Research Article

Generalized Jordan Semitriple Maps on Hilbert Space Effect Algebras

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Let $\mathcal{E}(H)$ be the Hilbert space effect algebra on a Hilbert space H with $\dim H \geq 3$, α, β two positive numbers with $2\alpha + \beta \neq 1$ and $\Phi : \mathcal{E}(H) \rightarrow \mathcal{E}(H)$ a bijective map. We show that if $\Phi(A^\alpha B^\beta A^\alpha) = \Phi(A)^\alpha \Phi(B)^\beta \Phi(A)^\alpha$ holds for all $A, B \in \mathcal{E}(H)$, then there exists a unitary or an antiunitary operator U on H such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{E}(H)$.

1. Introduction and Notations

Let H be a Hilbert space. Denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators acting on H . The operator interval in $\mathcal{B}(H)$: $\mathcal{E}(H) = \{T \in \mathcal{B}(H) \mid 0 \leq T \leq I\}$, where I is the identity, is called Hilbert space effect algebra on H , and the elements in $\mathcal{E}(H)$ are called effects. The concept of effect algebras plays a fundamental role in mathematical description of quantum measurement, and the range of general quantum observables (POV-measurement) consists of effects [1].

The Hilbert space effect algebra $\mathcal{E}(H)$ can be equipped with some algebraic operations. Denote the space of all projections on H by $\mathcal{P}(H)$. Clearly $\mathcal{P}(H) \subseteq \mathcal{E}(H)$. Let $A, B \in \mathcal{E}(H)$. $\mathcal{E}(H)$ is a partial ordered set with the order " $\leq$ ": $A \leq B$ if $B - A \geq 0$. Obviously, the sum $A + B$ may not be in $\mathcal{E}(H)$. In the case that $A + B \in \mathcal{E}(H)$, we can define an operation of conditional addition in $\mathcal{E}(H)$: $A \oplus B = A + B$. Particularly, since $P + Q \in \mathcal{P}(H) \Leftrightarrow PQ = 0$, this gives an orthogonal addition operation in $\mathcal{P}(H)$. Generally speaking, the product AB may not be in $\mathcal{E}(H)$ either. However, the Jordan semitriple product $A \circ B = ABA$ is a well-defined operation in $\mathcal{E}(H)$ since $ABA \in \mathcal{E}(H)$ for all $A, B \in \mathcal{E}(H)$. More generally, for any fixed positive real numbers α, β , the generalized Jordan triple product $\circ_{\alpha, \beta}$ induced by (α, β) is well defined, too, in $\mathcal{E}(H)$ and here $A \circ_{\alpha, \beta} B = A^\alpha B^\beta A^\alpha$.

Recently many mathematicians pay their attention to the problem of characterizing certain maps on Hilbert space effect algebras or other quantum structures [2–8]. Let H be a Hilbert space with $\dim H \geq 3$ and $\Phi : \mathcal{E}(H) \rightarrow \mathcal{E}(H)$ a bijective map. It seems that [5] is the first paper discussing the problem of characterizing the Jordan semitriple maps on Hilbert effect algebras. In [5] Molnár showed that if $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for all $A, B \in \mathcal{E}(H)$, then there exists a unitary or an antiunitary operator U on H such that $\Phi(A) = UAU^*$ for every $A \in \mathcal{E}(H)$; that is, Φ is implemented by a unitary or an antiunitary operator on H . Molnár and Šemrl proved in [7] that if Φ satisfies

$$\Psi(A \circ_{1/2, 1} B) = \Psi(A) \circ_{1/2, 1} \Psi(B) \quad \forall A, B \in \mathcal{E}(H), \quad (1)$$

where Ψ denotes Φ and Φ^{-1} , then there is a positive number p and a unitary or an antiunitary operator U such that $\Phi(A) = UA^p U^*$ for every $A \in \mathcal{E}(H)$ (we will show that $p = 1$ in fact). The generalized Jordan semitriple product $A \circ_{1/2, 1} B = A^{1/2} B A^{1/2}$ is called sequence product. Kim showed in [4] that if H is separable and Φ satisfies

