Research Article

Exact Solutions of the Time Fractional BBM-Burger Equation by Novel \((G'/G)\)-Expansion Method

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The fractional derivatives are used in the sense modified Riemann-Liouville to obtain exact solutions for BBM-Burger equation of fractional order. This equation can be converted into an ordinary differential equation by using a persistent fractional complex transform and, as a result, hyperbolic function solutions, trigonometric function solutions, and rational solutions are attained. The performance of the method is reliable, useful, and gives newer general exact solutions with more free parameters than the existing methods. Numerical results coupled with the graphical representation completely reveal the trustworthiness of the method.

1. Introduction

Differential equations of noninteger order are generalizations of conventional differential equations of integer order [1]. Exploration and applications of integrals and derivatives of random order are efficiently dealt with the field of mathematical analysis, called as fractional calculus, which has engrossed in considerable interest in many disciplines, nowadays. The behavior of many physical systems can be perfectly defined by the fractional theory. In recent years, we cannot rebuff the importance of fractional differential equations because of their numerous applications in the areas of physics and engineering. For example, the nonlinear fluctuation of earthquakes can be modeled with the help of fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eradicate the problems arising from the huge traffic flow [2, 3]. We apply a generalized fractional complex transform [4–7] to convert the given fractional order differential equation to ordinary differential equation. On account of development of the computer and its exact description of countless real-life problems, fractional calculus has touched the height of fame and success nowadays, although it was invented three centuries ago by Newton and Leibniz. Many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are better described by differential equations of noninteger order [8–11]. A physical interpretation of the fractional calculus was given in [12–14]. With the development of symbolic computation software, like Maple, many researchers developed and established many numerical and analytical methods to search for exact solutions of nonlinear evolution equations (NLEEs), for example, Cole-Hopf transformation [15], Tanh-function method [16–19], inverse scattering transform method [20], variational iteration method [21, 22], Exp-function method [23–26], homogeneous balance method [27, 28], and F-expansion method [29, 30], which are used for searching the exact solutions.

Lately, a straight and crisp method, called \((G'/G)\)-expansion method, was introduced by Wang et al. [31] and confirmed that it is a powerful method for seeking analytic solutions of nonlinear evolution equations (NLEEs). For additional references see the articles [32–37]. In order to establish the efficiency and diligence of \((G'/G)\)-expansion method and to extend the range of applicability, further research has been carried out by several researchers. For illustration, Zhang et al. [38] made an overview of \((G'/G)\)-expansion method for the evolution equations with variable coefficients. Zhang et al. [39] also presented an improved \((G'/G)\)-expansion method to look for more broad traveling wave solutions. Zayed [40] offered a new technique of
(G'/G)-expansion method where G(ξ) gratifies the Jacobi elliptic equation, \( [G'(ξ)]^2 = e_2 G^4(ξ) + e_1 G^2(ξ) + e_0 \), where \( e_2, e_1, \) and \( e_0 \) are random constants, and obtained new exact solutions. Zayed [41] for a second time offered a different approach of this method in which G(ξ) satisfies the Riccati equation \( G'(ξ) = AG(ξ) + BG^2(ξ) \), where \( A \) and \( B \) are casual constants.

The (G'/G)-expansion method and the transformed rational function method used by Ma and Lee [42] have a common idea. That is, we initially put the given nonlinear evolution equation (NLEE) into the equivalent ordinary differential equation (ODE), and then ODE can be transformed into a system of arithmetical polynomials with the influential constants. By the solutions of the ordinary differential equation, we can obtain the exact traveling solutions and rational solution of the nonlinear evolution equations.

In this article, we will apply novel (G'/G)-expansion method introduced by Alam et al. [43] to solve the time fractional BBM-Burger equation, whereas, the modified Riemann-Liouville derivative given by Jumarie [44] is used and abundant new families of exact solutions are found. The Jumarie's modified Riemann-Liouville derivative of order \( \alpha \) is defined by the following expression:

\[
D^\alpha_t f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^\alpha} dt', \quad 0 < \alpha < 1,
\]

\[
D^\alpha_\xi f(\xi) = \frac{1}{\Gamma(1+\gamma)} \frac{d}{d\xi} \int_0^\xi \frac{f(\xi')}{(\xi-\xi')^{\gamma-\alpha}} d\xi', \quad n \leq \alpha < n + 1, \quad n \geq 1.
\]

Some important properties of Jumarie's derivative are

\[
D^\alpha_t [f(t)g(t)] = g(t)D^\alpha_t f(t) + f(t)D^\alpha_t g(t),
\]

\[
D^\alpha_t [g(t)] = f^\alpha_g [g(t)] D^\alpha_t g(t) = D^\alpha_t f [g(t)] \left( g'(t) \right)^\alpha.
\]

\[ S(u, u_x, u_t, D^\alpha_t u, ...) = 0, \quad 0 < \alpha \leq 1, \quad (5) \]

where \( D^\alpha_t u \) is Jumarie's modified Riemann-Liouville derivatives of \( u, u(x, t) \) is an unknown function, and \( S \) is a polynomial in \( u \) and its various partial derivatives including fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the method are as follows.

