Research Article

A Weak Convergence to Hermite Process by Martingale Differences

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We consider the weak convergence to general Hermite process $Z_{H,k}$ of order k with index H . By applying martingale differences we construct a sequence $\{Z_{H,k}^{(n)}, n = 1, 2, \dots\}$ of multiple Wiener-Itô stochastic integrals such that it converges in distribution to the Hermite process $Z_{H,k}$.

1. Introduction

The self-similar stochastic processes, which are exactly of approximately similar to a part of themselves, have been widely studied since their practical applications to internet traffic, hydrology, turbulence, and finance. We refer to the work of Taqqu [1] for a guide on the appearance of the self-similarity in many applications and to the monographs by Embrechts and Maejima [2] and by Samorodnitsky and Taqqu [3] for complete expositions on self-similar processes. In this paper, we consider the so-called Hermite process with index $H \in (1/2, 1)$, which is a special class of self-similar processes with long range dependency. This process is given as limits of the so called *Non Central Limit Theorem* studied by Dobrushin and Major [4] and Taqqu [5].

Consider a stationary centered Gaussian sequence $(\xi_n)_{n \in \mathbb{N}}$ with $E(\xi_n^2) = 1$ such that

$$r(n) := E(\xi_0 \xi_n) = n^{(2H-2)/k} L(n), \quad (1)$$

where $k \geq 1$ are integers, $H \in (1/2, 1)$, and L is a slowly varying function at infinity. Denote by $H_m(x)$ the Hermite polynomial of degree m defined by

$$H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2}, \quad m = 1, 2, \dots \quad (2)$$

and $H_0(x) = 1$. Let the function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $E(g(\xi_0)) = 0$ and $E(g(\xi_0)^2) < \infty$. Suppose that g has Hermite

rank equal to k , that is, if g admits the following expansion in Hermite polynomials:

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x), \quad c_j = \frac{1}{j!} E(g(\xi_0) H_j(\xi_0)). \quad (3)$$

The Hermite rank of g is defined by

$$k = \min \{j; c_j \neq 0\}. \quad (4)$$

Since

$$E[g(\xi_0)] = 0, \quad (5)$$

we have $k \geq 1$. Then, the *Non Central Limit Theorem* (see [4, 5]) implies that the sequence of stochastic processes of the form

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\xi_j), \quad n = 1, 2, \dots \quad (6)$$

converges, as $n \rightarrow \infty$, in the sense of finite dimensional distributions to the process

$$\begin{aligned} & Z_{H,k}(t) \\ &= \int_{[0,t]^k} Q(t, s_1, \dots, s_k) dB(s_1) \cdots dB(s_k), \quad t \in [0, 1], \end{aligned} \quad (7)$$

where the integral is a multiple Wiener-Itô stochastic integral with respect to standard Brownian motion $B = \{B(t), t \geq 0\}$ and

$$\begin{aligned} Q(t, s_1, \dots, s_k) \\ = c(H, k) \left(\prod_{j=1}^k s_j^{(1/2)-H_0} \right) \\ \times \int_0^t u^{k(H_0-(1/2))} \prod_{j=1}^k (u - s_j)_+^{H_0-(3/2)} du \end{aligned} \quad (8)$$

with $H_0 = 1 - ((1 - H)/k) \in (1 - (1/2k), 1)$, $s_1, \dots, s_k \in [0, t]$ and a positive normal constant $c(H, k)$ such that $E[(Z_{H,k}(1))^2] = 1$.

Definition 1 (see [5]). Let $k \geq 1$ be integer and let $H \in (1/2, 1)$. The process $(Z_{H,k}(t))_{t \geq 0}$ defined by (7) is called the Hermite process of order k , with index H .

Clearly, when $k = 1$ the Hermite process is the fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. When $k = 2$ the Hermite process is called the Rosenblatt process (see [6]). It is important to note that the Hermite process is not Gaussian for $k \geq 2$. The simplest Hermite process is the fractional Brownian motion. The Rosenblatt process is the simplest non-Gaussian Hermite process. More work for the Hermite process and related processes can be found in Bardet and Tudor [7], Chronopoulou et al. [8], Li [9, 10], Li and Lim [11], Heydari et al. [12, 13], Maejima and Tudor [14], Pipiras and Taqqu [15], Torres and Tudor [16], Tudor [17], Tudor and Viens [18], and the references therein. The Hermite processes are neither a semimartingale nor a Markov process, and they admit the same properties as Gaussian fractional Brownian motion. We mention the following:

(i) it exhibits long-range dependence in the sense that

$$\sum_{n \geq \alpha} \rho_n(\alpha) = \infty \quad (9)$$

for any $\alpha > 0$, where

$$\begin{aligned} \rho_n(\alpha) \\ = E[(Z_{H,k}(\alpha + 1) - Z_{H,k}(\alpha))(Z_{H,k}(n + 1) - Z_{H,k}(n))] \end{aligned} \quad (10)$$