$$\begin{aligned} AB = BA &\implies \Phi(AB) = \Phi(A)\Phi(B) = \Phi(B)\Phi(A), \\ AB = BA, & \end{aligned} \quad (2)$$

$$A + B \in \mathcal{E}(H) \implies \Phi(A + B) = \Phi(A) + \Phi(B)$$

for $A, B \in \mathcal{E}(H)$, then Φ is also implemented by a unitary or an antiunitary operator on H . Our purpose in this paper is to generalize the results of [5, 7] to generalized Jordan semitriple maps, that is, the maps Φ satisfying

$$\Phi(A \circ_{\alpha, \beta} B) = \Phi(A) \circ_{\alpha, \beta} \Phi(B). \quad (3)$$

Let H be a Hilbert space with $\dim H \geq 3$, α, β two positive real numbers with $2\alpha + \beta \neq 1$, and $\Phi : \mathcal{E}(H) \rightarrow \mathcal{E}(H)$ a bijective map satisfying

$$\Phi(A^\alpha B^\beta A^\alpha) = \Phi(A)^\alpha \Phi(B)^\beta \Phi(A)^\alpha \quad (4)$$

for all $A, B \in \mathcal{E}(H)$. We first show that Φ is orthoadditive on $\mathcal{P}(H)$, that is, for any orthogonal projections P, Q , $\Phi(P+Q) = \Phi(P) + \Phi(Q)$. As an application, we prove that all bijective generalized Jordan semitriple maps on Hilbert space effect algebras, that is, the maps satisfying (4), are implemented by a unitary or an antiunitary operator on H .

2. Orthoadditivity of Generalized Jordan Semitriple Maps

Additivity of multiplicative maps is studied by many authors. In this section, we show that every generalized Jordan semitriple map on the effect algebra $\mathcal{E}(H)$ is orthoadditive on $\mathcal{P}(H)$. To do this we need a description which gives a description of the zero product preservers on $\mathcal{P}(H)$. Recall that a map Ψ preserves zero products (in both directions) if $\Psi(A)\Psi(B) = 0$, whenever (if and only if) $AB = 0$.

Lemma 1. *Let H be a real or complex Hilbert space with $\dim H \geq 3$. Assume that $\Phi : \mathcal{P}(H) \rightarrow \mathcal{P}(H)$ is a bijective map preserving zero products of operators in both directions.*

- (1) *If H is real, then there is a unitary operator U on H such that $\Phi(P) = UPU^*$ for all rank one $P \in \mathcal{P}(H)$.*
- (2) *If H is complex, then there is a unitary operator or an antiunitary operator U on H such that $\Phi(P) = UPU^*$ for all rank one $P \in \mathcal{P}(H)$.*

Proof. Denote by $\mathcal{P}_1(H)$ the set of all projections of rank one. For any $P \in \mathcal{P}(H)$, write $\{P\}^\perp = \{T \in \mathcal{P}(H) : TP = 0\}$.

Claim 1. For any $P \in \mathcal{P}(H)$, $\{P\}^\perp = \emptyset$ if and only if $P = I$; $\{P\}^\perp$ is singleton; that is, it contains exactly one element, if and only if $I - P$ is of rank one.

If $P = I \Rightarrow \{P\}^\perp = \emptyset$ is obvious, assume that $P \neq I$ and then $I - P \neq 0$ and $I - P \in \{P\}^\perp$.

If $\{P\}^\perp$ contains only one element, then $P \neq I$ and $\{P\}^\perp = \{I - P\}$. If $I - P$ is not of rank one, for any unit vector $x \in \text{ran}(I - P)$, one has $x \otimes x \in \{P\}^\perp$ but $x \otimes x \neq I - P$, a contradiction. So, $\text{rank}(I - P) = 1$. The converse is obvious.

Claim 2. Φ preserves rank one projections in both directions.

