Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2014, Article ID 181594, 15 pages
http://dx.doi.org/10.1155/2014/181594

Research Article

Exact Solutions of the Time Fractional BBM-Burger Equation by Novel (G'/G)-Expansion Method

Muhammad Shakeel, Qazi Mahmood Ul-Hassan, Jamshad Ahmad, and Tauseef Naqvi

Department of Mathematics, Faculty of Sciences, HITEC University Taxila Cantt, Taxila 47080, Pakistan

Correspondence should be addressed to Muhammad Shakeel; muhammadshakeel74@yahoo.com

Received 15 February 2014; Accepted 27 May 2014; Published 11 September 2014

Academic Editor: Hossein Jafari

The fractional derivatives are used in the sense modified Riemann-Liouville to obtain exact solutions for BBM-Burger equation of fractional order. This equation can be converted into an ordinary differential equation by using a persistent fractional complex transform and, as a result, hyperbolic function solutions, trigonometric function solutions, and rational solutions are attained. The performance of the method is reliable, useful, and gives newer general exact solutions with more free parameters than the existing methods. Numerical results coupled with the graphical representation completely reveal the trustworthiness of the method.

1. Introduction

Differential equations of noninteger order are generalizations of conventional differential equations of integer order [1]. Exploration and applications of integrals and derivatives of random order are efficiently dealt with the field of mathematical analysis, called as fractional calculus, which has engrossed in considerable interest in many disciplines, nowadays. The behavior of many physical systems can be perfectly defined by the fractional theory. In recent years, we cannot rebuff the importance of fractional differential equations because of their numerous applications in the areas of physics and engineering. For example, the nonlinear fluctuation of earthquakes can be modeled with the help of fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eradicate the problems arising from the huge traffic flow [2, 3]. We apply a generalized fractional complex transform [4–7] to convert the given fractional order differential equation to ordinary differential equation. On account of development of the computer and its exact description of countless real-life problems, fractional calculus has touched the height of fame and success nowadays, although it was invented three centuries ago by Newton and Leibniz. Many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are better described by differential equations of noninteger order [8–11]. A physical interpretation of the fractional calculus was given in [12–14]. With the development of symbolic computation software, likeMaple, many researchers developed and established many numerical and analytical methods to search for exact solutions of nonlinear evolution equations (NLEEs), for example, Cole-Hopf transformation [15], Tanh-function method [16–19], inverse scattering transform method [20], variational iteration method [21, 22], Exp-function method [23–26], homogeneous balance method [27, 28], and F-expansion method [29, 30], which are used for searching the exact solutions.

Lately, a straight and crisp method, called (G'/G)-expansion method, was introduced by Wang et al. [31] and confirmed that it is a powerful method for seeking analytic solutions of nonlinear evolution equations (NLEEs). For additional references see the articles [32–37]. In order to establish the efficiency and diligence of (G'/G)-expansion method and to extend the range of applicability, further research has been carried out by several researchers. For illustration, Zhang et al. [38] made an overview of (G'/G)-expansion method for the evolution equations with variable coefficients. Zhang et al. [39] also presented an improved (G'/G)-expansion method to look for more broad traveling wave solutions. Zayed [40] offered a new technique of

(G'/G)-expansion method where $G(\xi)$ gratifies the Jacobi elliptic equation, $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0$, where e_2, e_1, and e_0 are random constants, and obtained new exact solutions. Zayed [41] for a second time offered a different approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = AG(\xi) + BG^2(\xi)$, where A and B are casual constants.

The (G'/G)-expansion method and the transformed rational function method used by Ma and Lee [42] have a common idea. That is, we initially put the given nonlinear evolution equation (NLEE) into the equivalent ordinary differential equation (ODE), and then ODE can be transformed into a system of arithmetical polynomials with the influential constants. By the solutions of the ordinary differential equation, we can obtain the exact traveling solutions and rational solution of the nonlinear evolution equations.

In this article, we will apply novel (G'/G)-expansion method introduced by Alam et al. [43] to solve the time fractional BBM-Burger equation, whereas, the modified Riemann-Liouville derivative given by Jumarie [44] is used and abundant new families of exact solutions are found. The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)d\xi, \\ \quad 0<\alpha<1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, \\ \quad n\le\alpha<n+1, \quad n\ge 1. \end{cases} \tag{1}$$

Some important properties of Jumarie's derivative are

$$D_t^\alpha f(t) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}t^{\gamma-\alpha}, \tag{2}$$

$$D_t^\alpha\left(f(t)g(t)\right) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{3}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)]\left(g'(t)\right)^\alpha. \tag{4}$$

2. Description of the Method

Consider the fractional partial differential equation in the form,

$$S\left(u, u_x, u_t, D_t^\alpha u, \ldots\right) = 0, \quad 0<\alpha\le 1, \tag{5}$$

where $D_t^\alpha u$ is Jumarie's modified Riemann-Liouville derivatives of $u, u(x,t)$ is an unknown function, and S is a polynomial in u and its various partial derivatives including fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the method are as follows.

Step 1. Li and He [5] projected a fractional complex transformation to convert fractional partial differential equation into ordinary differential equation (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The following complex transformation:

$$u(x,t) = u(\xi), \qquad \xi = Lx + V\frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{6}$$

where L, V are arbitrary constants with $L, V \neq 0$, permits us to convert (5) into an ordinary differential equation of integer order in the form

$$P\left(u, u', u'', u''', \ldots\right) = 0, \tag{7}$$

where the primes stand for the ordinary derivatives with respect to ξ.

Step 2. Integrate (7) term by term one or more times, if possible; capitulate constant(s) of integration which can be calculated later on.

Step 3. Suppose that the solution of (7) can be expressed as

$$u(\xi) = \sum_{i=-m}^{m}\alpha_i(k+\Phi(\xi))^i, \tag{8}$$

where

$$\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}. \tag{9}$$

Herein α_{-m} or α_m may be zero, but both of them cannot be zero simultaneously. α_i $(i = 0, \pm1, \pm2, \ldots, \pm N)$ and k are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ordinary differential equation

$$GG'' = AGG' + BG^2 + C\left(G'\right)^2, \tag{10}$$

where primes denote the derivative with respect ξ; A, B, and C are genuine constants.

The Cole-Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\xi = G'(\xi)/G(\xi)$ reduces (10) into the Riccati equation

$$\Phi'(\xi) = B + A\Phi(\xi) + (C-1)\Phi^2(\xi). \tag{11}$$

Equation (11) has individual twenty-five solutions (see Zhu, [45] for details).

Step 4. The positive integer m can be calculated by balancing the highest order linear term with the highest order nonlinear term in (7).

Step 5. Inserting (8) together with (9) and (10) into (7), we obtain polynomials in $(k+(G'/G))^i$ and $(k+(G'/G))^{-i}$, $(i = 0, 1, 2, \ldots, N)$. Collecting each coefficient of the resulted polynomials to zero acquiesces an overdetermined set of algebraic equations for α_i $(i = 0, \pm1, \pm2, \ldots, \pm N)$, k, L, and V.

Step 6. After solving the system of algebraic equations obtained in Step 5, we can obtain the values of the constants. The solutions of (10) together with the obtained values of the constants yield abundant exact traveling wave solutions of the nonlinear evolution equation (5).