**Step 1.** Li and He [5] projected a fractional complex transformation to convert fractional partial differential equation into ordinary differential equation (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The following complex transformation:

\[
u(x, t) = u(\xi), \quad \xi = Lx + Vt + \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad (6)
\]

where \( L, V \) are arbitrary constants with \( L, V \neq 0 \), permits us to convert (5) into an ordinary differential equation of integer order in the form

\[
P(u, u', u'', u''', \ldots) = 0, \quad (7)
\]

where the primes stand for the ordinary derivatives with respect to \( \xi \).

**Step 2.** Integrate (7) term by term one or more times, if possible; capitulate constant(s) of integration which can be calculated later on.

**Step 3.** Suppose that the solution of (7) can be expressed as

\[
u(\xi) = \sum_{i=-m}^{m} \alpha_i (k + \Phi(\xi))^i, \quad (8)
\]

where

\[
\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}. \quad (9)
\]

Herein \( \alpha_m \) or \( \alpha_m \) may be zero, but both of them cannot be zero simultaneously. \( \alpha_i \) \( (i = 0, \pm 1, \pm 2, \ldots, \pm N) \) and \( k \) are constants to be determined later and \( G = G(\xi) \) satisfies the second order nonlinear ordinary differential equation

\[
GG'' = AG'G + BG^2 + C(G')^2, \quad (10)
\]

where \( G \) denotes the derivative with respect \( \xi \); \( A, B, \) and \( C \) are genuine constants.

The Cole-Hopf transformation \( \Phi(\xi) = \ln (G(\xi))_t = \frac{G'(\xi)}{G(\xi)} \) reduces (10) into the Riccati equation

\[
\Phi'(\xi) = B + A\Phi(\xi) + (C - 1)\Phi^2(\xi). \quad (11)
\]

Equation (11) has individual twenty-five solutions (see Zhu, [45] for details).

**Step 4.** The positive integer \( m \) can be calculated by balancing the highest order linear term with the highest order nonlinear term in (7).

**Step 5.** Inserting (8) together with (9) and (10) into (7), we obtain polynomials in \( (k + (G'/G))^i \) and \( (k + (G'/G))^{-i} \), \( (i = 0, 1, 2, \ldots, N) \). Collecting each coefficient of the resulted polynomials to zero acquiesces an overdetermined set of algebraic equations for \( \alpha_i \) \( (i = 0, \pm 1, \pm 2, \ldots, \pm N) \), \( k, L, \) and \( V \).

**Step 6.** After solving the system of algebraic equations obtained in Step 5, we can obtain the values of the constants. The solutions of (10) together with the obtained values of the constants yield abundant exact traveling wave solutions of the nonlinear evolution equation (5).
3. Application of the Method to the Time Fractional BBM-Burger Equation

Consider the following BBM-Burger equation in time fractional operator form as:

\[ D_\alpha^t u - u_{xx} + u_x + \left( \frac{u^2}{2} \right)_x = 0, \quad t > 0, \quad 0 < \alpha \leq 1. \quad (12) \]

S. Kumar and D. Kumar [46] use new fractional homotopy analysis transform method to time fractional BBM-Burger equation and found the series solution of this equation. Song and Zhang [47] and Fakhar et al. [48] find different type of solutions of the fractional BBM-Burger equation by using homotopy analysis method.

By the use of (4), (12) is converted into an ordinary differential equation of integer order and after integrating once, we obtain

\[ (L + V) u + \frac{L}{2} u^2 - VL^2 u'' + C_1 = 0, \quad (13) \]

where \( C_1 \) is an integration constant.

