ρ, V, β) be a representation of a Lie-type Hom-Lie algebra. If $\rho = \beta\rho'$, where (ρ', V) is a representation of the compatible Lie algebra, then (ρ, V, β) is called a Lie-type representation. It is easy to know that the representation in Proposition 6 is Lie-type. In addition, suppose that (ρ', V) is an irreducible representation of the compatible Lie algebra; then (ρ, V, β) is called a regular Lie-type representation.

Theorem 7. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a Lie-type Hom-Lie algebra with the compatible Lie algebra $(\mathfrak{g}, [\cdot, \cdot]')$.

(1) If (ρ, V, β) (β invertible) is a representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$, then

$$\beta(\rho'(x)v) = \rho'(\alpha(x))\beta(v), \quad \forall x \in \mathfrak{g}, v \in V, \quad (18)$$

where $(\rho', V) = (\beta^{-1}\rho, V)$ is a representation of the compatible Lie algebra.

(2) Suppose that (ρ', V) is a representation of $(\mathfrak{g}, [\cdot, \cdot]')$. If $\exists \beta \in \text{gl}(V)$ such that (18) is satisfied, let $\rho = \beta\rho'$; then (ρ, V, β) is a representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$.

Proof. (1) By the invertibility of β and (12), we can get (18) easily.

(2) for all $x \in \mathfrak{g}, v \in V$, (12) follows from (18) easily. for all $x, y \in \mathfrak{g}, v \in V$, we have

$$\rho([x, y])\beta(v) = \beta(\rho'(\alpha([x, y]'))\beta(v))$$

$$= \beta((\rho'(\alpha(x))\rho'(\alpha(y)) - \rho'(\alpha(y))\rho'(\alpha(x)))\beta(v))$$

$$= \beta(\rho'(\alpha(x))\beta(\rho'(y)v) - \rho'(\alpha(y))\beta(\rho'(x)v))$$

$$= \rho(\alpha(x))\rho(y)v - \rho(\alpha(y))\rho(x)v. \quad (19)$$

Now we can get (11). Therefore (ρ, V, β) is a representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. $\square$

4. Regular Lie-Type Representations of 3-Dimensional Simple Multiplicative Hom-Lie Algebras

Lemma 8. Let (ρ, V, β) be a representation of a multiplicative Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$; then $(\text{Ker}(\beta), \beta)$ is a Hom- $\mathfrak{g}$ -submodule of (V, β) .

Proof. for all $v \in \text{Ker}(\beta)$, by $\beta(\rho(x)v) = \rho(\alpha(x))\beta(v) = 0$, for all $x \in \mathfrak{g}$, it is easy to know that $\rho(x) \text{Ker}(\beta) \subseteq \text{Ker}(\beta)$; then the result follows easily. $\square$

Lemma 9. Let (ρ, V, β) be an irreducible or a completely reducible representation of a multiplicative Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$; then β is invertible.

Proof. By the reason of Lemma 8, if (V, β) is an irreducible Hom- $\mathfrak{g}$ module, we have that β is invertible. If (V, β) is a completely reducible Hom- $\mathfrak{g}$ module, then $V = V_1 \oplus \cdots \oplus V_s$, where $(V_i, \beta|_{V_i})$ ($i = 1, \dots, s$) are irreducible Hom- $\mathfrak{g}$ submodules. By the irreducibility of $(V_i, \beta|_{V_i})$, we have that $\beta|_{V_i}$ ($i = 1, \dots, s$) is invertible, so β is an invertible linear map of V . $\square$

Lemma 10. Let (ρ, V, β) with β invertible be a nontrivial finite dimensional regular Lie-type representation of a Lie-type Hom-Lie algebra; then (V, β) is an irreducible Hom- $\mathfrak{g}$ -module.