For any rank one projection $x \otimes x$, let $P = I - x \otimes x$; then $\{x \otimes x\} = \{P\}^\perp$. By the property of Φ that it preserves zero products in both directions, one has $\Phi(\{x \otimes x\}) = \Phi(\{P\}^\perp) = \{\Phi(P)\}^\perp$.

Thus $\{\Phi(P)\}^\perp$ is singleton, which forces that $\Phi(x \otimes x)$ is of rank one by Claim 1, and vice versa.

Claim 3. There exists a unitary or antiunitary operator U on H such that

$$\Phi(P) = UPU^* \quad (5)$$

holds for all $P \in \mathcal{P}_1(H)$. U is unitary in real case.

By Claim 3, for every unit vector $x \in H$, there is a unit vector $y_x \in H$ such that $\Phi(x \otimes x) = y_x \otimes y_x$.

Let $\phi : PH \rightarrow PH$ be the map on the projective space PH defined by $\phi([x]) = [y_x]$, where $[x]$ stands for the 1-dimensional linear subspace spanned by x . Then it is clear that ϕ is bijective since Φ preserves rank one projection in both directions. For any linearly independent unit vectors x, u and any scalars γ, δ , let $v = \|\gamma x + \delta u\|^{-1}(\gamma x + \delta u)$. For any unit $y \in H$, there is a unit vector w such that $\Phi(w \otimes w) = y \otimes y$. If $\langle y, y_x \rangle = \langle y, y_u \rangle = 0$, then $(y \otimes y)(y_x \otimes y_x) = (y \otimes y)(y_u \otimes y_u) = 0$. This implies that $\langle w, x \rangle = \langle w, u \rangle = 0$ and consequently $\langle w, v \rangle = 0$. Hence $\langle y, y_v \rangle = 0$. It follows from the arbitrariness of y that $y_v \in [y_x] + [y_u]$; that is, $[v] \in [x] + [u] \Rightarrow \phi(v) \in \phi(x) + \phi(u)$. Now we can use the fundamental theorem of projective geometry to get a semilinear bijection $A : H \rightarrow H$ such that $\phi([x]) = [Ax]$. Therefore, for any unit $x \in H$, $\Phi(x \otimes x) = \|Ax\|^{-2} Ax \otimes Ax$. It is obvious that A preserves orthogonality in both directions; that is, $\langle x, u \rangle = 0 \Leftrightarrow \langle Ax, Au \rangle = 0$. Hence $A = cU$ for some unitary or antiunitary operator U on H . In the case that H is real, U is unitary. Clearly, $|c| = 1$, so $\Phi(x \otimes x) = U(x \otimes x)U^*$ for all rank one projection $x \otimes x$. $\square$

Theorem 2. *Let H be a Hilbert space with $\dim H \geq 3$, α, β positive numbers with $2\alpha + \beta \neq 1$, and $\Phi : \mathcal{E}(H) \rightarrow \mathcal{E}(H)$ a bijective map. If Φ satisfies*

$$\Phi(A^\alpha B^\beta A^\alpha) = \Phi(A)^\alpha \Phi(B)^\beta \Phi(A)^\alpha, \quad (6)$$

for all $A, B \in \mathcal{E}(H)$, then for any projections P, Q , $P \perp Q$ implies that

$$\Phi(P + Q) = \Phi(P) + \Phi(Q). \quad (7)$$

Proof. For $P \in \mathcal{P}(H)$, we have

$$\Phi(P)^\gamma = \Phi(P), \quad (8)$$

where $\gamma > 0$. Note $\Phi(P) \geq 0$. For $\gamma \neq 1$, the spectral solution theorem implies that $\Phi(P)^2 = \Phi(P)$. By considering Φ^{-1} , we see that Φ preserves projections in both directions.