3. Application of the Method to the Time Fractional BBM-Burger Equation

Consider the following BBM-Burger equation in time fractional operator form as:

$$D_t^{\alpha} u - u_{xxt} + u_x + \left(\frac{u^2}{2}\right)_x = 0, \quad t > 0,\ 0 < \alpha \le 1. \tag{12}$$

S. Kumar and D. Kumar [46] used new fractional homotopy analysis transform method to time fractional BBM-Burger equation and found the series solution of this equation. Song and Zhang [47] and Fakhari et al. [48] find different type of solutions of the fractional BBM-Burger equation by using homotopy analysis method.

By the use of (4), (12) is converted into an ordinary differential equation of integer order and after integrating once, we obtain

$$(L + V)\, u + \frac{L}{2} u^2 - VL^2 u'' + C_1 = 0, \tag{13}$$

where C_1 is an integration constant.

Considering the homogeneous balance between u'' and u^2 in (13), we obtain $m = 2$. Therefore, the trial solution (8) becomes

$$\begin{aligned} u(\xi) &= \alpha_{-2}(k + \Phi(\xi))^{-2} + \alpha_{-1}(k + \Phi(\xi))^{-1} \\ &\quad + \alpha_0 + \alpha_1 (k + \Phi(\xi)) + \alpha_2 (k + \Phi(\xi))^2. \end{aligned} \tag{14}$$

Using (14) into (13), left-hand side is converted into polynomials in $(k + (G'/G))^i$ and $(k + (G'/G))^{-i}$ $(i = 0, 1, 2, \ldots, N)$. Equating the coefficients of same power of the resulted polynomials to zero, we attain a set of algebraic equations (which are omitted for the sake of simplicity) for α_0, α_1, α_2, α_{-1}, α_{-2}, k, C_1, L, and V. Solving the overdetermined set of algebraic equations by using the symbolic computation software, such as Maple 13, we obtain the following solution sets:

The Set 1. We consider that

$$\begin{aligned} \alpha_0 &= \Big(V\Big(12(C-1)^2 k^2 - 12A(C-1)k \\ &\qquad - 8B + 8BC + A^2\Big) L^2 - V - L\Big)(L)^{-1} \\ \alpha_1 &= 12VL(C-1)(-2kC + 2k + A), \\ \alpha_2 &= 12VL(C-1)^2, \\ V &= V, \quad L = L, \quad k = k, \quad \alpha_{-1} = 0, \quad \alpha_{-2} = 0, \\ C_1 &= -\frac{1}{2L}\Big(\Big(\Big((-4C+4)B + A^2\Big) VL^2 - V - L\Big) \\ &\qquad \times \Big(\Big((-4C+4)B + A^2\Big) VL^2 + V + L\Big)\Big), \end{aligned} \tag{15}$$

where k, L, V, A, B, and C are arbitrary constants.

The Set 2. We consider that

$$\begin{aligned} \alpha_0 &= \Big(V\Big(12(C-1)^2 k^2 - 12A(C-1)k \\ &\qquad - 8B + 8BC + A^2\Big) L^2 - V - L\Big)(L)^{-1} \\ \alpha_{-1} &= -12\Big((1-C)k^2 + kA - B\Big) VL((2-2C)k + A), \\ \alpha_{-2} &= 12VL\Big((1-C)k^2 + kA - B\Big)^2, \\ V &= V, \quad L = L, \quad k = k, \quad \alpha_2 = 0, \quad \alpha_1 = 0, \\ C_1 &= -\frac{1}{2L}\Big(V\Big((-4C+4)B + A^2\Big) L^2 - V - L\Big) \\ &\qquad \times \Big(V\Big((-4C+4)B + A^2\Big) L^2 + V + L\Big), \end{aligned} \tag{16}$$

where k, L, V, A, B, and C are arbitrary constants.

The Set 3. We consider that

$$\begin{aligned} \alpha_2 &= 12VL(C-1)^2, \\ \alpha_0 &= \frac{-2VL^2\Big((4-4C)B + A^2\Big) - V - L}{L}, \\ k &= \frac{A}{2(C-1)}, \\ \alpha_{-2} &= \frac{3VL\Big((4-4C)B + A^2\Big)^2}{4(1-C)^2}, \\ \alpha_1 &= 0, \quad \alpha_{-1} = 0, \quad V = V, \quad L = L, \\ C_1 &= -\frac{8}{L}\Big(VL^2\Big((-4C+4)B + A^2\Big) - \frac{V}{4} - \frac{L}{4}\Big) \\ &\qquad \times \Big(VL^2\Big((-4C+4)B + A^2\Big) + \frac{V}{4} + \frac{L}{4}\Big), \end{aligned} \tag{17}$$

where V, L, A, B, and C are arbitrary constants.

Substituting (15)–(17) into (14), we obtain

$$\begin{aligned} &u_1(\xi) \\ &= \Big(V\Big(12(C-1)^2 k^2 - 12A(C-1)k \\ &\qquad - 8B + 8BC + A^2\Big) L^2 - V - L\Big)(L)^{-1} \\ &\quad + 12VL(C-1)(-2kC + 2k + A) \\ &\quad \times \left(k + \left(\frac{G'}{G}\right)\right) + 12VL(C-1)^2 \times \left(k + \left(\frac{G'}{G}\right)\right), \end{aligned} \tag{18}$$

$$\begin{aligned}
&u_2(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad \left.\times L^2-V-L\right)(L)^{-1}\\
&\quad -12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)\\
&\quad \times\left(k+\left(\frac{G'}{G}\right)\right)^{-1}\\
&\quad +12VL\left((1-C)k^2+kA-B\right)^2\times\left(k+\left(\frac{G'}{G}\right)\right)^{-2},
\end{aligned} \tag{19}$$

$$\begin{aligned}
&u_3(\xi)\\
&= -\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L}\\
&\quad +12VL(C-1)^2\times\left(\frac{A}{2(C-1)}+\left(\frac{G'}{G}\right)\right)^2\\
&\quad +\frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2}\\
&\quad \times\left(\frac{A}{2(C-1)}+\left(\frac{G'}{G}\right)\right)^{-2},
\end{aligned} \tag{20}$$

where

$$\xi = Lx + V\frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{21}$$

Substituting the solutions $G(\xi)$ of (10) into (18) and simplifying, we obtain the following solutions.