Considering the homogeneous balance between \( u'' \) and \( u^2 \) in (13), we obtain \( m = 2 \). Therefore, the trial solution (8) becomes

\[ u(\xi) = \alpha_{-2}(k + \Phi(\xi))^{-2} + \alpha_{-1}(k + \Phi(\xi))^{-1} + \alpha_{0} + \alpha_{1}(k + \Phi(\xi)) + \alpha_{2}(k + \Phi(\xi))^2. \quad (14) \]

Using (14) into (13), left-hand side is converted into polynomials in \( (k + (G'/G))^i \) and \( (k + (G'/G))^{-i} \) (\( i = 0, 1, 2, \ldots, N \)). Equating the coefficients of same power of the resulted polynomials to zero, we attain a set of algebraic equations (which are omitted for the sake of simplicity) for \( \alpha_0, \alpha_1, \alpha_2, \alpha_{-1}, \alpha_{-2}, k, C_1, L, \) and \( V \). Solving the overdetermined set of algebraic equations by using the symbolic computation software, such as Maple 13, we obtain the following solution sets:

**The Set 1.** We consider that

\[ \alpha_0 = \left( V \left( 12(C - 1)^2 k^2 - 12A \left( C - 1 \right) k \right) \right. \]
\[ \left. -8B + 8BC + A^2 \right) L^2 - V - L \right) \right) \right) (L)^{-1} \]
\[ \alpha_1 = -12 \left( (1 - C) k^2 + kA - B \right) VL \left( (2 - 2C) k + A \right), \]
\[ \alpha_{-1} = -12 \left( (1 - C) k^2 + kA - B \right) VL \left( (2 - 2C) k + A \right), \]
\[ \alpha_{-2} = 12VL \left( (1 - C) k^2 + kA - B \right)^2, \]
\[ V = V, \quad L = L, \quad k = k, \quad \alpha_{-1} = 0, \quad \alpha_{-2} = 0, \]
\[ C_1 = -\frac{1}{2L} \left( V \left( (4C + 4) B + A^2 \right) L^2 - V - L \right) \]
\[ \times \left( V \left( (4C + 4) B + A^2 \right) L^2 + V + L \right), \]

where \( k, L, V, A, B, \) and \( C \) are arbitrary constants.

**The Set 2.** We consider that

\[ \alpha_0 = \left( V \left( 12(C - 1)^2 k^2 - 12A \left( C - 1 \right) k \right) \right. \]
\[ \left. -8B + 8BC + A^2 \right) L^2 - V - L \right) \right) (L)^{-1} \]
\[ \alpha_{-1} = -12 \left( (1 - C) k^2 + kA - B \right) VL \left( (2 - 2C) k + A \right), \]
\[ \alpha_{-2} = 12VL \left( (1 - C) k^2 + kA - B \right)^2, \]
\[ V = V, \quad L = L, \quad k = k, \quad \alpha_{-1} = 0, \quad \alpha_{-2} = 0, \]
\[ C_1 = -\frac{1}{2L} \left( V \left( (4C + 4) B + A^2 \right) L^2 - V - L \right) \]
\[ \times \left( V \left( (4C + 4) B + A^2 \right) L^2 + V + L \right), \]

where \( k, L, V, A, B, \) and \( C \) are arbitrary constants.

**The Set 3.** We consider that

\[ \alpha_0 = \left( V \left( 12(C - 1)^2 k^2 - 12A \left( C - 1 \right) k \right) \right. \]
\[ \left. -8B + 8BC + A^2 \right) L^2 - V - L \right) \right) (L)^{-1} \]
\[ k = \frac{A}{2 \left( C - 1 \right)}, \]
\[ \alpha_{-2} = \frac{3VL \left( (4C + 4) B + A^2 \right)^2}{4 \left( 1 - C \right)^2}, \]
\[ \alpha_{-1} = 0, \quad \alpha_{-1} = 0, \quad V = V, \quad L = L, \]
\[ C_1 = -\frac{8}{L} \left( V \left( (4C + 4) B + A^2 \right) \right. \]
\[ \left. -\frac{V}{4} - \frac{L}{4} \right) \]
\[ \times \left( V \left( (4C + 4) B + A^2 \right) + \frac{V}{4} + \frac{L}{4} \right), \]

where \( V, L, A, B, \) and \( C \) are arbitrary constants.

Substituting (15)–(17) into (14), we obtain

\[ u_1(\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A \left( C - 1 \right) k \right) \right. \]
\[ \left. -8B + 8BC + A^2 \right) L^2 - V - L \right) \right) (L)^{-1} \]
\[ + 12VL \left( (1 - C) k^2 + kA - B \right)^2, \]
\[ V = V, \quad L = L, \quad k = k, \quad \alpha_{-1} = 0, \quad \alpha_{-2} = 0, \]
\[ C_1 = -\frac{1}{2L} \left( V \left( (4C + 4) B + A^2 \right) \right. \]
\[ \left. -\frac{V}{4} - \frac{L}{4} \right) \]
\[ \times \left( V \left( (4C + 4) B + A^2 \right) + \frac{V}{4} + \frac{L}{4} \right), \]