Proof. Suppose to the contrary that (V, β) is reducible. Then we assume that (V_1, β) is a nontrivial Hom- $\mathfrak{g}$ -submodule of (V, β) . Let $(\rho', V) = (\beta^{-1}\rho, V)$ be the representation of the compatible Lie algebra. Then

$$\rho'(x)V_1 = \beta^{-1}(\rho(x)V_1) \subseteq \beta^{-1}(V_1) \subseteq V_1. \quad (20)$$

That is, (ρ', V_1) is a nontrivial subrepresentation of the compatible Lie algebra, which is a contradiction. Therefore (V, β) is an irreducible Hom- $\mathfrak{g}$ -module. $\square$

It is natural to ask the question: are there nontrivial finite dimensional regular Lie-type representations in 3-dimensional simple multiplicative Hom-Lie algebras? Let us see the following theorem.

Theorem 11. Let $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ be a 3-dimensional simple multiplicative Hom-Lie algebra; then there exist nontrivial finite dimensional regular Lie-type representations (ρ, V, β) , and these representations are irreducible. For every such representation, $\rho(h)$ is a semisimple linear transformation of V . In addition, there is a basis $\{v_0, v_1, \dots, v_m\}$ of V such that

- (a) $\rho(h)v_i = (m - 2i)a^{-i}b_0v_i$, $\beta(v_i) = a^{-i}b_0v_i$, $i = 0, \dots, m$;
- (b) $\rho(f)v_i = (i + 1)a^{-i-1}b_0v_{i+1}$, $i = 0, \dots, m - 1$; $\rho(f)v_m = 0$;
- (c) $\rho(e)v_0 = 0$, $\rho(e)v_i = (m - i + 1)a^{-i+1}b_0v_{i-1}$, $i = 1, \dots, m$.

Proof. Let (ρ', V) be an $m + 1$ dimensional irreducible representation of the compatible Lie algebra A_1 . Take

$\{v_0, v_1, \dots, v_m\}$ as a basis of V , where $v_i = 1/i\rho'(f)v_{i-1}$ ($i = 1, \dots, m$) satisfies

$$\begin{aligned} \rho'(h)v_i &= (m - 2i)v_i, \\ \rho'(e)v_i &= (m - i + 1)v_{i-1} \quad (i = 1, \dots, m), \\ \rho'(e)v_0 &= 0. \end{aligned} \quad (21)$$

Now we prove that when $\beta \in \text{gl}(V)$ is defined by

$$\beta \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix}^T = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix}^T \begin{pmatrix} b_0 & & & \\ & a^{-1}b_0 & & \\ & & \ddots & \\ & & & a^{-m}b_0 \end{pmatrix}, \quad (22)$$

$$b_0 \neq 0,$$

(18) is always satisfied. Because

$$\begin{aligned} \beta(\rho'(h)v_i) &= (m - 2i)\beta(v_i) = (m - 2i)a^{-i}b_0v_i, \\ \rho'(\alpha(h))\beta(v_i) &= \rho'(h)a^{-i}b_0v_i \\ &= a^{-i}b_0(m - 2i)v_i = \beta(\rho'(h)v_i), \\ & \quad i = 0, \dots, m, \\ \beta(\rho'(f)v_i) &= (i + 1)\beta(v_{i+1}) = (i + 1)a^{-i-1}b_0v_{i+1}, \\ \rho'(\alpha(f))\beta(v_i) &= a^{-1}\rho'(f)a^{-i}b_0v_i \\ &= a^{-i-1}b_0(i + 1)v_{i+1} \\ &= \beta(\rho'(f)v_i), \quad i = 0, \dots, m, \\ \beta(\rho'(e)v_i) &= \beta((m - i + 1)v_{i-1}) \\ &= (m - i + 1)\beta(v_{i-1}) \\ &= (m - i + 1)a^{-(i-1)}b_0v_{i-1}, \\ \rho'(\alpha(e))\beta(v_i) &= a\rho'(e)\beta(v_i) \\ &= \rho'(e)a^{-(i-1)}b_0v_i \\ &= a^{-(i-1)}b_0(m - i + 1)v_{i-1} \\ &= \beta(\rho'(e)v_i), \quad i = 0, \dots, m. \end{aligned} \quad (23)$$

Thus for β defined by (22), (18) is always established. Let $\rho = \beta\rho'$; then it follows from Theorem 7 (2) that (ρ, V, β) is a nontrivial finite dimensional regular Lie-type representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. By Lemma 10, we have that (ρ, V, β) is irreducible.