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

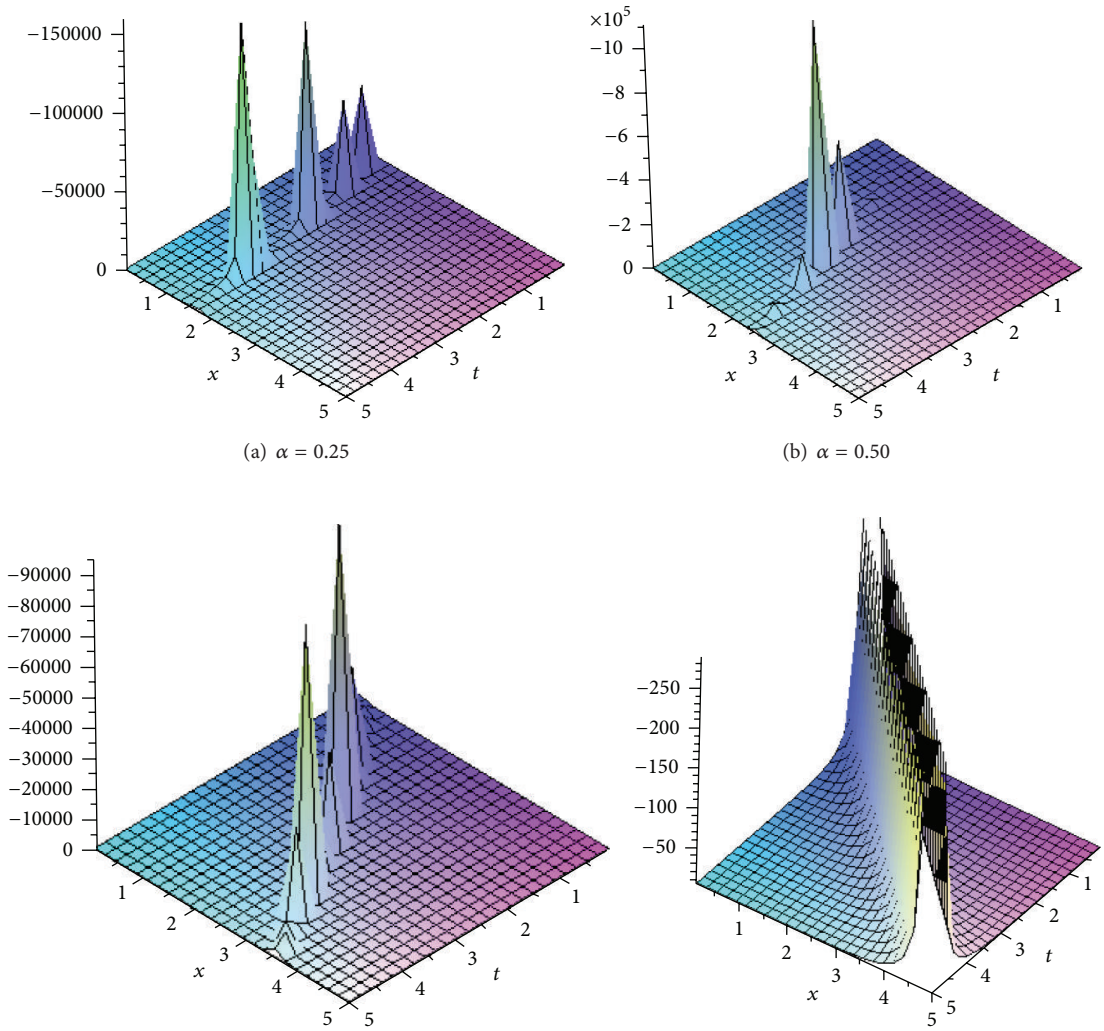
$$\begin{aligned}
&u_1^1(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad \left.\times L^2-V-L\right)(L)^{-1}\\
&\quad +12VL(C-1)(-2kC+2k+A)\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}\\
&\quad +12VL(C-1)^2\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^2,
\end{aligned}$$

$$\begin{aligned}
&u_1^1(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad \left.\times L^2-V-L\right)(L)^{-1}\\
&\quad +12VL(C-1)(-2kC+2k+A)\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}\\
&\quad +12VL(C-1)^2\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^2,
\end{aligned} \tag{22}$$

and its graph is shown in Figure 1. Consider

$$\begin{aligned}
&u_1^3(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k\right.\right.\\
&\quad \left.\left.-8B+8BC+A^2\right)L^2-V-L\right)(L)^{-1}\\
&\quad +12VL(C-1)(-2kC+2k+A)\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\right.\\
&\quad \left.\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}\\
&\quad +12VL(C-1)^2\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\right.\\
&\quad \left.\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}^2,
\end{aligned}$$

$$\begin{aligned}
&u_1^4(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k\right.\right.\\
&\quad \left.\left.-8B+8BC+A^2\right)L^2-V-L\right)(L)^{-1}\\
&\quad +12VL(C-1)(-2kC+2k+A)\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\right.\\
&\quad \left.\times\left(A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}\\
&\quad +12VL(C-1)^2\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\right.\\
&\quad \left.\times\left(A+\sqrt{\Delta}\left(\coth\left(\sqrt{\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}^2,
\end{aligned}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

FIGURE 1: (a)–(d) show the singular soliton solution for u_1^2 for different values of parameters.

$$
\begin{aligned}
&u_1^5(\xi) \\
&= \Big(V\Big(12(C-1)^2k^2 - 12A(C-1)k \\
&\qquad - 8B + 8BC + A^2\Big)L^2 - V - L\Big)(L)^{-1} \\
&\quad + 12VL(C-1)(-2kC + 2k + A) \\
&\quad \times \left\{k - \frac{1}{4(C-1)}\right. \\
&\qquad \times \Big(2A + \sqrt{\Delta} \\
&\qquad\quad \left.\times \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right) + \coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\Big)\right\} \\
&\quad + 12VL(C-1)^2 \\
&\quad \times \left\{k - \frac{1}{4(C-1)}\right. \\
&\qquad \times \Big(2A + \sqrt{\Delta} \\
&\qquad\quad \left.\times \left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right) + \coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\Big)\right\}^2,
\end{aligned}
$$

$$
\begin{aligned}
&u_1^6(\xi) \\
&= \Big(V\Big(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\Big) \\
&\qquad \times L^2 - V - L\Big)(L)^{-1} \\
&\quad + 12VL(C-1)(-2kC + 2k + A)
\end{aligned}
$$

$$\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\frac{\pm\sqrt{\Delta(F^2+H^2)}-F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{F\sinh\left(\sqrt{\Delta}\xi\right)+B}\right\}\right]^2$$

$$+12VL(C-1)^2\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\frac{\pm\sqrt{\Delta(F^2+H^2)}-F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{F\sinh\left(\sqrt{\Delta}\xi\right)+B}\right\}\right]^2,$$

$$u_1^7(\xi)=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}+12VL(C-1)(-2kC+2k+A)\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\frac{\pm\sqrt{\Delta(F^2+H^2)}+F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{F\sinh\left(\sqrt{\Delta}\xi\right)+B}\right\}\right]^2+12VL(C-1)^2\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\frac{\pm\sqrt{\Delta(F^2+H^2)}+F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)}{F\sinh\left(\sqrt{\Delta}\xi\right)+B}\right\}\right]^2, \tag{23}$$

where F and H are real constants. Consider the following:

$$u_1^8(\xi)=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}+12VL(C-1)(-2kC+2k+A)\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi/2\right)-A\cosh\left(\sqrt{\Delta}\xi/2\right)}\right\}+12VL(C-1)^2\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi/2\right)-A\cosh\left(\sqrt{\Delta}\xi/2\right)}\right\}^2,$$

$$u_1^9(\xi)=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}+12VL(C-1)(-2kC+2k+A)\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi/2\right)-A\sinh\left(\sqrt{\Delta}\xi/2\right)}\right\}+12VL(C-1)^2\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi/2\right)-A\sinh\left(\sqrt{\Delta}\xi/2\right)}\right\}^2,$$

$$u_1^{10}(\xi)=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}+12VL(C-1)(-2kC+2k+A)\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)-A\cosh\left(\sqrt{\Delta}\xi\right)\pm i\sqrt{\Delta}}\right\}+12VL(C-1)^2\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)-A\cosh\left(\sqrt{\Delta}\xi\right)\pm i\sqrt{\Delta}}\right\}^2,$$

$$u_1^{11}(\xi)=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}+12VL(C-1)(-2kC+2k+A)$$

$$\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)-A\sinh\left(\sqrt{\Delta}\xi\right)\pm\sqrt{\Delta}}\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)-A\sinh\left(\sqrt{\Delta}\xi\right)\pm\sqrt{\Delta}}\right\}^2. \tag{24}$$