(ii) it is H -selfsimilar in the sense that for any $c > 0$,

$$\{Z_{H,k}(ct), t \geq 0\} \stackrel{d}{=} \{c^H Z_{H,k}(t), t \geq 0\}, \quad (11)$$

where $\stackrel{d}{=}$ indicates here equality of the finite-dimensional distributions;

(iii) it has stationary increments, that is, the joint distribution of $(Z_{H,k}(t + h) - Z_{H,k}(t), h \geq 0)$ is independent of every $t \geq 0$;

(iv) its covariance function is

$$E[Z_{H,k}(t)Z_{H,k}(s)] = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}]; \quad (12)$$

(v) it is Hölder continuous of order $\gamma < H$.

These properties of the Hermite process motivate our interesting to study them. Among the applications of the Hermite processes in statistics or econometrics, for example, see Hall et al. [19], Leonenko and Woyczynski [20], and Tudor [17]. Besides these more or less practical applications of the Hermite processes, our motivation is also theoretical; it comes from the recent intensive interest to push further the stochastic calculus with respect to more and more general integrator processes. We believe that such class of processes constitutes an interesting and instructive example where the recent developed techniques of the generalized stochastic calculus can find a significant test bench. On the other hand, the fractional Brownian motion has become an object of intense study in recent some years, due to its interesting properties and its applications in various scientific areas including telecommunications, turbulence, image processing, and finance. Some surveys and complete literature for fractional Brownian motion could be found in Alós et al. [21], Biagini et al. [22], Hu [23], Mishura [24], Nieminen [25], Nualart [26], and the references therein. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on the general Hermite process. The main reasons for this, in our opinion, are the complexity of dependence structures and property of non-Gaussian. Therefore, it seems interesting to study the Hermite process. In this short paper we will prove a weak convergence theorem to the Hermite process based on martingale differences. Our main object is to explain and prove the following theorem.

Theorem 2. Let $(\xi^{(n)})_{n \geq 1}$ be a sequence of square integrable martingale differences satisfying the following conditions:

$$\lim_{n \rightarrow \infty} \frac{(\xi_i^{(n)})^2}{1/n} = 1 \quad a.s. \quad (13)$$

for every $i \geq 1$ and

$$\max_{1 \leq i \leq n} |\xi_i^{(n)}| \leq \frac{C}{\sqrt{n}} \quad a.s. \quad (14)$$

for some $C \geq 1$. Define the two sequences $\{W^n\}$ and $\{Z^n\}$ of processes as follows:

$$W_t^n := \sum_{i=1}^{[nt]} \xi_i^{(n)}, \quad 0 \leq t \leq 1, \quad (15)$$

$$Z_t^n := \int_{\mathbb{R}^k} Q^n(t, s_1, \dots, s_k) dW_{s_1}^n \cdots dW_{s_k}^n, \quad 0 \leq t \leq 1,$$

where $Q^n(t, s_1, \dots, s_k)$ is given by

$$\begin{aligned} Q^n(t, s_1, \dots, s_k) &= c(H, k) n^k \\ &\times \int_{((s_1-1)/n)}^{(s_1/n)} \cdots \int_{((s_k-1)/n)}^{(s_k/n)} Q\left(\frac{[nt]}{n}, r_1, \dots, r_k\right) \\ &\times dr_1 \cdots dr_k, \end{aligned} \quad (16)$$

with $s_1, \dots, s_k \in [1/n, 1]$. Then the process Z^n converges in distribution to the Hermite process $Z_{H,k}$ as n tends to infinity.

Remark 3. One eliminates the diagonal “ $s_1 \neq \dots \neq s_k$ ” because the Hermite process is defined as a multiple Wiener-Itô integral as a consequence it has zero mean.

This paper is organized as follows. In Section 2 we present some preliminaries for the Hermite process. In Section 3 we will give the proof of Theorem 2.

2. Hermite Processes

In this section, we briefly recall some basic properties of the Hermite process. We refer to Maejima and Tudor [14], Peccati and Taqqu [27], and Pipiras and Taqqu [15] for some complete descriptions of the Hermite processes. For simplicity we let C stand for a positive constant depending only on subscripts and its value may be different in different appearances. Throughout this paper we assume that $1/2 < H < 1$, and let $Z_{H,k} = \{Z_{H,k}(t)\}_{t \in \mathbb{R}}$ be a Hermite process of order k with index $H \in (1/2, 1)$. As we pointed out before, we shall use Hermite processes to illustrate the preceding discussion. The Hermite processes $\{Z_{H,k}(t), t \geq 0\}$ are self-similar processes with parameter $H \in (1/2, 1)$, that is, for any $c > 0$,

$$\{Z_{H,k}(ct), t \geq 0\} \stackrel{d}{=} \{c^H Z_{H,k}(t), t \geq 0\}, \quad (17)$$

where $\stackrel{d}{=}$ indicates here equality of the finite-dimensional distributions. They have also stationary increments and are represented by multiple integrals of order k and hence are non-Gaussian when $k \geq 2$. The simplest Hermite process is fractional Brownian motion. The Rosenblatt process is the simplest non-Gaussian Hermite process. The Hermite processes are neither a semimartingale nor a Markov process. They have the four representations, which are as follows (see [15, 27]):