Furthermore, we have

$$\begin{aligned} \rho(h)v_i &= \beta(\rho'(h)v_i) = (m - 2i)a^{-i}b_0v_i, \\ & \quad i = 0, \dots, m, \end{aligned} \quad (24)$$

and thus we can get (a). For some fixed b_0 , it is obvious that $\rho(h)$ is a semisimple linear transformation of V . Take $x = e, f$ in $\rho(x)v_i = \beta(\rho'(x)v_i)$ ($i = 0, \dots, m$); we can get (b) and (c) easily. $\square$

5. Irreducible and Completely Reducible Representations of a 3-Dimensional Simple Multiplicative Hom-Lie Algebra

In this section, $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ denotes a 3-dimensional simple multiplicative Hom-Lie algebra.

By Theorem 11, we know that there exist nontrivial finite dimensional irreducible regular Lie-type representations of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. However, are there other nontrivial irreducible representations of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$? Is any finite dimensional representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ completely reducible? We will study these questions in this section.

By Lemma 9, we only need to consider the case when β is invertible.

Let (ρ, V, β) with β invertible be a finite dimensional representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$. By Proposition 6, we have that (ρ, V, β) is of Lie-type, and $(\rho', V) = (\beta^{-1}\rho, V)$ is a finite dimensional representation of the compatible Lie algebra A_1 . By Weyl theorem, we know that (ρ', V) is completely reducible. That is, $V = V_1 \oplus \dots \oplus V_s$, where V_i ($i = 1, \dots, s$) are irreducible A_1 -modules. Suppose that $\dim V_i = m_i + 1$. Denote that a tuple $\tau = (m_1, \dots, m_s)$. By the representation theory of A_1 , we have that V_j ($j = 1, \dots, s$) is a highest weight module with highest weight vectors v_{j0} and the highest weight m_j , respectively. Take $\{v_{j0}, v_{j1}, \dots, v_{jm_j}\}$ as a basis of V_j satisfying

$$\begin{aligned} v_{ji} &= \frac{1}{i\rho'(f)v_{j,i-1}}, \quad (i = 1, \dots, m_j), \\ \rho'(h)v_{ji} &= (m_j - 2i)v_{ji}, \quad (i = 0, \dots, m_j), \\ \rho'(e)v_{ji} &= (m_j - i + 1)v_{j,i-1}, \quad (i = 1, \dots, m_j), \\ \rho'(e)v_0 &= 0. \end{aligned} \quad (25)$$

Theorem 12. When $\tau = (m_1, \dots, m_s)$, $m_i \neq m_j$ for $1 \leq i \neq j \leq s$, then

- (1) $\beta = \text{diag}(\dots, b_{i0}, b_{i0}/a, \dots, b_{i0}/a^{m_i}, \dots)$, $i = 1, \dots, s$;
- (2) (V, β) is a completely reducible Hom- $\mathfrak{g}$ -module with a decomposition $V = V_1 \oplus \dots \oplus V_s$, where (V_i, β_i) , ($i = 1, \dots, s$) are irreducible Hom- $\mathfrak{g}$ -submodules; here $\beta_i = \beta|_{V_i}$.

Proof. (1) According to (18), we get the result.