When $\Delta = A^2-4BC+4B<0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$),

$$u_1^{12}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k+\frac{1}{2(C-1)}\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k+\frac{1}{2(C-1)}\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^2,$$

$$u_1^{13}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^2,$$

$$u_1^{14}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k-\frac{1}{2(C-1)}\times\left(-A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{1}{2(C-1)}\times\left(-A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}^2,$$

$$u_1^{15}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k-\frac{1}{2(C-1)}\times\left(A+\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{1}{2(C-1)}\times\left(A+\sqrt{-\Delta}\left(\cot\left(\sqrt{-\Delta}\xi\right)\pm\operatorname{csch}\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}^2,$$

$$u_1^{16}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\frac{\pm\sqrt{-\Delta\left(F^2-H^2\right)}-F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{F\sin\left(\sqrt{-\Delta}\xi\right)+B}\right\}\right]$$
$$+12VL(C-1)^2$$
$$\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\frac{\pm\sqrt{-\Delta(F^2-H^2)}-F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{F\sin\left(\sqrt{-\Delta}\xi\right)+B}\right\}\right]^2,$$

$$
\begin{aligned}
&u_1^{17}(\xi)\\
&= \left(V\left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right)\right.\\
&\quad \left.\times L^2 - V - L\right)(L)^{-1}\\
&\quad + 12VL(C-1)(-2kC + 2k + A)\\
&\quad \times\left[k + \frac{1}{2(C-1)}\right.\\
&\quad \times\left\{-A\right.\\
&\quad \left.\left.+\frac{\pm\sqrt{-\Delta\left(F^2-H^2\right)} - F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{F\sin\left(\sqrt{-\Delta}\xi\right)+B}\right\}\right]\\
&\quad + 12VL(C-1)^2\\
&\quad \times\left[k + \frac{1}{2(C-1)}\right.\\
&\quad \left.\times\left\{-A + \frac{\pm\sqrt{-\Delta(F^2-H^2)} - F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{F\sin(\sqrt{-\Delta}\xi)+B}\right\}\right]^2,
\end{aligned}
$$

$$
\begin{aligned}
&u_1^{18}(\xi)\\
&= \left(V\left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right)\right.\\
&\quad \left.\times L^2 - V - L\right)(L)^{-1}\\
&\quad + 12VL(C-1)(-2kC + 2k + A)\\
&\quad \times\left[k + \frac{1}{2(C-1)}\right.\\
&\quad \times\left\{-A\right.\\
&\quad \left.\left.+\frac{\pm\sqrt{-\Delta\left(F^2-H^2\right)} + F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{F\sin\left(\sqrt{-\Delta}\xi\right)+B}\right\}\right]\\
&\quad + 12VL(C-1)^2\\
&\quad \times\left[k + \frac{1}{2(C-1)}\right.\\
&\quad \left.\times\left\{-A + \frac{\pm\sqrt{-\Delta(F^2-H^2)} + F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)}{F\sin\left(\sqrt{-\Delta}\xi\right)+B}\right\}\right]^2,
\end{aligned}
\tag{25}
$$

where F and H are real constants such that $F^2 - H^2 > 0$. Consider the following:

$$
\begin{aligned}
&u_1^{19}(\xi)\\
&= \left(V\left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right)\right.\\
&\quad \left.\times L^2 - V - L\right)(L)^{-1}\\
&\quad + 12VL(C-1)(-2kC + 2k + A)\\
&\quad \times\left\{k - \frac{2B\cos\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi/2\right) + A\cos\left(\sqrt{-\Delta}\xi/2\right)}\right\}\\
&\quad + 12VL(C-1)^2\\
&\quad \times\left\{k - \frac{2B\cos\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi/2\right) + A\cos\left(\sqrt{-\Delta}\xi/2\right)}\right\}^2,
\end{aligned}
$$

$$
\begin{aligned}
&u_1^{20}(\xi)\\
&= \left(V\left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right)\right.\\
&\quad \left.\times L^2 - V - L\right)(L)^{-1}\\
&\quad + 12VL(C-1)(-2kC + 2k + A)\\
&\quad \times\left\{k + \frac{2B\sin\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right) - A\sin\left(\sqrt{-\Delta}\xi/2\right)}\right\}\\
&\quad + 12VL(C-1)^2\\
&\quad \times\left\{k + \frac{2B\sin\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right) - A\sin\left(\sqrt{-\Delta}\xi/2\right)}\right\}^2,
\end{aligned}
$$

$$
\begin{aligned}
&u_1^{21}(\xi)\\
&= \left(V\left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right)\right.\\
&\quad \left.\times L^2 - V - L\right)(L)^{-1}\\
&\quad + 12VL(C-1)(-2kC + 2k + A)
\end{aligned}
$$

$$\times\left\{k-\frac{2B\cos\left(\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)+A\cos\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}}\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{2B\cos\left(\sqrt{-\Delta}\xi\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right)+A\cos\left(\sqrt{-\Delta}\xi\right)\pm\sqrt{-\Delta}}\right\}^2,$$

$$u_1^{22}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.$$
$$\left.\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k+\frac{2B\sin\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right)-A\sin\left(\sqrt{-\Delta}\xi/2\right)\pm\sqrt{-\Delta}}\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k+\frac{2B\sin\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right)-A\sin\left(\sqrt{-\Delta}\xi/2\right)\pm\sqrt{-\Delta}}\right\}^2. \tag{26}$$

When $B=0$ and $A(C-1)\neq 0$,

$$u_1^{23}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.$$
$$\left.\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k-\frac{Ac_1}{(C-1)\{c_1+\cosh(A\xi)-\sinh(A\xi)\}}\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{Ac_1}{(C-1)\{c_1+\cosh(A\xi)-\sinh(A\xi)\}}\right\}^2, \tag{27}$$

and its graph is shown in Figure 2. Consider

$$u_1^{24}(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.$$
$$\left.\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k-\frac{A(\cosh(A\xi)+\sinh(A\xi))}{(C-1)\{c_1+\cosh(A\xi)+\sinh(A\xi)\}}\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{A(\cosh(A\xi)+\sinh(A\xi))}{(C-1)\{c_1+\cosh(A\xi)+\sinh(A\xi)\}}\right\}^2, \tag{28}$$

where c_1 is an arbitrary constant.