(i) time domain representation: for $t \in \mathbb{R}$

$$Z_{H,k}(t) = \int_{\mathbb{R}^k} f(t, s_1, \dots, s_k) dB(s_1) \cdots dB(s_k), \quad (18)$$

where the integral is multiple Wiener-Itô integral of order k with respect to the standard Brownian motion $B((t))_{t \in \mathbb{R}}$ and the kernel f is defined as

$$f(t, s_1, \dots, s_k) := c(H, k) \int_0^t \left(\prod_{j=1}^k (u - s_j)_+^{H_0 - (3/2)} \right) du, \quad (19)$$

with $H_0 = 1 - ((1-H)/k) \in (1 - (1/2k), 1)$, $s_1, \dots, s_k \in \mathbb{R}$, $x_+ = \max\{x, 0\}$, and

$$\begin{aligned} c(H, k) &= \left(\frac{H(2H-1) [\Gamma((1/2) + ((1-H)/k))]^k}{k! [\Gamma((1/2) - ((1-H)/k)) \Gamma(2(1-H)/k)]^k} \right)^{1/2}; \end{aligned} \quad (20)$$

(ii) positive half-axis representation: for $t \geq 0$

$$Z_{H,k}(t) = \int_{[0, \infty)^k} g(t, s_1, \dots, s_k) dB(s_1) \cdots dB(s_k), \quad (21)$$

where the kernel g is defined as

$$\begin{aligned} g(t, s_1, \dots, s_k) &:= c(H, k) \int_0^t \left(\prod_{j=1}^k s_j^{(1/2) - H_0} (1 - us_j)_+^{H_0 - (3/2)} \right) du \end{aligned} \quad (22)$$

with $s_1, \dots, s_k \in \mathbb{R}^+$;

(iii) spectral domain representation: for $t \in \mathbb{R}$

$$\begin{aligned} Z_{H,k}(t) &= \hat{c}(H, k) \int_{\mathbb{R}^d} \frac{e^{i(\lambda_1 + \dots + \lambda_k)t} - 1}{i(\lambda_1 + \dots + \lambda_k)} \prod_{j=1}^k |\lambda_j|^{(1/2) - H_0} \\ &\times d\widehat{W}(\lambda_1) \cdots d\widehat{W}(\lambda_k), \end{aligned} \quad (23)$$

where $\widehat{W}$ is a suitable complex-valued Brownian motion on $\mathbb{R}$ and

$$\begin{aligned} \hat{c}(H, k) &= \left(kH(2H-1) \right. \\ &\times \left(k! \left[2\Gamma\left(\frac{2(1-H)}{k}\right) \right. \right. \\ &\times \left. \left. \sin\left(\left(\frac{1}{2} - \frac{1-H}{k}\right)\pi\right) \right]^k \right)^{-1} \Big)^{1/2}; \end{aligned} \quad (24)$$

(iv) finite time interval representation: for $t \geq 0$,

$$\begin{aligned} Z_{H,k}(t) &= \int_{[0, t]^k} Q(t, s_1, \dots, s_k) dB(s_1) \cdots dB(s_k), \quad t \in [0, 1], \end{aligned} \quad (25)$$

where the integral is a multiple Wiener-Itô stochastic integral with respect to standard Brownian motion $B = \{B(t), t \geq 0\}$ and

$$Q(t, s_1, \dots, s_k) = c(H, k) \left(\prod_{j=1}^k s_j^{(1/2)-H_0} \right) \times \int_0^t u^{k(H_0-(1/2))} \prod_{j=1}^k (u - s_j)_+^{H_0-(3/2)} du \quad (26)$$

with $s_1, \dots, s_k \in [0, t]$.

3. Proof of Theorem 2

For simplicity we let C stand for a positive constant depending only on subscripts and its value may be different in different appearances. Denote by $D = D(0, 1)$ the Skorohod space of right continuous functions on the interval $[0, 1]$ with left-hand limits and equip D with the metric

$$d(u, v) := \inf \left\{ \varepsilon > 0 : \exists \lambda \in \Lambda \text{ such that } \|\lambda\| \leq \varepsilon \right. \\ \left. \text{and } \sup_t |u(t) - v(\lambda(t))| \leq \varepsilon \right\}, \quad (27)$$

where $\|\lambda\| := \sup_{s \neq t} |\log(\lambda(t) - \lambda(s))/(t - s)|$ and

$$\Lambda := \{\lambda : [0, 1] \rightarrow [0, 1],$$

a strictly increasing and continuous mapping\}.