(2) Because $(V_j) \subseteq V_j$, $j = 1, \dots, s$,

$$\rho(x)V_j = \beta(\rho'(x)V_j) \subseteq V_j. \quad (26)$$

Let $\beta_j = \beta|_{V_j}$, combined with Lemma 10; we have that (V_j, β_j) ($j = 1, \dots, s$) are irreducible Hom- $\mathfrak{g}$ -submodules. $\square$

When $\tau = (m, \dots, m)$, by (18) it can be checked that

$$\beta = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1s} \\ B_{21} & B_{22} & \dots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \dots & B_{ss} \end{pmatrix}, \quad (27)$$

where $B_{ij} = \text{diag}(b_{ij}, b_{ij}/a, \dots, b_{ij}/a^m)$.

Take

$$B_1 = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1s} \\ b_{21} & b_{22} & \dots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s1} & b_{s2} & \dots & b_{ss} \end{pmatrix}. \quad (28)$$

By the theory of linear algebra, there is an invertible matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1s} \\ p_{21} & p_{22} & \dots & p_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \dots & p_{ss} \end{pmatrix} \quad (29)$$

such that $P^{-1}B_1P$ is a Jordan canonical form; that is,

$$P^{-1}B_1P = \begin{pmatrix} J_{11} & & & \\ & J_{22} & & \\ & & \ddots & \\ & & & J_{tt} \end{pmatrix}, \quad (30)$$

where $J_{ii} = \begin{pmatrix} \lambda_i & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ ($i = 1, \dots, t$) are Jordan blocks.

Theorem 13. The condition is the same as the previous remark.

- (1) If $t = 1$, then (V, β) is a reducible but not completely reducible Hom- $\mathfrak{g}$ -module.
- (2) If $s > t > 1$, then (V, β) is a reducible but not completely reducible Hom- $\mathfrak{g}$ -module.
- (3) If $t = s$, then (V, β) is a completely reducible Hom- $\mathfrak{g}$ -module.

Proof. Let

$$v'_{i0} = p_{1i}v_{10} + p_{2i}v_{20} + \dots + p_{si}v_{s0}, \quad i = 1, \dots, s, \quad (31)$$

$$v'_{ij} = \frac{1}{j}\rho'(f)v'_{i,j-1}, \quad i = 1, \dots, s; \quad j = 1, \dots, m. \quad (32)$$

That is,

$$\begin{aligned} v'_{ij} &= p_{1i}v_{1j} + p_{2i}v_{2j} + \dots + p_{si}v_{sj}, \\ i &= 1, \dots, s; \quad j = 1, \dots, m. \end{aligned} \quad (33)$$

Then

$$\begin{aligned} & (v'_{10}, \dots, v'_{1m}, \dots, v'_{s0}, \dots, v'_{sm}) \\ &= (v_{10}, \dots, v_{1m}, \dots, v_{s0}, \dots, v_{sm})Q, \end{aligned} \quad (34)$$

where

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1s} \\ Q_{21} & Q_{22} & \cdots & Q_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{s1} & Q_{s2} & \cdots & Q_{ss} \end{pmatrix}; \quad (35)$$

here Q_{ij} are $m \times m$ matrices satisfying

$$Q_{ij} = \text{diag}(p_{ij}, \dots, p_{ij}), \quad i, j = 1, \dots, s. \quad (36)$$

Because P is invertible, it is easy to check that the matrix Q is invertible; therefore $\{v'_{10}, \dots, v'_{1m}, \dots, v'_{s0}, \dots, v'_{sm}\}$ is a basis of V .