When $A=B=0$ and $(C-1)\neq 0$, the solution of (12) is

$$u_1^{25}(\xi)=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.$$
$$\left.\times L^2-V-L\right)(L)^{-1}$$
$$+12VL(C-1)(-2kC+2k+A)$$
$$\times\left\{k-\frac{1}{(C-1)\xi+c_2}\right\}$$
$$+12VL(C-1)^2$$
$$\times\left\{k-\frac{1}{(C-1)\xi+c_2}\right\}^2, \tag{29}$$

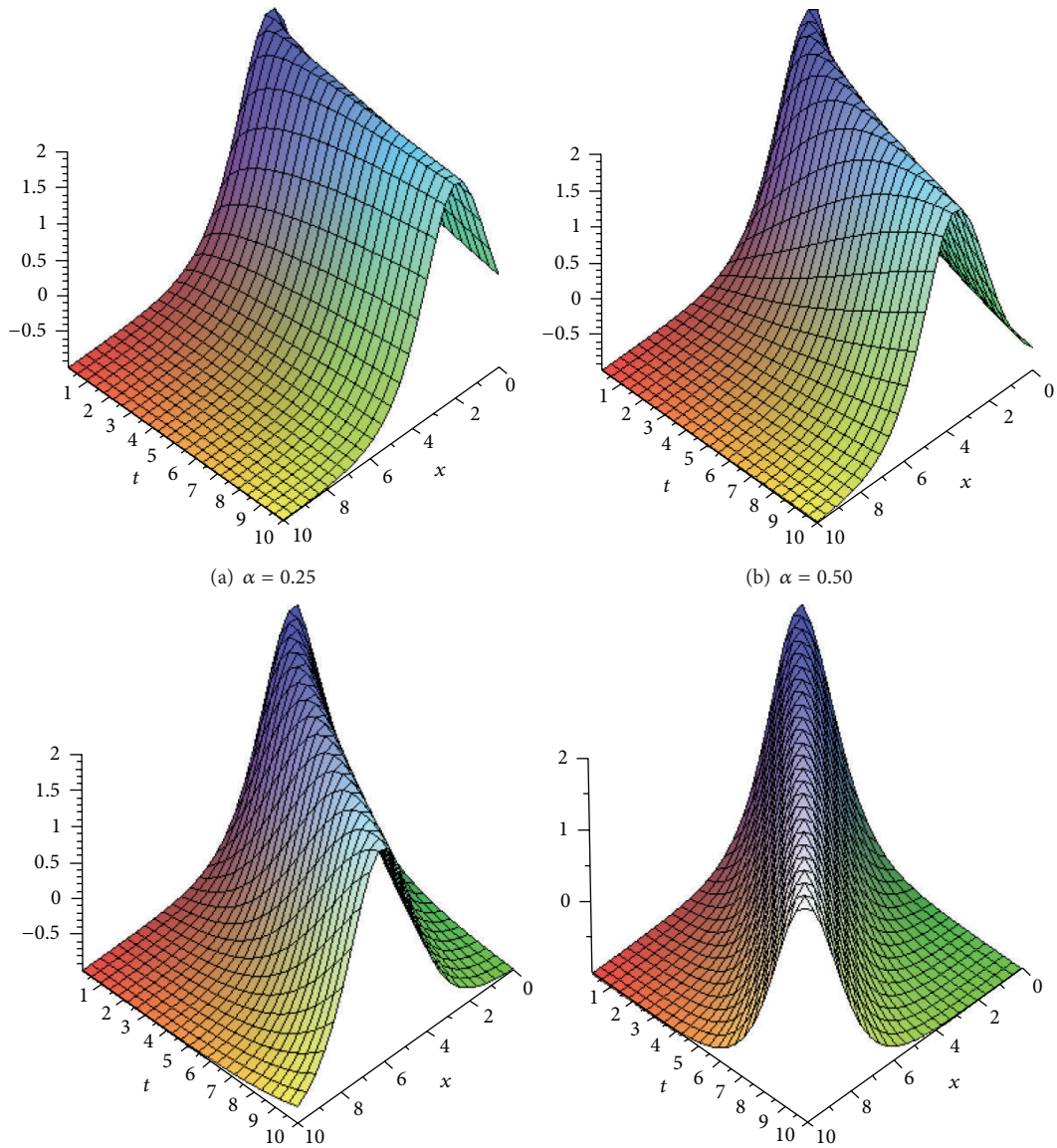
where c_2 is an arbitrary constant.

Substituting the solutions $G(\xi)$ of (10) into (19) and simplifying, we obtain the following solutions.

When $\Delta=A^2-4BC+4B>0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$),

$$u_2^1(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.$$
$$\left.\times L^2-V-L\right)(L)^{-1}$$
$$-12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)$$
$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-1}$$
$$+12VL\left((1-C)k^2+kA-B\right)^2$$
$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-2},$$

$$u_2^2(\xi)$$
$$=\left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.$$
$$\left.\times L^2-V-L\right)(L)^{-1}$$
$$-12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)$$
$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-1}$$
$$+12VL\left((1-C)k^2+kA-B\right)^2$$
$$\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-2},$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

FIGURE 2: (a)–(d) show the bell-shape sech^2 solitary traveling wave solution for u_1^{23} for different values of parameters.

$$
\begin{aligned}
&u_2^3(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad \left.\times L^2-V-L\right)(L)^{-1}\\
&\quad -12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\right.\\
&\qquad \left.\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}^{-1}\\
&\quad +12VL\left((1-C)k^2+kA-B\right)^2\\
&\quad \times\left\{k-\frac{1}{2(C-1)}\right.\\
&\qquad \left.\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}^{-2};
\end{aligned}
\tag{30}
$$

the other families of exact solutions of (12) are omitted for convenience.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$
\begin{aligned}
&u_2^{12}(\xi)\\
&= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad \left.\times L^2-V-L\right)(L)^{-1}
\end{aligned}
$$

$$- 12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)$$
$$\times\left\{k+\frac{1}{2(C-1)}\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-1}$$
$$+ 12VL\left((1-C)k^2+kA-B\right)^2$$
$$\times\left\{k+\frac{1}{2(C-1)}\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-2}, \tag{31}$$

and its graph is given in Figure 3. Consider

$$\begin{aligned}
u_2^{13}(\xi) &= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad\left.\times L^2-V-L\right)(L)^{-1}\\
&\quad- 12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)\\
&\quad\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-1}\\
&\quad+ 12VL\left((1-C)k^2+kA-B\right)^2\\
&\quad\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-2},\\
u_2^{14}(\xi) &= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad\left.\times L^2-V-L\right)(L)^{-1}\\
&\quad- 12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)\\
&\quad\times\left\{k+\frac{1}{2(C-1)}\right.\\
&\qquad\left.\times\left(-A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}^{-1}\\
&\quad+ 12VL\left((1-C)k^2+kA-B\right)^2\\
&\quad\times\left\{k+\frac{1}{2(C-1)}\right.\\
&\qquad\left.\times\left(-A+\sqrt{-\Delta}\left(\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}^{-2}.
\end{aligned} \tag{32}$$