where \( V, L, A, B, \) and \( C \) are arbitrary constants.
\[ u_2(\xi) = \left(V \left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right) \right. \]
\[ \times L^2 - V - L \right) (L)^{-1} \]
\[ -12VL \left((1-C)k^2 + kA - B \right) ((2-2C)k + A) \]
\[ \times \left( k + \left( \frac{G'}{G} \right) \right)^{-1} \]
\[ + 12VL(1-C)k^2 + kA - B \times k + \left( \frac{G'}{G} \right) ^{-2}, \]
\[ u_3(\xi) = \frac{2VL^2 \left(4 - 4C \right) B + A^2 - V - L}{L} \]
\[ + 12VL(C-1)^2 \times \left( \frac{A}{2(C-1)} + \left( \frac{G'}{G} \right) \right) ^2 \]
\[ + \frac{4(1-C)^2}{4(1-C)^2} \]
\[ \times \left( \frac{A}{2(C-1)} + \left( \frac{G'}{G} \right) \right) ^{-2}, \]
where
\[ \xi = Lx + V \frac{t^\alpha}{\Gamma(1+\alpha)}. \] (21)

Substituting the solutions \( G(\xi) \) of (10) into (18) and simplifying, we obtain the following solutions.

When \( \Delta = A^2 - 4BC + 4B > 0 \) and \( A(C-1) \neq 0 \) (or \( B(C-1) \neq 0 \)),

\[ u_1^1(\xi) = \left(V \left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right) \right. \]
\[ \times L^2 - V - L \right) (L)^{-1} \]
\[ + 12VL(C-1)^2 \left(-2kC + 2k + A \right) \]
\[ \times \left\{ k - \frac{1}{2(C-1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\} \]
\[ + 12VL \left((1-C)k^2 + kA - B \right) \times \left\{ k - \frac{1}{2(C-1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) ^2 \right\}, \]

and its graph is shown in Figure 1. Consider

\[ u_1^2(\xi) = \left(V \left(12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2\right) \right. \]
\[ \times L^2 - V - L \right) (L)^{-1} \]
\[ + 12VL(C-1)^2 \left(-2kC + 2k + A \right) \]
\[ \times \left\{ k - \frac{1}{2(C-1)} \left( A - \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\} \]
\[ + 12VL \left((1-C)k^2 + kA - B \right) \times \left\{ k - \frac{1}{2(C-1)} \left( A - \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) ^2 \right\}, \]
Figure 1: (a)–(d) show the singular soliton solution for $u_1^2$ for different values of parameters.

\[ u_1^2 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) \right) \right. \]
\[ + 12VL(C - 1)^2 \]
\[ \left. \times \left\{ k - \frac{1}{4(C - 1)} \right. \right. \]
\[ \times \left( 2A + \sqrt{\Delta} \right) \]
\[ \times \left( \tanh \left( \frac{\sqrt{\Delta} \xi}{4} \right) + \coth \left( \frac{\sqrt{\Delta} \xi}{4} \right) \right) \left( \frac{\sqrt{\Delta} \xi}{4} \right) \right\}^2, \]

\[ u_1^6 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) \right) \right. \]
\[ + 12VL(C - 1)^2 \]
\[ \times \left( 2A + \sqrt{\Delta} \right) \]
\[ \times \left( \tanh \left( \frac{\sqrt{\Delta} \xi}{4} \right) + \coth \left( \frac{\sqrt{\Delta} \xi}{4} \right) \right) \left( \frac{\sqrt{\Delta} \xi}{4} \right) \right\}^2, \]
\( k + \frac{1}{2} (C - 1) \)

\( \times \left\{ -A \pm \sqrt{\frac{\Delta (F^2 + H^2) - F \sqrt{\Delta x}}{F \sinh\left(\sqrt{\Delta x}\right) + B}} \right\}^2 \),

+ 12VL(C - 1)^2

\( u_7^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)

\( \times \left\{ k + \frac{2B \cosh\left(\sqrt{\Delta x}/2\right)}{\sqrt{\Delta} \sinh\left(\sqrt{\Delta x}/2\right)} \right\}^2 \),

\( u_8^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)

\( \times \left\{ k + \frac{2B \sinh\left(\sqrt{\Delta x}/2\right)}{\sqrt{\Delta} \cosh\left(\sqrt{\Delta x}/2\right) - A \cosh\left(\sqrt{\Delta x}/2\right)} \right\}^2 \),

\( u_9^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)