Let $V'_i = \mathbb{C}\{v'_{i0}, v'_{i1}, \dots, v'_{im}\}$, because

$$\rho'(h)v'_{i0} = \rho'(h)(p_{1i}v'_{10} + p_{2i}v'_{20} + \cdots + p_{si}v'_{s0}) = mv'_{i0}. \quad (37)$$

By (32) and the representation of Lie algebra A_1 , we have

$$\begin{aligned} \rho'(h)(v'_{i0}, \dots, v'_{im}) &= (v'_{i0}, \dots, v'_{im}) \begin{pmatrix} m & & & \\ & m-2 & & \\ & & \ddots & \\ & & & -m \end{pmatrix}, \\ \rho'(f)(v'_{i0}, \dots, v'_{im}) &= (v'_{i0}, \dots, v'_{im}) \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & m & 0 \end{pmatrix}, \\ \rho'(e)(v'_{i0}, \dots, v'_{im}) &= (v'_{i0}, \dots, v'_{im}) \\ &\times \begin{pmatrix} 0 & m & 0 & \cdots & 0 \\ 0 & 0 & m-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \quad (38)$$

Through (38) it is easy to get that $V'_1, \dots, V'_s$ are A_1 -irreducible-modules.

(1) In this case, by (30), we have

$$\beta(v'_{10}, v'_{20}, \dots, v'_{s0}) = (v'_{10}, v'_{20}, \dots, v'_{s0}) \begin{pmatrix} \lambda & 0 & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}. \quad (39)$$

Let

$$V'_i = \mathbb{C}\{v'_{i0}, \dots, v'_{im}\}, \quad i = 1, \dots, s, \quad (40)$$

By (32), it is easy to check that

$$\beta(V'_i) \subseteq V'_i + V'_{i+1}, \quad i = 1, \dots, s-1; \quad \beta(V'_s) \subseteq V'_s. \quad (41)$$

Then $\rho(x)V'_s = \beta(\rho'(x)V'_s) \subseteq V'_s$. Let $\beta_s = \beta|_{V'_s}$; then (V'_s, β_s) is an irreducible Hom- $\mathfrak{g}$ -submodule, but (V, β) is not completely reducible.

(2) Suppose (omit the order of $v'_{10}, \dots, v'_{s0}$)

$$\begin{aligned} &\beta(v'_{10}, v'_{20}, \dots, v'_{s0}) \\ &= (v'_{10}, v'_{20}, \dots, v'_{s0}) \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_j & \\ & & & B_{j+1} & \\ & & & & \ddots & \\ & & & & & B_t \end{pmatrix}; \end{aligned} \quad (42)$$

here $\lambda_i \in \mathbb{C}$, $i = 1, \dots, j$, $B_{j+1}, \dots, B_t$ are $k_p \times k_p$ nondiagonal Jordan blocks ($k_p \in \mathbb{Z}^+$, $p = 1, \dots, t-j$).

Let

$$V'_i = \mathbb{C}\{v'_{i0}, \dots, v'_{im}\}, \quad i = 1, \dots, j, \quad (43)$$

denoted by $\beta_i = \beta|_{V'_i}$. It is easy to check that

$$\begin{aligned} &\beta_i(V'_i) \subseteq V'_i, \\ &\rho(x)V'_i = \beta(\rho'(x)V'_i) \subseteq \beta(V'_i) \subseteq V'_i, \quad i = 1, \dots, j. \end{aligned} \quad (44)$$

By the statement of the proof and Lemma 10, we have that (V'_i, β_i) ($i = 1, \dots, j$) are irreducible Hom- $\mathfrak{g}$ -submodules.

Let

$$\begin{aligned} V'_{j+p} &= \mathbb{C}\{v'_{j+k_1+\dots+k_{p-1}+1,0}, \dots, \\ &\quad v'_{j+k_1+\dots+k_{p-1}+1,m}, \dots, \\ &\quad v'_{j+k_1+\dots+k_p,0}, \dots, \\ &\quad v'_{j+k_1+\dots+k_p,m}\}, \\ &p = 1, \dots, t-j. \end{aligned} \quad (45)$$

Then

$$\beta(V'_{j+p}) \subseteq V'_{j+p}, \quad p = 1, \dots, t-j. \quad (46)$$

Let $\beta_i = \beta|_{V'_i}$, ($i = j+1, \dots, t$); then (V_i, β_i) ($i = j+1, \dots, t$) are Hom- $\mathfrak{g}$ -submodules. As (1) of the theorem, we can prove that (V_i, β_i) ($i = j+1, \dots, t$) are reducible but not completely reducible Hom- $\mathfrak{g}$ -modules. We have the conclusion.