For any $A \in \mathcal{E}(H)$,

$$\begin{aligned} \Phi(I) \Phi(A) &= \Phi(I) \Phi(|A|) = \Phi(I) \Phi\left(I^\alpha (A^{1/\beta})^\beta I^\alpha\right) \\ &= \Phi(I) \Phi(I)^\alpha \Phi(A^{1/\beta})^\beta \Phi(I)^\alpha \\ &= \Phi(I) \Phi(A^{1/\beta})^\beta \Phi(I). \end{aligned} \quad (9)$$

Similarly, we have

$$\Phi(I)\Phi(A) = \Phi(I)\Phi(A^{1/\beta})^\beta\Phi(I). \quad (10)$$

So

$$\Phi(A)\Phi(I) = \Phi(I)\Phi(A) \quad (11)$$

holds for all $A \in \mathcal{E}(H)$. By the surjectivity of Φ and $\Phi(I) = \Phi(I)^2$, we must have $\Phi(I) = I$. Since Φ is bijective, there is $A_0 \in \mathcal{E}(H)$ such that $\Phi(A_0) = 0$. Thus

$$\Phi(0) = \Phi(A_0^\alpha 0 A_0^\alpha) = \Phi(A_0)^\alpha \Phi(0)^\beta \Phi(A_0)^\alpha = 0; \quad (12)$$

that is, $\Phi(0) = 0$.

Now we show that Φ preserves zero product in both directions on $\mathcal{P}(H)$, that it preserves orthogonality in both directions. For any P, Q in $\mathcal{P}(H)$,

$$\begin{aligned} PQ = 0 &\iff P^{\beta/2}Q^\alpha = 0 \iff (P^{\beta/2}Q^\alpha)^* P^{\beta/2}Q^\alpha \\ &= 0 \iff Q^\alpha P^\beta Q^\alpha = 0. \end{aligned} \quad (13)$$

Then

$$\begin{aligned} PQ = 0 &\iff Q^\alpha P^\beta Q^\alpha = 0 \iff \Phi(Q^\alpha P^\beta Q^\alpha) = 0 \\ &\iff \Phi(Q)^\alpha \Phi(P)^\beta \Phi(Q)^\alpha = 0 \iff \Phi(P)\Phi(Q) = 0. \end{aligned} \quad (14)$$

Therefore, by Lemma 1, there exists a unitary or an antiunitary operator U on H such that $\Phi(P) = UPU^*$ holds for all rank one projections P . Without loss of generality we may assume that $\Phi(P) = P$ for every rank one projection P .

For any $P, Q \in \mathcal{P}(H)$, if $P \perp Q$, let $R = \Phi(P + Q) \in \mathcal{P}(H)$, $S = \Phi(P) \in \mathcal{P}(H)$, and $T = \Phi(Q) \in \mathcal{P}(H)$. We will show that $R = S + T$. To do this, it is enough to show that $\text{ran}(R) = \text{ran}(S + T)$. Note that for any x in H with norm 1,

$$\begin{aligned} x \otimes x R x \otimes x &= (x \otimes x)^\alpha R^\beta (x \otimes x)^\alpha \\ &= \Phi((x \otimes x)^\alpha (P + Q)^\beta (x \otimes x)^\alpha) \\ &= \Phi(x \otimes x (P + Q) x \otimes x), \\ x \otimes x \perp S &\iff x \otimes x \perp P, \\ x \otimes x \perp T &\iff x \otimes x \perp Q. \end{aligned} \quad (15)$$

Thus we have

$$\begin{aligned} x \otimes x R x \otimes x = 0 &\iff x \otimes x (P + Q) x \otimes x \\ &= 0 \iff x \otimes x (S + T) x \otimes x = 0. \end{aligned} \quad (16)$$

This forces $\text{ran}(R) = \text{ran}(S + T)$. So $\Phi(P + Q) = \Phi(P) + \Phi(Q)$ whenever $P \perp Q$. $\square$

3. Characterization of Generalized Jordan Semitriple Maps

The following is our main result.