Under this metric, D is a separable and complete metric space. Let $X, X^n : \Omega \rightarrow D$ be random variables and

$$T_X := \{t \in (0, 1) : P(X_t \neq X_{t-}) = 0\} \cup \{0, 1\}. \quad (29)$$

Recall that X^n converges in distribution to X if, for every bounded and continuous $\phi : D \rightarrow \mathbb{R}$,

$$E(\phi(X^n)) \rightarrow E(\phi(X)), \quad (30)$$

as n tends to infinity. In order to prove Theorem 2 we need the three basic results. The first one is in Billingsley [28] which basically states that weak convergence in D can be proved by using the convergence of finite-dimensional distributions and the tightness of a sequence.

Theorem 4. Suppose that $X, X^n : \Omega \rightarrow D$ are random variables and

$$(X_{t_1}^n, \dots, X_{t_k}^n) \xrightarrow{d} (X_{t_1}, \dots, X_{t_k}) \quad (31)$$

holds whenever $t_1, \dots, t_k \in T_X$, where $\xrightarrow{d}$ means standard convergence in distribution. Assume that $P\{X_1 \neq X_{1-}\} = 0$ and that

$$E \left\{ \left| X_t^n - X_{t_1}^n \right|^C \left| X_{t_2}^n - X_{t_1}^n \right|^C \right\} \leq [F(t_2) - F(t_1)]^{2\alpha}, \quad (32)$$

for $t_1 \leq t \leq t_2$ and $n \geq 1$, where $C > 0$, $\alpha > (1/2)$ and F is a nondecreasing, continuous function on $[0, 1]$. Then

$$X^n \xrightarrow{d} X \quad (33)$$

in the sense of (30).

Secondly, we need the Lindeberg condition (see [29]).

Theorem 5. Let $t \in (0, 1]$, $\sigma_t^2 \geq 0$ and let $(\xi_i^{(n)})_{n \geq 1}$ be a sequence of square integrable martingale differences satisfying the Lindeberg condition:

$$\sum_{i=1}^{[nt]} E \left[(\xi_i^{(n)})^2 I_{\{|\xi_i^{(n)}| > \varepsilon\}} \mid \mathcal{F}_{i-1}^n \right] \xrightarrow{P} 0 \quad (34)$$

for $\varepsilon > 0$. Then $\sum_{i=1}^{[nt]} (\xi_i^{(n)})^2 \xrightarrow{P} \sigma_t^2$ implies that the convergence

$$W_t^n := \sum_{i=1}^{[nt]} \xi_i^{(n)} \xrightarrow{d} \mathcal{N} \sim N(0, \sigma_t^2) \quad (35)$$

holds, as n tends to infinity.

Thirdly, we need the following auxiliary lemma.

Lemma 6. Let Q be as in (8) and Let $(\xi_i^{(n)})_{n \geq 1}$ satisfy conditions (13) and (14). Then, as n tends to infinity

$$\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt_u]}{n}, s_1, \dots, s_k\right) ds_1 \dots ds_k \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt_v]}{n}, s_1, \dots, s_k\right) \\ \times ds_1 \dots ds_k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \\ \rightarrow \int_0^1 \dots \int_0^1 Q(t_u, s_1, \dots, s_k) Q(t_v, s_1, \dots, s_k) ds_1 \dots ds_k \quad (36)$$

holds almost surely, for $t_u, t_v \in [0, 1]$.

Proof. Let us take $t_u = t_v$ and prove the first the case, that is,

$$\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \left(\int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \dots ds_k \right)^2 \\ \times (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \\ \rightarrow \int_0^1 \dots \int_0^1 Q^2(t_u, s_1, \dots, s_k) ds_1 \dots ds_k, \quad (37)$$

holds a.s. as $n \rightarrow \infty$. For every $t_1, \dots, t_k \in [0, 1]$, define

$$\begin{aligned} g_n(t_1, \dots, t_k) &:= n^k \\ &\times \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n I_{(((i_1-1)/n), (i_1/n)]}(t_1) \cdots I_{(((i_k-1)/n), (i_k/n)]}(t_k) \\ &\times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \\ &\times \left(\frac{\xi_{i_1}^{(n)}}{1/\sqrt{n}} \right) \cdots \left(\frac{\xi_{i_k}^{(n)}}{1/\sqrt{n}} \right). \end{aligned} \quad (38)$$