\( \times \left\{ k + \frac{2B \cosh\left(\sqrt{\Delta x}/2\right)}{\sqrt{\Delta} \sinh\left(\sqrt{\Delta x}/2\right) - A \cosh\left(\sqrt{\Delta x}/2\right)} \right\}^2 \),

\( u_{10}^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)

\( \times \left\{ k + \frac{2B \sinh\left(\sqrt{\Delta x}/2\right)}{\sqrt{\Delta} \cosh\left(\sqrt{\Delta x}/2\right) - A \sinh\left(\sqrt{\Delta x}/2\right)} \right\}^2 \),

\( u_{11}^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)

where \( F \) and \( H \) are real constants. Consider the following:

\( u_{10}^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)

\( \times \left\{ k + \frac{2B \cosh\left(\sqrt{\Delta x}/2\right)}{\sqrt{\Delta} \sinh\left(\sqrt{\Delta x}/2\right) - A \cosh\left(\sqrt{\Delta x}/2\right)} \right\}^2 \),

\( u_{11}^0 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1}

+ 12VL(C - 1) (-2kC + 2k + A)
\[
\times \left\{ k + \frac{2B \sinh(\sqrt{\Delta \xi})}{\sqrt{\Delta} \cosh(\sqrt{\Delta \xi}) - A \sinh(\sqrt{\Delta \xi}) \pm \sqrt{\Delta}} \right\} + 12VL(C - 1)^2 \times \left\{ k - \frac{1}{2(C - 1)} \right\} \\
\times \left\{ -A + \sqrt{-\Delta} \left( \tan\left( \sqrt{-\Delta \xi} \right) \pm \sec\left( \sqrt{-\Delta \xi} \right) \right) \right\}^2,
\]

When \( \Delta = A^2 - 4BC + 4B < 0 \) and \( A(C - 1) \neq 0 \) (or \( B(C - 1) \neq 0 \)),

\[
u_{12}(\xi) = \left( V \left( 12(C - 1)^3 k^2 - 12A(C - 1)k - 8B + 8BC + A^2 \right) \times L^2 - V - L \right) (L)^{-1} + 12VL(C - 1)(-2kC + 2k + A) \times \left\{ k + \frac{1}{2(C - 1)} \left( -A + \sqrt{-\Delta} \tan\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\} + 12VL(C - 1)^2 \times \left\{ k + \frac{1}{2(C - 1)} \left( -A + \sqrt{-\Delta} \tan\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\}^2,
\]

\[
u_{13}(\xi) = \left( V \left( 12(C - 1)^3 k^2 - 12A(C - 1)k - 8B + 8BC + A^2 \right) \times L^2 - V - L \right) (L)^{-1} + 12VL(C - 1)(-2kC + 2k + A) \times \left\{ k + \frac{1}{2(C - 1)} \left( A + \sqrt{-\Delta} \cot\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\} + 12VL(C - 1)^2 \times \left\{ k + \frac{1}{2(C - 1)} \left( A + \sqrt{-\Delta} \cot\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\}^2,
\]

\[
u_{14}(\xi) = \left( V \left( 12(C - 1)^3 k^2 - 12A(C - 1)k - 8B + 8BC + A^2 \right) \times L^2 - V - L \right) (L)^{-1} + 12VL(C - 1)(-2kC + 2k + A) \times \left\{ k + \frac{1}{2(C - 1)} \right\} \times \left\{ k + \frac{1}{2(C - 1)} \left( A + \sqrt{-\Delta} \cot\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\}^2,
\]

\[
u_{15}(\xi) = \left( V \left( 12(C - 1)^3 k^2 - 12A(C - 1)k - 8B + 8BC + A^2 \right) \times L^2 - V - L \right) (L)^{-1} + 12VL(C - 1)(-2kC + 2k + A) \times \left\{ k + \frac{1}{2(C - 1)} \right\} \times \left\{ k + \frac{1}{2(C - 1)} \left( A + \sqrt{-\Delta} \cot\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\}^2,
\]