When $A = B = 0$ and $(C-1) \neq 0$, the solution of (12) is

$$\begin{aligned}
u_2^{25}(\xi) &= \left(V\left(12(C-1)^2k^2-12A(C-1)k-8B+8BC+A^2\right)\right.\\
&\quad\left.\times L^2-V-L\right)(L)^{-1}\\
&\quad- 12VL\left((1-C)k^2+kA-B\right)((2-2C)k+A)\\
&\quad\times\left\{k-\frac{1}{(C-1)\xi+c_2}\right\}^{-1}\\
&\quad+ 12VL\left((1-C)k^2+kA-B\right)^2\\
&\quad\times\left\{k-\frac{1}{(C-1)\xi+c_2}\right\}^{-2},
\end{aligned} \tag{33}$$

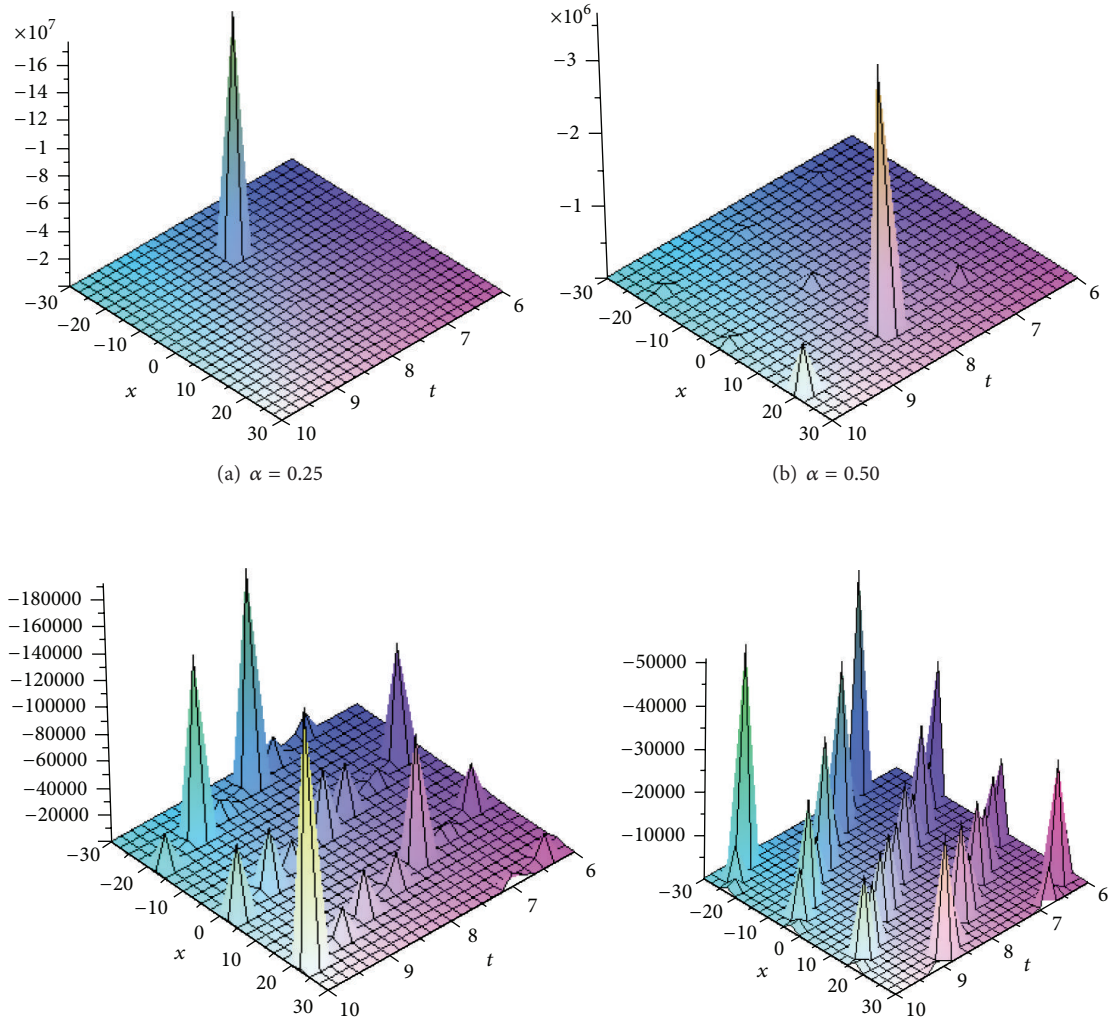
where c_2 is an arbitrary constant.

We can write down the other families of exact solutions of (12) which are omitted for practicality.

Finally, substituting the solutions $G(\xi)$ of (10) into (20) and simplifying, we obtain the following solutions.

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$\begin{aligned}
u_3^1(\xi) &= -\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L}\\
&\quad+ 12VL(C-1)^2\\
&\quad\times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^2\\
&\quad+\frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2}\\
&\quad\times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-2},\\
u_3^2(\xi) &= -\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L}\\
&\quad+ 12VL(C-1)^2\\
&\quad\times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^2\\
&\quad+\frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2}\\
&\quad\times\left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-2},\\
u_3^3(\xi) &= -\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L}\\
&\quad+ 12VL(C-1)^2
\end{aligned}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

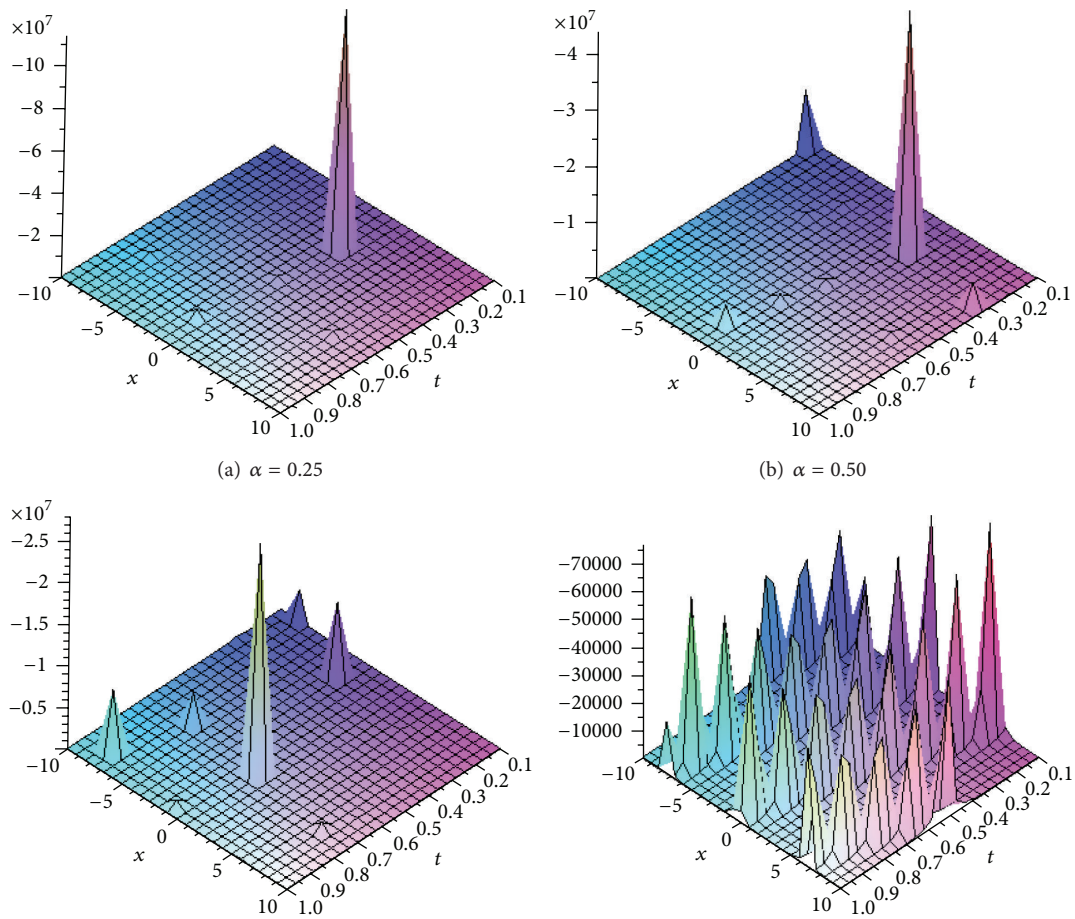
Figure 3: (a)–(d) show singular periodic traveling wave solution of u_2^{12} for different values of parameters.