(3) In this case, by (30) and Theorem 12 (1) we get

$$\beta \begin{pmatrix} v'_{10} \\ \vdots \\ v'_{1m} \\ \vdots \\ v'_{s0} \\ \vdots \\ v'_{sm} \end{pmatrix}^t = \begin{pmatrix} v'_{10} \\ \vdots \\ v'_{1m} \\ \vdots \\ v'_{s0} \\ \vdots \\ v'_{sm} \end{pmatrix}^t \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \frac{\lambda_1}{a^m} & \\ & & & \ddots \\ & & & & \lambda_s & \\ & & & & & \ddots \\ & & & & & & \frac{\lambda_s}{a^m} \end{pmatrix}. \quad (47)$$

Let

$$V'_i = \mathbb{C} \{v'_{i0}, \dots, v'_{im}\}, \quad i = 1, \dots, s; \quad (48)$$

then $V'_1, \dots, V'_s$ are irreducible A_1 -modules and

$$\beta(V'_i) \subseteq V'_i, \quad i = 1, \dots, s, \quad (49)$$

$$\rho(x)V'_i = \beta(\rho'(x)V'_i) \subseteq V'_i, \quad i = 1, \dots, s, \quad \forall x \in \mathfrak{g}.$$

Denoting by $\beta_i = \beta|_{V'_i}$, $i = 1, \dots, s$, thanks to Lemma 10 we have that (V'_i, β_i) ($i = 1, \dots, s$) are irreducible Hom- $\mathfrak{g}$ -submodules. So (V, β) is a completely reducible Hom- $\mathfrak{g}$ -module. $\square$

When $\tau = (m_1, \dots, m_1, \dots, m_i, \dots, m_i, m_{i+1}, \dots, m_l)$, suppose that the multiplicity of m_t ($t = 1, \dots, l$) is k_t and $k_1 + \dots + k_i + l - i = s$. By (18) we can get β as follows:

$$\beta \begin{pmatrix} v_{10} \\ \vdots \\ v_{1m_1} \\ \vdots \\ v_{s0} \\ \vdots \\ v_{sm_l} \end{pmatrix} = \begin{pmatrix} v_{10} \\ \vdots \\ v_{1m_1} \\ \vdots \\ v_{s0} \\ \vdots \\ v_{sm_l} \end{pmatrix} \begin{pmatrix} B_1 & & & \\ & \ddots & & \\ & & B_i & \\ & & & B_{i+1} & \\ & & & & \ddots \\ & & & & & B_l \end{pmatrix}, \quad (50)$$

where $B_1, \dots, B_l$ are matrices of the form (27),

$$B_j = \text{diag} \left(\lambda_j, \frac{\lambda_j}{a}, \dots, \frac{\lambda_j}{a^{m_j}} \right), \quad j = i+1, \dots, l. \quad (51)$$

Theorem 14. The condition is as the previous remark. (V, β) is completely reducible if there exist invertible matrices P_j ($j = 1, \dots, i$) such that $P_j^{-1} B_j P_j$ ($j = 1, \dots, i$) are diagonal matrices. Otherwise (V, β) is reducible but not completely reducible.

Proof. It can be got from Theorems 12 and 13 directly. $\square$

Proposition 15. Let (ρ, V, β) be a finite dimensional representation of $(\mathfrak{g}, [\cdot, \cdot], \alpha)$; then (ρ, V, β) is not always completely reducible, and (ρ, V, β) is irreducible if and only if it is of regular Lie-type.

Proof. The claim follows from Theorems 11, 12, 13, and 14 directly. $\square$

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