Theorem 3. *Let H be a Hilbert space with $\dim H \geq 3$. Assume that α, β are positive numbers with $2\alpha + \beta \neq 1$ and $\Phi : \mathcal{E}(H) \rightarrow \mathcal{E}(H)$ a bijective map. If*

$$\Phi(A^\alpha B^\beta A^\alpha) = \Phi(A)^\alpha \Phi(B)^\beta \Phi(A)^\alpha, \quad (17)$$

for all $A, B \in \mathcal{E}(H)$, then there exists a unitary or an antiunitary operator U on H such that $\Phi(A) = UAU^$ for all $A \in \mathcal{E}(H)$.*

The following corollary gives a better form of sequence product multiplicative maps, comparing with [7].

Corollary 4. *Let H be a Hilbert space with $\dim H \geq 3$ and $\Phi : \mathcal{E}(H) \rightarrow \mathcal{E}(H)$ a bijective map. If*

$$\Phi(\sqrt{AB}\sqrt{A}) = \sqrt{\Phi(A)}\Phi(B)\sqrt{\Phi(A)} \quad (18)$$

for all $A, B \in \mathcal{E}(H)$, then there exists an unitary or an antiunitary operator U on H such that $\Phi(A) = UAU^$ for every $A \in \mathcal{E}(H)$.*

Proof of Theorem 3. By Theorem 2 we know that Φ preserves projections in both directions, $\Phi(I) = I$, $\Phi(0) = 0$, and Φ is orthoadditive on $\mathcal{P}(H)$.

Furthermore, Φ preserves the order of projections in both directions. To see this, let $P, Q \in \mathcal{P}(H)$. It is clear that

$$\begin{aligned} P \leq Q &\iff PQ = QP = P \iff QPQ = P, \quad PQP = P \\ &\iff Q^\alpha P^\beta Q^\alpha = P^\beta = P, \quad P^\alpha Q^\beta P^\alpha = P^\alpha = P. \end{aligned} \quad (19)$$

So, $P \leq Q$ implies that

$$\begin{aligned} \Phi(P) &= \Phi(Q^\alpha P^\beta Q^\alpha) = \Phi(Q)^\alpha \Phi(P)^\beta \Phi(Q)^\alpha, \\ \Phi(P) &= \Phi(P^\alpha Q^\beta P^\alpha) = \Phi(P)^\alpha \Phi(Q)^\beta \Phi(P)^\alpha. \end{aligned} \quad (20)$$

Consequently, $\Phi(P) \leq \Phi(Q)$. Similarly, using Φ^{-1} , one can check that $\Phi(P) \leq \Phi(Q)$ implies that $P \leq Q$.

So, by Theorem 2 and [9], the restriction of Φ on $\mathcal{P}(H)$ can be extended to a bounded linear map $\Psi : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$. Note that Ψ preserves projections in both directions; it must be a Jordan star-isomorphism. Since every Jordan star-automorphism of $\mathcal{B}(H)$ is either a star-automorphism or star-antiautomorphism, there exists a unitary operator U or an antiunitary operator V such that

$$\Psi(A) = UAU^* \quad \forall A \in \mathcal{B}(H) \quad (21)$$

or

$$\Psi(A) = VA^*V^* \quad \forall A \in \mathcal{B}(H). \quad (22)$$

It follows that there is a unitary or an antiunitary operator U such that $\Phi(P) = UPU^*$ for all $P \in \mathcal{P}(H)$.

Without loss of generality, we can assume that $\Phi(P) = P$ holds for every projection P .

Next we show that $\Phi(\lambda P) = \lambda P$ holds for every $\lambda \in [0, 1]$ and $P \in \mathcal{P}(H)$.