$$\times\left(\frac{1}{2(C-1)}\times\left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right)^2 + \frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2}\times\left(\frac{1}{2(C-1)}\times\left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right)\pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right)^{-2}. \tag{34}$$

Others families of exact solutions are omitted for the sake of simplicity.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_3^{12}(\xi) = -\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L} + 12VL(C-1)^2\times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^2 + \frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2}\times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-2}, \tag{35}$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

FIGURE 4: (a)–(d) show singular periodic traveling wave solution of u_3^{12} for different values of parameters.

and its graph is given in Figure 4. Consider

$$\begin{aligned} u_3^{13}(\xi) = &-\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L} \\ &+12VL(C-1)^2 \\ &\times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^2 \\ &+\frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2} \\ &\times\left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-2}, \\ u_3^{14}(\xi) = &-\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L} \\ &+12VL(C-1)^2 \\ &\times\left(\frac{1}{2(C-1)}\right. \\ &\left.\times\left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right\}\right)^2 \\ &+\frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2} \\ &\times\left(\frac{1}{2(C-1)}\right. \\ &\left.\times\left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right)\pm\sec\left(\sqrt{-\Delta}\xi\right)\right\}\right)^{-2}. \end{aligned} \tag{36}$$

When $(C-1)\neq 0$ and $A=B=0$, the solution of (12) is

$$\begin{aligned} u_3^{25}(\xi) = &-\frac{2VL^2\left((4-4C)B+A^2\right)-V-L}{L} \\ &+12VL(C-1)^2 \end{aligned}$$

$$\times\left(\frac{A}{2(C-1)}-\frac{1}{(C-1)\xi+c_2}\right)^2 + \frac{3VL\left((4-4C)B+A^2\right)^2}{4(1-C)^2} \times\left(\frac{A}{2(C-1)}-\frac{1}{(C-1)\xi+c_2}\right)^{-2}, \tag{37}$$

where c_2 is an arbitrary constant.

Other exact solutions of (12) are omitted here for convenience.

4. Discussion

If we replace A by $-A$ and B by $-B$ and put $C = 0$ in (10), then the novel (G'/G)-expansion method coincides with Akbar et al.'s generalized and improved (G'/G)-expansion method [36]. On the other hand if we put $k = 0$ in (8) and $C = 0$ in (10) then the method is identical to the improved (G'/G)-expansion method presented by Zhang et al. [39]. Again if we set $k = 0$, $C = 0$ and the negative exponents of (G'/G) are zero in (8), then the method turn out into the basic (G'/G)-expansion method introduced by Wang et al. [31]. At the end, if we put $C = 0$ in (10) and α_i $(i = 1, 2, 3, \ldots, N)$ are functions of x and t instead of constants then the method is transformed into the generalized (G'/G)-expansion method developed by Zhang et al. [38]. Thus the methods presented in [31, 36, 38, 39] are only special cases of the novel (G'/G)-expansion method. So our method is more general having more free parameters than the existing methods.

5. Conclusions

A novel (G'/G)-expansion method is applied to fractional partial differential equation successfully. As applications, abundant new exact solutions for time fractional BBM-Burger equation have been successfully obtained. The nonlinear fractional complex transformation for ξ is very important, which ensures that a certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. The obtained solutions are more general with more free parameters. Thus novel (G'/G)-expansion method would be a powerful mathematical tool for solving nonlinear evolution equations. To the best of our knowledge, the solutions obtained in this article are not reported in literature previously.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] A. G. Nikitin and T. A. Barannyk, "Solitary wave and other solutions for nonlinear heat equations," *Central European Journal of Mathematics*, vol. 2, no. 5, pp. 840–858, 2004.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[3] J. H. He, "Some applications of nonlinear fractional differential equations and their applications," *Bulletin of Science, Technology & Society*, vol. 15, no. 2, pp. 86–90, 1999.

[4] Z. B. Li and J. H. He, "Application of the fractional complex transform to fractional differential equations," *Nonlinear Science Letter A*, vol. 2, no. 3, pp. 121–126, 2011.

[5] Z. Li and J. He, "Fractional complex transform for fractional differential equations," *Mathematical & Computational Applications*, vol. 15, no. 2, pp. 970–973, 2010.

[6] J. He, S. K. Elagan, and Z. B. Li, "Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus," *Physics Letters A*, vol. 376, no. 4, pp. 257–259, 2012.

[7] R. W. Ibrahim, "Fractional complex transforms for fractional differential equations," *Advances in Difference Equations*, vol. 2012, article 192, 2012.

[8] H. Jafari, A. Kadem, D. Baleanu, and T. Yilmaz, "Solutions of the fractional davey-stewartson equations with variational iteration method," *Romanian Reports in Physics*, vol. 64, no. 2, pp. 337–346, 2012.

[9] H. Jafari and S. Seifi, "Solving a system of nonlinear fractional partial differential equations using homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 1962–1969, 2009.

[10] S. Das, R. Kumar, P. K. Gupta, and H. Jafari, "Approximate analytical solutions for fractional space- and time-partial differential equations using homotopy analysis method," *Applications and Applied Mathematics*, vol. 5, no. 10, pp. 1641–1659, 2010.

[11] H. Jafari and C. M. Khalique, "Analytical solutions of nonlinear fractional differential equations using variational iteration method," *Journal of Nonlinear Systems and Applications*, vol. 2, no. 3-4, pp. 148–151, 2011.

[12] Y. Hao, H. M. Srivastava, and H. Jafari, "Helmholtz and diffusion equations associated with local fractional derivative operators involving the CANtorian and CANtor-type cylindrical coordinates," *Advances in Mathematical Physics*, vol. 2013, Article ID 754248, 5 pages, 2013.

[13] X. J. Yang, "Local fractional integral transforms," *Progress in Nonlinear Science*, vol. 4, pp. 1–225, 2011.

[14] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science, New York, NY, USA, 2012.