Then one has

$$\begin{aligned} g_n^2(t_1, \dots, t_k) &= \left(n^k \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \right)^2 \\ &\times \left(\frac{\xi_{i_1}^{(n)}}{1/\sqrt{n}} \right)^2 \cdots \left(\frac{\xi_{i_k}^{(n)}}{1/\sqrt{n}} \right)^2, \end{aligned} \quad (39)$$

for $t_j \in ((i_j-1)/n, i_j/n]$, $j = 1, \dots, k$ and

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 g_n^2(s_1, \dots, s_k) ds_1 \cdots ds_k \\ &= \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \\ &\times \left(\int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \right)^2 \\ &\times (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2. \end{aligned} \quad (40)$$

Firstly, we prove that $g_n^2(t_1, \dots, t_k)$ is a.s. uniformly integrable. In fact, together with Hölder's inequality, condition (14), and Fubini theorem, we get

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 g_n^2(s_1, \dots, s_k) ds_1 \cdots ds_k \\ &= \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \\ &\times \left(\int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \right)^2 \\ &\leq C \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q^2(t_u, s_1, \dots, s_k) \\ &\quad \times ds_1 \cdots ds_k \end{aligned}$$

$$\begin{aligned} &\leq C \int_{[0,1]^k} \left(\prod_{j=1}^k s_j^{1-2H_0} \int_0^{t_u} \int_0^{t_u} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \right. \\ &\quad \times \prod_{j=1}^k (x-s_j)_+^{H_0-(3/2)} (y-s_j)_+^{H_0-(3/2)} dx dy \Big) \\ &\quad \times ds_1 \cdots ds_k \\ &= C \int_0^{t_u} \int_0^{t_u} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \\ &\quad \times \left[\int_{[0,1]^k} \prod_{j=1}^k s_j^{1-2H_0} (x-s_j)_+^{H_0-(3/2)} (y-s_j)_+^{H_0-(3/2)} \right. \\ &\quad \times ds_1 \cdots ds_k \Big] dx dy \\ &= C \int_0^{t_u} \int_0^{t_u} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \\ &\quad \times \left[\int_0^1 s^{1-2H_0} (x-s)_+^{H_0-(3/2)} (y-s)_+^{H_0-(3/2)} ds \right]^k dx dy \\ &\leq C \int_0^{t_u} \int_0^{t_u} |x-y|^{(2H_0-2)k} dx dy \\ &= C \int_0^{t_u} \int_0^{t_u} |x-y|^{2H-2} dx dy \\ &\leq C \int_0^1 \int_0^1 |x-y|^{2H-2} dx dy < \infty, \end{aligned} \quad (41)$$

since $1/2 < H < 1$, where we have used the fact

$$\begin{aligned} &\int_0^{x \wedge y} s^{1-2H_0} (x-s)_+^{H_0-(3/2)} (y-s)_+^{H_0-(3/2)} ds \\ &= \frac{\beta(2-2H_0, H_0-(1/2))}{(xy)^{H_0-(1/2)}} |x-y|^{2H_0-2}. \end{aligned} \quad (42)$$

In addition, for $t_j \in ((i_j-1)/n, i_j/n]$, $j = 1, \dots, k$,

$$\begin{aligned} &g_n^2(t_1, \dots, t_k) \\ &= \left(n^k \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \right)^2 \\ &\times \left(\frac{\xi_{i_1}^{(n)}}{1/\sqrt{n}} \right)^2 \cdots \left(\frac{\xi_{i_k}^{(n)}}{1/\sqrt{n}} \right)^2 \\ &\rightarrow Q^2(t_u, s_1, \dots, s_k) \end{aligned} \quad (43)$$

holds a.s., as n tends to infinity, because

$$\begin{aligned} n^k \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \\ \longrightarrow Q(t_u, s_1, \dots, s_k), \\ \left(\frac{\xi_{i_j}^{(n)}}{1/\sqrt{n}} \right)^2 \longrightarrow 1, \quad j = 1, 2, \dots, k \end{aligned} \quad (44)$$

holds a.s., as n tends to infinity. Thus

$$\begin{aligned} \int_0^1 \cdots \int_0^1 g_n^2(s_1, \dots, s_k) ds_1 \cdots ds_k \\ \longrightarrow \int_0^1 \cdots \int_0^1 Q^2(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \end{aligned} \quad (45)$$

holds a.s., as n tends to infinity, and (37) follows.