By the definition of Φ that we have for any $\lambda \in [0, 1]$ and any rank one projection P , we have

$$\Phi(\lambda P) = \Phi\left(P^\alpha (\lambda^{1/\beta} P)^\beta P^\alpha\right) = P\Phi(\lambda^{1/\beta} P)^\beta P. \quad (23)$$

So there is a $f_P(\lambda) \in [0, 1]$ such that $\Phi(\lambda P) = f_P(\lambda)P$. Denote f_P by f ; that is, $f(\lambda) = f_P(\lambda)$. $f : [0, 1] \rightarrow [0, 1]$ is a function such that $\Phi(\lambda P) = f(\lambda)P$. It follows that for every $\mu \in [0, 1]$

$$\begin{aligned} \Phi(\lambda\mu P) &= f(\lambda\mu)P, \\ \Phi(\lambda\mu P) &= \Phi\left(\lambda^{(1/2\alpha)\alpha} P \mu^{(1/\beta)\beta} P \lambda^{(1/2\alpha)\alpha} P\right) \\ &= \Phi\left(\lambda^{1/2\alpha} P\right)^\alpha \Phi\left(\mu^{1/\beta} P\right)^\beta \Phi\left(\lambda^{1/2\alpha} P\right)^\alpha \\ &= f\left(\lambda^{1/2\alpha}\right)^{2\alpha} f\left(\mu^{1/\beta}\right)^\beta P. \end{aligned} \quad (24)$$

On the other hand,

$$\begin{aligned} f(\lambda)P &= \Phi(\lambda P) = \Phi\left(\lambda^{(1/2\alpha)\alpha} P P P \lambda^{(1/2\alpha)\alpha}\right) = f\left(\lambda^{1/2\alpha}\right)^{2\alpha} P, \\ f(\mu)P &= \Phi(\mu P) = \Phi\left(I \mu^{(1/\beta)\beta} P I\right) = f\left(\mu^{1/\beta}\right)^\beta P. \end{aligned} \quad (25)$$

Thus $f(\lambda^{1/2\alpha})^{2\alpha} = f(\lambda)$ and $f(\mu^{1/\beta})^\beta = f(\mu)$. Hence, we have $f(\lambda\mu) = f(\lambda)f(\mu)$ holds for all $\lambda, \mu \in [0, 1]$; that is, f is multiplicative.

Obviously, by the surjectivity of Φ , $f([0, 1]) = [0, 1]$. So f is continuous on $[0, 1]$. It is well known that every multiplicative continuous bijection $g : [0, 1] \rightarrow [0, 1]$ is of the form $g(\lambda) = \lambda^\rho$ ($\rho > 0$). Hence there is a $\rho_P > 0$ such that $f(\lambda) = \lambda^{\rho_P}$ for every $\lambda \in [0, 1]$. It follows that

$$\Phi(\lambda P) = \lambda^{\rho_P} P \quad (26)$$

for every rank one projection P .

We will show that ρ_P is independent of P . For any rank one projections P, Q and $\lambda, \mu \in [0, 1]$, we have

$$\begin{aligned} \Phi(\lambda P \mu^2 Q \lambda P) &= \Phi\left(\lambda^{(1/\alpha)\alpha} P \mu^{(2/\beta)\beta} Q \lambda^{(1/\alpha)\alpha} P\right) \\ &= f\left(\lambda^{1/\alpha}\right)^{2\alpha} f\left(\mu^{2/\beta}\right)^\beta P Q P = \lambda^{2\rho_P} \mu^{2\rho_Q} P Q P, \\ \Phi(\mu P \lambda^2 Q \mu P) &= \Phi\left(\mu^{(1/\alpha)\alpha} P \lambda^{(2/\beta)\beta} Q \mu^{(1/\alpha)\alpha} P\right) \\ &= \mu^{2\rho_P} \lambda^{2\rho_Q} P Q P. \end{aligned} \quad (27)$$

If $PQ \neq 0$, we get

$$\lambda^{2\rho_P} \mu^{2\rho_Q} = \mu^{2\rho_P} \lambda^{2\rho_Q}. \quad (28)$$

Letting $\mu = 1$ and comparing two sides of the above equation yield $\lambda^{\rho_P} = \lambda^{\rho_Q}$. So $\rho_P = \rho_Q$. If $PQ = 0$, pick a rank one projection R so that $PRP \neq 0$ and $QRQ \neq 0$. Then $\Phi(\lambda R) = \lambda^{\rho_R} R$. By the fact we just checked, we get $\rho_P = \rho_R = \rho_Q$. It follows that there is a common $\rho \geq 0$ such that

$$\Phi(\lambda P) = \lambda^\rho P \quad (29)$$

holds for all rank one projection P and $\lambda \in [0, 1]$.