\[
u_{16}(\xi) = \left( V \left( 12(C - 1)^3 k^2 - 12A(C - 1)k - 8B + 8BC + A^2 \right) \times L^2 - V - L \right) (L)^{-1} + 12VL(C - 1)(-2kC + 2k + A) \times \left\{ k + \frac{1}{2(C - 1)} \right\} \times \left\{ k + \frac{1}{2(C - 1)} \right\} \times \left\{ k + \frac{1}{2(C - 1)} \left( A + \sqrt{-\Delta} \cot\left( \frac{\sqrt{-\Delta \xi}}{2} \right) \right) \right\}^2,
\]
\[ u_{17}(\xi) = \left( V \left( \frac{1}{2}(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) \left( L \right)^{-1} + 12VL(C-1)(-2kC + 2k + A) \]
\[
\times \left[ k + \frac{1}{2}(C-1) \times \left\{ -A + \frac{\pm \sqrt{-\Delta(F^2 - H^2)} - F \sqrt{-\Delta} \cos \left( \frac{\sqrt{-\Delta}\xi}{2} \right)}{F \sin \left( \frac{\sqrt{-\Delta}\xi}{2} \right) + B} \right\} \right]^2 \]
\[ \begin{align*}
&\times \left\{ k - \frac{2B \cos \left( \sqrt{-\Delta} \xi \right)}{\sqrt{-\Delta} \sin \left( \sqrt{-\Delta} \xi \right) + A \cos \left( \sqrt{-\Delta} \xi \right) \pm \sqrt{-\Delta}} \right\} \\
&+ 12VL(C - 1)^2 \\
&\times \left\{ k - \frac{2B \cos \left( \sqrt{-\Delta} \xi \right)}{\sqrt{-\Delta} \sin \left( \sqrt{-\Delta} \xi \right) + A \cos \left( \sqrt{-\Delta} \xi \right) \pm \sqrt{-\Delta}} \right\}^2, \\
\end{align*} \]

When \( A = B = 0 \) and \( (C - 1) \neq 0 \), the solution of (12) is

\[ u_1^{25} (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \\
\times L^2 - V - L \right) (L)^{-1} \\
- 12VL \left( (1 - C) k^2 + kA - B \right) ((2 - 2C) k + A) \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-1} \\
+ 12VL \left( 1 - C \right)^2 \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-2}, \]

(29)

where \( c_2 \) is an arbitrary constant.

Substituting the solutions \( G(\xi) \) of (10) into (19) and simplifying, we obtain the following solutions.

When \( \Delta = A^2 - 4BC + 4B > 0 \) and \( A(C - 1) \neq 0 \) (or \( B(C - 1) \neq 0 \)),

\[ u_1 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \\
\times L^2 - V - L \right) (L)^{-1} \\
- 12VL \left( (1 - C) k^2 + kA - B \right) ((2 - 2C) k + A) \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-1} \\
+ 12VL \left( (1 - C) k^2 + kA - B \right)^2 \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-2} \\
+ 12VL \left( (1 - C) k^2 + kA - B \right)^2 \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-2}, \]

(27)

\[ u_2^2 (\xi) = \left( V \left( 12(C - 1)^2 k^2 - 12A (C - 1) k - 8B + 8BC + A^2 \right) \right) \\
\times L^2 - V - L \right) (L)^{-1} \\
- 12VL \left( (1 - C) k^2 + kA - B \right) ((2 - 2C) k + A) \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-1} \\
+ 12VL \left( (1 - C) k^2 + kA - B \right)^2 \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-2} \\
+ 12VL \left( (1 - C) k^2 + kA - B \right)^2 \\
\times \left\{ k - \frac{1}{2(C - 1)} \left( A + \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right\}^{-2}, \]

(28)

where \( c_1 \) is an arbitrary constant.
Figure 2: (a)–(d) show the bell-shape sech^2 solitary traveling wave solution for \( u_{23} \) for different values of parameters.

\[
\begin{align*}
\alpha &= 0.25 \\
\alpha &= 0.50 \\
\alpha &= 0.75 \\
\alpha &= 1
\end{align*}
\]

\[ u_2^3 (\xi) = \left( V \left( 12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2 \right) \times L^2 - V - L \right) (L)^{-1} \\
&- 12VL \left( (1 - C)^k^2 + kA - B \right) (2 - 2C) k + A \\
&\times \left\{ k - \frac{1}{2(C-1)} \right\}^{-1} \\
&\times \left( A + \sqrt{\Delta} \left( \tanh \left( \sqrt{\Delta} \xi \right) \pm i \text{sech} \left( \sqrt{\Delta} \xi \right) \right) \right)^{-1} \\
&+ 12VL \left( (1 - C)^k^2 + kA - B \right)^2 \\
&\times L^2 - V - L \right) (L)^{-1}
\]
and its graph is given in Figure 3. Consider

\[ u_{13}^2(\xi) = \left( V \left( 12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1} \]

\[ -12VL \left( (1-C)k^2 + kA - B \right) \left( (2-C)k + A \right) \times \left\{ k + \frac{1}{2(C-1)} \left( -A + \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-1} \]