Next we will prove the original claim. Denote that $t_u^n := [nt_u]/n$ and $t_v^n := [nt_v]/n$. From the previous discussion we know that when $n \rightarrow \infty$,

$$\begin{aligned} \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_v, s_1, \dots, s_k) ds_1 \cdots ds_k \\ \times (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \\ \longrightarrow \int_0^1 \cdots \int_0^1 Q(t_u, s_1, \dots, s_k) \\ \times Q(t_v, s_1, \dots, s_k) ds_1 \cdots ds_k, \quad \text{a.s..} \end{aligned} \quad (46)$$

It turns into show that the difference

$$\begin{aligned} \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \\ \times \left(\int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \right. \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_v, s_1, \dots, s_k) ds_1 \cdots ds_k \\ \left. - \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u^n, s_1, \dots, s_k) ds_1 \cdots ds_k \right. \end{aligned}$$

$$\begin{aligned} \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_v^n, s_1, \dots, s_k) ds_1 \cdots ds_k \Big) \\ \times (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \end{aligned} \quad (47)$$

tends to zero a.s. as n tends to infinity.

Since Q is increasing with respect to t and $t_u \geq t_u^n, t_v \geq t_v^n$, we can get

$$\begin{aligned} 0 \leq \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \\ \times \left(\int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k \right. \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_v, s_1, \dots, s_k) ds_1 \cdots ds_k \\ \left. - \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u^n, s_1, \dots, s_k) ds_1 \cdots ds_k \right. \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_v^n, s_1, \dots, s_k) ds_1 \cdots ds_k \Big) \\ \times (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \\ \leq \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_u, s_1, \dots, s_k) ds_1 \cdots ds_k (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} (Q(t_v, s_1, \dots, s_k) - Q(t_v^n, s_1, \dots, s_k)) \\ \times ds_1 \cdots ds_k \\ + \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} |Q(t_v^n, s_1, \dots, s_k) - Q(t_v, s_1, \dots, s_k)| \\ \times ds_1 \cdots ds_k \\ \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} \end{aligned}$$

$$\begin{aligned}
& \cdots \int_{((i_k-1)/n)}^{(i_k/n)} |Q(t_u, s_1, \dots, s_k) - Q(t_u^n, s_1, \dots, s_k)| \\
& \quad \times ds_1 \cdots ds_k \\
& + \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \int_{((i_1-1)/n)}^{(i_1/n)} \\
& \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q(t_v, s_1, \dots, s_k) ds_1 \cdots ds_k (\xi_{i_1}^n \cdots \xi_{i_k}^n)^2 \\
& \times \int_{((i_1-1)/n)}^{(i_1/n)} \\
& \cdots \int_{((i_k-1)/n)}^{(i_k/n)} |Q(t_u, s_1, \dots, s_k) - Q(t_u^n, s_1, \dots, s_k)| \\
& \quad \times ds_1 \cdots ds_k \\
& \equiv \text{I} + \text{II} + \text{III}.
\end{aligned} \tag{48}$$

Notice that

$$\begin{aligned}
& \int_0^{t_u} \int_{t_v^n}^{t_v} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \\
& \quad \times \left[\int_0^1 s^{1-2H_0} (x-s)_+^{H_0-(3/2)} (y-s)_+^{H_0-(3/2)} ds \right]^k dx dy \\
& \leq \int_0^{t_u} \int_{t_v^n}^{t_v} |x-y|^{(2H_0-2)k} dx dy \\
& = \int_0^{t_u} \int_{t_v^n}^{t_v} |x-y|^{2H-2} dx dy \\
& \leq \int_0^1 \int_0^1 |x-y|^{2H-2} I_{[0, t_u]}(x) I_{[nt_v]/n, t_v]}(y) dx dy,
\end{aligned} \tag{49}$$

$t_u \geq t_u^n$, $t_v \geq t_v^n$. We can get, by using the first part of this proof,

$$\begin{aligned}
I & \longrightarrow \int_0^1 \cdots \int_0^1 Q(t_u, s_1, \dots, s_k) \\
& \quad \times (Q(t_v, s_1, \dots, s_k) - Q(t_v^n, s_1, \dots, s_k)) \\
& \quad \times ds_1 \cdots ds_k \\
& \leq C \int_{[0,1]^k} \left(\prod_{j=1}^k s_j^{1-2H_0} \int_0^{t_u} \int_{t_v^n}^{t_v} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \right. \\
& \quad \times \prod_{j=1}^k (x-s_j)_+^{H_0-(3/2)} (y-s_j)_+^{H_0-(3/2)} dx dy \Big) \\
& \quad \times ds_1 \cdots ds_k
\end{aligned}$$

$$\begin{aligned}
& = C \int_0^{t_u} \int_{t_v^n}^{t_v} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \\
& \quad \times \left[\int_{[0,1]^k} \prod_{j=1}^k s_j^{1-2H_0} (x-s_j)_+^{H_0-(3/2)} \right. \\
& \quad \times (y-s_j)_+^{H_0-(3/2)} ds_1 \cdots ds_k \Big] dx dy \\
& = C \int_0^{t_u} \int_{t_v^n}^{t_v} x^{k(H_0-(1/2))} y^{k(H_0-(1/2))} \\
& \quad \times \left[\int_0^1 s^{1-2H_0} (x-s)_+^{H_0-(3/2)} (y-s)_+^{H_0-(3/2)} ds \right]^k \\
& \quad \times dx dy \longrightarrow 0,
\end{aligned} \tag{50}$$