We claim that $\rho = 1$. To see this, it is enough to check that $f(\lambda + \mu) = f(\lambda) + f(\mu)$ whenever $\lambda + \mu \leq 1$, where $f(\lambda) = \lambda^\rho$. Pick unit vectors x, y with $\langle x, y \rangle = 0$ and let $P = x \otimes x$, $Q = y \otimes y$, and $A = (\lambda^{1/2} x + \mu^{1/2} y) \otimes (\lambda^{1/2} x + \mu^{1/2} y)$. Then

$$\begin{aligned} APA &= (\lambda^{1/2} x + \mu^{1/2} y) \otimes (\lambda^{1/2} x + \mu^{1/2} y) x \\ &\quad \otimes x (\lambda^{1/2} x + \mu^{1/2} y) \otimes (\lambda^{1/2} x + \mu^{1/2} y) \\ &= \lambda^{1/2} (\lambda^{1/2} x + \mu^{1/2} y) \otimes x (\lambda^{1/2} x + \mu^{1/2} y) \\ &\quad \otimes (\lambda^{1/2} x + \mu^{1/2} y) \\ &= \lambda (\lambda^{1/2} x + \mu^{1/2} y) \otimes (\lambda^{1/2} x + \mu^{1/2} y) = \lambda A. \end{aligned} \quad (30)$$

Similarly, we have

$$\begin{aligned} AQA &= \mu A, \\ A(P + Q)A &= A^2 = (\lambda + \mu)A. \end{aligned} \quad (31)$$

Since Φ is orthoadditive on $\mathcal{P}(H)$, we have

$$\begin{aligned} \Phi(A(P + Q)A) &= \Phi\left(A^{(1/\alpha)\alpha} (P + Q)^{(1/\beta)\beta} A^{(1/\alpha)\alpha}\right) \\ &= \Phi\left(A^{1/\alpha}\right)^\alpha \Phi\left((P + Q)^{1/\beta}\right)^\beta \Phi\left(A^{1/\alpha}\right)^\alpha \\ &= \Phi\left(A^{1/\alpha}\right)^\alpha \Phi(P + Q) \Phi\left(A^{1/\alpha}\right)^\alpha \\ &= \Phi\left(A^{1/\alpha}\right)^\alpha (\Phi(P) + \Phi(Q)) \Phi\left(A^{1/\alpha}\right)^\alpha \\ &= \Phi\left(A^{1/\alpha}\right)^\alpha \Phi\left(P^{1/\beta}\right)^\beta \Phi\left(A^{1/\alpha}\right)^\alpha \\ &\quad + \Phi\left(A^{1/\alpha}\right)^\alpha \Phi\left(Q^{1/\beta}\right)^\beta \Phi\left(A^{1/\alpha}\right)^\alpha \\ &= \Phi(APA) + \Phi(AQA). \end{aligned} \quad (32)$$

Note that $A_0 = A/\|A\|$ is a rank one projection, and for any $\delta \in [0, 1]$, we have

$$\begin{aligned} \Phi(\delta A) &= \Phi(\delta \|A\| A_0) = f(\delta \|A\|) A_0 \\ &= f(\delta) f(\|A\|) \Phi(A_0) = f(\delta) \Phi(A). \end{aligned} \quad (33)$$

Thus $\Phi(\delta A) = f(\delta)\Phi(A)$ and

$$\begin{aligned} f(\lambda + \mu) \Phi(A) &= \Phi((\lambda + \mu)A) = \Phi(A(P + Q)A) \\ &= \Phi(APA) + \Phi(AQA) \\ &= \Phi(\lambda A) + \Phi(\mu A) \\ &= f(\lambda) \Phi(A) + f(\mu) \Phi(A). \end{aligned} \quad (34)$$

This forces that $f(\lambda + \mu) = f(\lambda) + f(\mu)$ whenever $\lambda + \mu \leq 1$, and hence, $\rho = 1$.