\[ + 12VL \left( (1-C)k^2 + kA - B \right)^2 \times \left\{ k + \frac{1}{2(C-1)} \left( -A + \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-2}, \quad (31) \]

\[ u_{14}^2(\xi) = \left( V \left( 12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1} \]

\[ -12VL \left( (1-C)k^2 + kA - B \right) \left( (2-C)k + A \right) \times \left\{ k - \frac{1}{2(C-1)} \left( A + \sqrt{-\Delta} \cot \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-1} \]

\[ + 12VL \left( (1-C)k^2 + kA - B \right)^2 \times \left\{ k - \frac{1}{2(C-1)} \left( A + \sqrt{-\Delta} \cot \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-2}, \]

\[ u_{15}^3(\xi) = \left( V \left( 12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1} \]

\[ \left. \begin{array}{l}
\times \left\{ k + \frac{1}{2(C-1)} \left( -A + \sqrt{-\Delta} \pm \sec \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-1} \\
+ 12VL \left( (1-C)k^2 + kA - B \right)^2 \times \left\{ k + \frac{1}{2(C-1)} \left( -A + \sqrt{-\Delta} \pm \sec \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-2}. \end{array} \right\} \quad (32) \]

When \( A = B = 0 \) and \( (C-1) \neq 0 \), the solution of (12) is

\[ u_{25}^3(\xi) = \left( V \left( 12(C-1)^2k^2 - 12A(C-1)k - 8B + 8BC + A^2 \right) \right) \times L^2 - V - L \right) (L)^{-1} \]

\[ -12VL \left( (1-C)k^2 + kA - B \right) \left( (2-C)k + A \right) \times \left\{ k - \frac{1}{2(C-1)} \left( A + \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-1} \]

\[ + 12VL \left( (1-C)k^2 + kA - B \right)^2 \times \left\{ k - \frac{1}{2(C-1)} \left( A + \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right\}^{-2}, \quad (31) \]

where \( c_2 \) is an arbitrary constant.

We can write down the other families of exact solutions of (12) which are omitted for practicality.

Finally, substituting the solutions \( G(\xi) \) of (10) into (20) and simplifying, we obtain the following solutions.

When \( \Delta = A^2 - 4BC + 4B > 0 \) and \( A(C-1) \neq 0 \) (or \( B(C-1) \neq 0 \)),

\[ u_{13}^2(\xi) = -\frac{2VL^2 \left( (4-4C)B + A^2 \right) - V - L}{L} \]

\[ + 12VL(C-1)^2 \]

\[ \times \left( \frac{1}{2(C-1)} \left( \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right)^2 + \frac{3VL \left( (4-4C)B + A^2 \right)^2}{4(1-C)^2} \]

\[ \times \left( \frac{1}{2(C-1)} \left( \sqrt{\Delta} \tanh \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right)^{-2}, \]

\[ u_{14}^2(\xi) = -\frac{2VL^2 \left( (4-4C)B + A^2 \right) - V - L}{L} \]

\[ + 12VL(C-1)^2 \]

\[ \times \left( \frac{1}{2(C-1)} \left( \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right)^2 \]

\[ + \frac{3VL \left( (4-4C)B + A^2 \right)^2}{4(1-C)^2} \]

\[ \times \left( \frac{1}{2(C-1)} \left( \sqrt{\Delta} \coth \left( \frac{\sqrt{\Delta} \xi}{2} \right) \right) \right)^{-2}, \]

\[ u_{15}^3(\xi) = -\frac{2VL^2 \left( (4-4C)B + A^2 \right) - V - L}{L} \]

\[ + 12VL(C-1)^2 \]
Equations (34) and (35) express the singular periodic traveling wave solutions of $u_{12}^3$ for different values of parameters. The authors have derived these solutions using specific mathematical techniques to simplify the process of solving partial differential equations. The solutions are presented in a parametric form that allows for the visualization of the wave behavior in different parameter regimes.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C - 1) \neq 0$ (or $B(C - 1) \neq 0$), the solution is given by

$$u_{12}^3 (\xi) = \frac{2VL^2 (4 - 4C) B + A^2)}{L} - V - L$$

$$\times \left( \frac{1}{2(C - 1)} \left( \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right)^2$$

$$+ 12VL(C - 1)^2$$

$$\times \left( \frac{1}{2(C - 1)} \left( \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right)^2$$

$$+ 3VL (4 - 4C) B + A^2)^2$$

$$\times \left( \frac{1}{2(C - 1)} \left( \sqrt{-\Delta} \tan \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right)^2$$

(35)