[15] A. H. Salas and C. A. Gómez S., "Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation," *Mathematical Problems in Engineering*, vol. 2010, Article ID 194329, 14 pages, 2010.

[16] W. Malfliet, "The tanh method: a tool for solving certain classes of nonlinear evolution and wave equations," *Journal of Computational and Applied Mathematics*, vol. 164-165, pp. 529–541, 2004.

[17] M. A. Abdou, "The extended tanh method and its applications for solving nonlinear physical models," *Applied Mathematics and Computation*, vol. 190, no. 1, pp. 988–996, 2007.

[18] A. M. Wazwaz, "The extended tanh method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations," *Chaos, Solitons and Fractals*, vol. 38, no. 5, pp. 1505–1516, 2008.

[19] E. G. Fan, "Extended tanh-function method and its applications to nonlinear equations," *Physics Letters A*, vol. 277, no. 4-5, pp. 212–218, 2000.

[20] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, vol. 149 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, UK, 1991.

[21] H. Jafari and H. Tajadodi, "He's variational iteration method for solving fractional Riccati differential equation," *International Journal of Differential Equations*, vol. 2010, Article ID ID 764738, 8 pages, 2010.

[22] N. Faraz, Y. Khan, H. Jafari, A. Yildirim, and M. Madani, "Fractional variational iteration method via modified Riemann-Liouville derivative," *Journal of King Saud University-Science*, vol. 23, no. 4, pp. 413–417, 2011.

[23] J. H. He and X. H. Wu, "Exp-function method for nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 30, no. 3, pp. 700–708, 2006.

[24] H. Naher, F. A. Abdullah, and M. Ali Akbar, "New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the exp-function method," *Journal of Applied Mathematics*, vol. 2012, Article ID 575387, 14 pages, 2012.

[25] S. T. Mohyud-Din, "Solutions of nonlinear differential equations by Exp-function method," *World Applied Sciences Journal*, vol. 7, pp. 116–147, 2009.

[26] S. T. Mohyud-Din, M. A. Noor, and A. Waheed, "Exp-function method for generalized travelling solutions of calogero-degasperis-fokas equation," *Zeitschrift fur Naturforschung A: Journal of Physical Sciences*, vol. 65, no. 1, pp. 78–84, 2010.

[27] X. Zhao, L. Wang, and W. Sun, "The repeated homogeneous balance method and its application to nonlinear partial differential equations," *Chaos, Solitons and Fractals*, vol. 28, no. 2, pp. 448–453, 2006.

[28] F. Zhaosheng, "Comment on 'On the extended applications of homogeneous balance method'," *Applied Mathematics and Computation*, vol. 158, no. 2, pp. 593–596, 2004.

[29] M. Wang and X. Li, "Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation," *Chaos, Solitons and Fractals*, vol. 24, no. 5, pp. 1257–1268, 2005.

[30] M. A. Abdou, "The extended F-expansion method and its application for a class of nonlinear evolution equations," *Chaos, Solitons & Fractals*, vol. 31, no. 1, pp. 95–104, 2007.

[31] M. Wang, X. Li, and J. Zhang, "The $(G^{\hat{E}\acute{z}}/G)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[32] M. A. Akbar, N. H. M. Ali, and S. T. Mohyud-Din, "The alternative G'/G-expansion method with generalized Riccati equation: application to fifth order (1 + 1)-dimensional Caudrey-Dodd-Gibbon equation," *International Journal of Physical Sciences*, vol. 7, no. 5, pp. 743–752, 2012.

[33] H. Jafari, N. Kadkhoda, and E. Salehpour, "Application of (G'/G)-expansion method to nonlinear Lienard equation," *Indian Journal of Science and Technology*, vol. 5, no. 4, pp. 2554–2556, 2012.

[34] E. J. Parkes, "Observations on the basic $(G\prime/G)$-expansion method for finding solutions to nonlinear evolution equations," *Applied Mathematics and Computation*, vol. 217, no. 4, pp. 1759–1763, 2010.

[35] M. Ali Akbar and N. H. M. Ali, "The modified alternative $(G\prime/G)$-expansion method for finding the exact solutions of nonlinear PDEs in mathematical physics," *International Journal of Physical Sciences*, vol. 6, no. 35, pp. 7910–7920, 2011.

[36] M. A. Akbar, N. H. M. Ali, and E. M. E. Zayed, "A generalized and improved (G'/G)-expansion method for nonlinear evolution equations," *Mathematical Problems in Engineering*, vol. 2012, Article ID 459879, 22 pages, 2012.

[37] E. Salehpour, H. Jafari, and N. Kadkhoda, "Application of (G'/G)-expansion method to nonlinear Lienard equation," *Indian Journal of Science and Technology*, vol. 5, no. 4, pp. 2554–2556, 2012.

[38] J. Zhang, X. Wei, and Y. Lu, "A generalized G'/G-expansion method and its applications," *Physics Letters A*, vol. 372, no. 20, pp. 3653–3658, 2008.

[39] J. Zhang, F. Jiang, and X. Zhao, "An improved $(G\prime/G)$-expansion method for solving nonlinear evolution equations," *International Journal of Computer Mathematics*, vol. 87, no. 8, pp. 1716–1725, 2010.

[40] E. M. E. Zayed, "New traveling wave solutions for higher dimensional nonlinear evolution equations using a generalized (G'/G)-expansion method," *Journal of Physics A: Mathematical and Theoretical*, vol. 42, no. 19, Article ID 195202, 2009.

[41] E. M. E. Zayed, "The G'/G-expansion method combined with the Riccati equation for finding exact solutions of nonlinear PDEs," *Journal of Applied Mathematics Informatics*, vol. 29, no. 1-2, pp. 351–367, 2011.

[42] W. Ma and J. Lee, "A transformed rational function method and exact solutions to the 3 + 1 dimensional Jimbo-Miwa equation," *Chaos, Solitons & Fractals*, vol. 42, no. 3, pp. 1356–1363, 2009.

[43] M. N Alam, M. A. Akbar, and S. T. Mohyud-Din, "A novel (G'/G)-expansion method and its application to the Boussinesq equation," *Chinese Physics B*, vol. 23, no. 2, Article ID 020203, 2013.

[44] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results," *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1367–1376, 2006.

[45] S. Zhu, "The generalizing Riccati equation mapping method in non-linear evolution equation: application to (2 + 1)-dimensional Boiti-Leon-Pempinelle equation," *Chaos, Solitons and Fractals*, vol. 37, no. 5, pp. 1335–1342, 2008.

[46] S. Kumar and D. Kumar, "Fractional modelling for BBM-Burger equation by using new homotopy analysis transform method," *Journal of the Association of Arab Universities for Basic and Applied Sciences*, 2013.

[47] L. Song and H. Zhang, "Solving the fractional BBM-Burgers equation using the homotopy analysis method," *Chaos, Solitons and Fractals*, vol. 40, no. 4, pp. 1616–1622, 2009.

[48] A. Fakhari, G. Domairry, and E. Brahimpour, "Approximate explicit solutions of nonlinear BBMB equations by homotopy analysis method and comparison with the exact solution," *Physics Letters A*, vol. 368, no. 1-2, pp. 64–68, 2007.