as n tends to infinity because $0 \leq t_v - [nt_v]/n \leq 1/n$. Similarly the last two summands II and III also tend to zero and we have that when $n \rightarrow \infty$,

$$\begin{aligned}
& \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \int_{((i_1-1)/n)}^{(i_1/n)} \\
& \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt_u]}{n}, s_1, \dots, s_k\right) ds_1 \cdots ds_k \\
& \times \int_{((i_1-1)/n)}^{(i_1/n)} \cdots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt_v]}{n}, s_1, \dots, s_k\right) ds_1 \cdots ds_k \\
& \times (\xi_{i_1}^{(n)} \cdots \xi_{i_k}^{(n)})^2 \\
& \longrightarrow \int_0^1 \cdots \int_0^1 Q(t_u, s_1, \dots, s_k) Q(t_v, s_1, \dots, s_k) ds_1 \cdots ds_k
\end{aligned} \tag{51}$$

holds a.s., and the lemma follows. $\square$

We can prove the main result in this paper.

Proof of Theorem 2. We prove this theorem by using Theorem 4. We first have to prove that the finite-dimensional distributions of Z^n converge to those of $Z_{H,k}$.

For any $a_1, \dots, a_d \in \mathbb{R}$ and $t_1, \dots, t_d \in [0, 1]$, we want to show that the linear combination

$$Y^n := \sum_{u=1}^d a_u Z_{t_u}^n \tag{52}$$

converges in distribution to a normally distributed random variable with expectation zero and variance

$$E\left(\sum_{u=1}^d a_u Z_{t_u}\right)^2. \tag{53}$$

The fact that the expectation is zero is trivial. Let us write Y^n as

$$\begin{aligned} Y^n &= \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \\ &\quad \times \sum_{u=1}^d a_u \int_{((i_1-1)/n)}^{(i_1/n)} \\ &\quad \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) ds_1 \dots ds_k \\ &:= \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n Y_{i_1, \dots, i_k}^n. \end{aligned} \quad (54)$$

The Lindeberg condition is satisfied if for any $\varepsilon > 0$, we have

$$\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n E\left[\left(Y_{i_1, \dots, i_k}^n\right)^2 I_{\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right] \xrightarrow{p} 0, \quad (55)$$

as n tends to infinity. Consider the set

$$\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\} = \left\{\left(Y_{i_1, \dots, i_k}^n\right)^2 > \varepsilon^2\right\}. \quad (56)$$

We get an upper bound to $Y_{i_1, \dots, i_k}^n$ by noticing that $Q(t, s_1, \dots, s_k)$ is increasing with respect to t and decreasing with respect to $s_1, \dots, s_k$. Consider the following:

$$\begin{aligned} &\left(Y_{i_1, \dots, i_k}^n\right)^2 \\ &= \left(n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \sum_{u=1}^d a_u \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) ds_1 \dots ds_k\right)^2 \\ &\leq n^{2k} (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \left(\sum_{u=1}^d |a_u|\right)^2 \\ &\quad \times \left(\int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q(1, s_1, \dots, s_k) ds_1 \dots ds_k\right)^2 \\ &\leq n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \left(\sum_{u=1}^d |a_u|\right)^2 \\ &\quad \times \int_0^{(1/n)} \dots \int_0^{(1/n)} Q^2(1, s_1, \dots, s_k) ds_1 \dots ds_k \\ &=: n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 A \delta^n, \end{aligned} \quad (57)$$

where $A := (\sum_{u=1}^d |a_u|)^2$ and $\delta^n := \int_0^{1/n} \dots \int_0^{1/n} Q^2(1, s_1, \dots, s_k) ds_1 \dots ds_k$. Thus we obtain

$$\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\} \subset \left\{n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 A \delta^n > \varepsilon^2\right\}. \quad (58)$$