Others families of exact solutions are omitted for the sake of simplicity.
and its graph is given in Figure 4. Consider

\[ u_{13}^{(3)}(\xi) = -\frac{2VL^2 \left( (4-4C)B + A^2 \right) - V - L}{L} \]
\[ + 12VL(C-1)^2 \]
\[ \times \left( \frac{1}{2(C-1)} \left( \sqrt{-\Delta} \cot \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right)^2 \]
\[ + \frac{3VL \left( (4-4C)B + A^2 \right)^2}{4(1-C)^2} \]
\[ \times \left( \frac{1}{2(C-1)} \left( \sqrt{-\Delta} \cot \left( \frac{\sqrt{-\Delta} \xi}{2} \right) \right) \right)^{-2} , \]
\[ u_{14}^{(3)}(\xi) = -\frac{2VL^2 \left( (4-4C)B + A^2 \right) - V - L}{L} \]
\[ + 12VL(C-1)^2 \]
\[ \times \left( \frac{1}{2(C-1)} \right) \]
\[ \times \left\{ \sqrt{-\Delta} \tan \left( \sqrt{-\Delta} \xi \right) \pm \sec \left( \sqrt{-\Delta} \xi \right) \right\}^2 \]
\[ + \frac{3VL \left( (4-4C)B + A^2 \right)^2}{4(1-C)^2} \]
\[ \times \left( \frac{1}{2(C-1)} \right) \]
\[ \times \left\{ \sqrt{-\Delta} \tan \left( \sqrt{-\Delta} \xi \right) \pm \sec \left( \sqrt{-\Delta} \xi \right) \right\}^{-2} . \]

(36)

When \((C-1) \neq 0\) and \(A = B = 0\), the solution of (12) is

\[ u_{25}^{(3)}(\xi) = -\frac{2VL^2 \left( (4-4C)B + A^2 \right) - V - L}{L} \]
\[ + 12VL(C-1)^2 \]
\[ \times \left( \frac{1}{2(C-1)} \right) \]
\[ \times \left\{ \sqrt{-\Delta} \tan \left( \sqrt{-\Delta} \xi \right) \pm \sec \left( \sqrt{-\Delta} \xi \right) \right\}^2 \]
\[ + \frac{3VL \left( (4-4C)B + A^2 \right)^2}{4(1-C)^2} \]
\[ \times \left( \frac{1}{2(C-1)} \right) \]
\[ \times \left\{ \sqrt{-\Delta} \tan \left( \sqrt{-\Delta} \xi \right) \pm \sec \left( \sqrt{-\Delta} \xi \right) \right\}^{-2} . \]
\[
\frac{A}{2(C - 1)} - \frac{1}{(C - 1)\xi + c_2}^2 \\
+ \frac{3VL(4 - 4C)B + A^2}{4(1 - C)^2} \\
+ \left(\frac{A}{2(C - 1)} - \frac{1}{(C - 1)\xi + c_2}^2 \right)^{-2},
\]

where \(c_2\) is an arbitrary constant.

Other exact solutions of (12) are omitted here for convenience.

4. Discussion

If we replace \(A\) by \(-A\) and \(B\) by \(-B\) and put \(C = 0\) in (10), then the novel \((G'/G)\)-expansion method coincides with Akbar et al.'s generalized and improved \((G'/G)\)-expansion method [36]. On the other hand if we put \(k = 0\) in (8) and \(C = 0\) in (10) then the method is identical to the improved \((G'/G)\)-expansion method presented by Zhang et al. [39]. Again if we set \(k = 0, C = 0\) and the negative exponents of \((G'/G)\) are zero in (8), then the method turn out into the basic \((G'/G)\)-expansion method introduced by Wang et al. [31]. At the end, if we put \(C = 0\) in (10) and \(\alpha_i (i = 1, 2, 3, \ldots, N)\) are functions of \(x\) and \(t\) instead of constants then the method is transformed into the generalized \((G'/G)\)-expansion method developed by Zhang et al. [38]. Thus the methods presented in [31, 36, 38, 39] are only special cases of the novel \((G'/G)\)-expansion method. So our method is more general having more free parameters than the existing methods.

5. Conclusions

A novel \((G'/G)\)-expansion method is applied to fractional partial differential equation successfully. As applications, abundant new exact solutions for time fractional BBM-Burger equation have been successfully obtained. The nonlinear fractional complex transformation for \(\xi\) is very important, which ensures that a certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. The obtained solutions are more general with more free parameters. Thus novel \((G'/G)\)-expansion method would be a powerful mathematical tool for solving nonlinear evolution equations. To the best of our knowledge, the solutions obtained in this article are not reported in literature previously.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References


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