Now, let us show that $\Phi(\lambda P) = \lambda P$ holds for all $\lambda \in [0, 1]$ and all $P \in \mathcal{P}(H)$. By the orthoadditivity of Φ , $\Phi(\lambda P) = \lambda P$ holds for every $\lambda \in [0, 1]$ and every finite rank projection

$P \in \mathcal{P}(H)$. Assume P have infinite rank. For any $x \in \text{ran}(P)$ with $\|x\| = 1$, we have

$$\begin{aligned}\Phi(\lambda^\alpha Px \otimes x \lambda^\alpha P) &= \Phi(\lambda P)^\alpha x \otimes \Phi(\lambda P)^\alpha x, \\ \Phi(\lambda^\alpha Px \otimes x \lambda^\alpha P) &= \Phi(\lambda^{2\alpha} x \otimes x) = \lambda^2 x \otimes x.\end{aligned}\quad (35)$$

It follows that $\Phi(\lambda P)^\alpha x$ and x are linearly dependent; that is, there is a scalar $f_x(\lambda)$, dependent on x such that $\Phi(\lambda P)^\alpha x = f_x(\lambda)x$ with $f_x(0) = 0$ and $f_x(1) = 1$. Similarly, for every $y \in \text{ran}(I - P)$, we have $\Phi(\lambda P)^\alpha y = 0$. These entail that there exists a function $g_P(x, \lambda)$ having values in $[0, 1]$ and satisfying $\Phi(\lambda P)x = f_x(\lambda)^{1/\alpha}x = g_P(x, \lambda)Px$ for all $x \in \text{ran } P$ and all $x \in (\text{ran } P)^\perp$. Hence, $g_P(x, 0) = 0$ and $g_P(x, 1) = 1$. As before, for a fixed $x \in \text{ran } P$, we have that $g_P(x, \lambda) = \lambda^\delta$ for some positive δ . Next we claim that $g(x, \lambda) = \lambda$. Taking the rank one projection $Q = x \otimes x$ for $x \in \text{ran } P$, so $Q \leq P$, we have

$$\lambda^\delta Q = \Phi(Q(\lambda P)Q) = \Phi(\lambda^{1/2}QP(\lambda^{1/2}Q)) = \lambda Q, \quad (36)$$

which entails that $\lambda^\delta = \lambda$. In sum, now we have that $\Phi(\lambda P)x = 0$ for $x \in (\text{ran } P)^\perp$ and $\Phi(\lambda P)x = \lambda Px = \lambda x$ for $x \in \text{ran } P$. Let $H = \text{ran } P \oplus (\text{ran } P)^\perp$, since $\Phi(\lambda P) \geq 0$,

$$\Phi(\lambda P) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} = \lambda P. \quad (37)$$

Finally, we show that $\Phi(A) = A$ for every $A \in \mathcal{E}(H)$. In fact, letting $A \in \mathcal{E}(H)$, for every unit vector $x \in H$, we have

$$\begin{aligned}\Phi(A)^\alpha x \otimes \Phi(A)^\alpha x &= \Phi(A^\alpha) x \otimes x \Phi(A^\alpha) \\ &= \Phi(A^\alpha)(x \otimes x)^\beta \Phi(A^\alpha) \\ &= \Phi(A^\alpha(x \otimes x)^\beta A^\alpha) \\ &= A^\alpha x \otimes A^\alpha x.\end{aligned}\quad (38)$$

Thus there exists a scalar λ_x with $|\lambda_x| = 1$ such that $\Phi(A)^\alpha x = \lambda_x A x$. This entails that $\Phi(A)^\alpha = cA$ for some scalar c with $|c| = 1$. Since both A and $\Phi(A)$ are positive, we see that $c = 1$. Hence $\Phi(A)^\alpha = A^\alpha$ and the uniqueness of the positive root of a positive operator implies that $\Phi(A) = A$, as desired. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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