Combining this with the Cauchy-Schwartz inequality, we can get that

$$\begin{aligned} &E\left[\left(Y_{i_1, \dots, i_k}^n\right)^2 I_{\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right] \\ &= \left(n^k \sum_{u=1}^d a_u \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) ds_1 \dots ds_k\right)^2 \\ &\quad \times E\left[\left(\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)}\right)^2 I_{\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right] \\ &\leq \left(n^k \sum_{u=1}^d a_u \int_0^{1/n} \dots \int_0^{1/n} Q(1, s_1, \dots, s_k) ds_1 \dots ds_k\right)^2 \\ &\quad \times \left(\frac{C^2}{n}\right)^k E\left[I_{\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right] \\ &\leq C^{2k} A \delta^n E\left[I_{\{n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 A \delta^n > \varepsilon^2\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right], \end{aligned} \quad (59)$$

a.s., which deduces that the Lindeberg condition is satisfied:

$$\begin{aligned} &\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n E\left[\left(Y_{i_1, \dots, i_k}^n\right)^2 I_{\{|Y_{i_1, \dots, i_k}^n| > \varepsilon\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right] \\ &\leq \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n C^{2k} A \delta^n E\left[I_{\{n^k (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 A \delta^n > \varepsilon^2\}} \mid \mathcal{F}_{i_1-1, \dots, i_k-1}^n\right] \\ &\leq C^{2k} A \delta^n \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n I_{\{C^{2k} A \delta^n > \varepsilon^2\}} \longrightarrow 0, \end{aligned} \quad (60)$$

as n tends to infinity, because of $\delta^n \rightarrow 0$ and $I_{\{C^{2k} A \delta^n > \varepsilon^2\}} = 0$ for large n . Moreover, we have

$$\begin{aligned} &\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n \left(Y_{i_1, \dots, i_k}^n\right)^2 \\ &= \sum_{u=1, v=1}^d a_u a_v \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n n^{2k} \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt_u]}{n}, s_1, \dots, s_k\right) ds_1 \dots ds_k \\ &\quad \times \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} Q\left(\frac{[nt_v]}{n}, s_1, \dots, s_k\right) ds_1 \dots ds_k \\ &\quad \times (\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2, \end{aligned} \quad (61)$$

for all $n \geq 1$. By Lemma 6 and the fact that

$$\begin{aligned} EZ_{t_u} Z_{t_v} &= \int_0^{\min(t_u, t_v)} \dots \int_0^{\min(t_u, t_v)} Q(t_u, s_1, \dots, s_k) Q(t_v, s_1, \dots, s_k) \\ &\quad \times ds_1 \dots ds_k, \end{aligned} \quad (62)$$

it follows that the sum $\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^n (Y_{i_1, \dots, i_k}^n)^2$ converges to

$$\begin{aligned} &\sum_{u=1, v=1}^d a_u a_v \int_0^1 \dots \int_0^1 Q(t_u, s_1, \dots, s_k) Q(t_v, s_1, \dots, s_k) ds_1 \dots ds_k \\ &= E \left(\sum_{u=1}^d a_u Z_{t_u} \right)^2. \end{aligned} \quad (63)$$

So by Theorem 5, the finite-dimensional distributions of Z^n converge to those of $Z_{H,k}$.

Finally, we need to prove the tightness of the sequence Z^n in D . Let $s, t \in [0, 1]$, $s < t$. It is easy to prove that

$$E \left[\xi_{i_1}^{(n)} \xi_{j_1}^{(n)} \dots \xi_{i_k}^{(n)} \xi_{j_k}^{(n)} \right] = 0 \quad (64)$$

provided $i_l \neq j_l$ for some $l = 1, 2, \dots, k$. It follows from the Cauchy-Schwartz inequality that

$$\begin{aligned} &E(Z_t^n - Z_s^n)^2 \\ &= E \left(\sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^{[nt]} n^k \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} \xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)} \right. \\ &\quad \times \left[Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) \right. \\ &\quad \left. \left. - Q\left(\frac{[ns]}{n}, s_1, \dots, s_k\right) \right] ds_1 \dots ds_k \right)^2 \\ &= \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^{[nt]} E(\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \\ &\quad \times \left(n^k \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} \left[Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) \right. \right. \\ &\quad \left. \left. - Q\left(\frac{[ns]}{n}, s_1, \dots, s_k\right) \right] ds_1 \dots ds_k \right)^2 \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^{[nt]} n^k E(\xi_{i_1}^{(n)} \dots \xi_{i_k}^{(n)})^2 \\ &\quad \times \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} \left[Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) \right. \\ &\quad \left. - Q\left(\frac{[ns]}{n}, s_1, \dots, s_k\right) \right]^2 ds_1 \dots ds_k \\ &\leq \sum_{i_1, \dots, i_k=1, i_1 \neq \dots \neq i_k}^{[nt]} C^{2k} \\ &\quad \times \int_{((i_1-1)/n)}^{(i_1/n)} \dots \int_{((i_k-1)/n)}^{(i_k/n)} \left[Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) \right. \\ &\quad \left. - Q\left(\frac{[ns]}{n}, s_1, \dots, s_k\right) \right]^2 ds_1 \dots ds_k \\ &\leq C^{2k} \int_0^1 \dots \int_0^1 \left[Q\left(\frac{[nt]}{n}, s_1, \dots, s_k\right) \right. \\ &\quad \left. - Q\left(\frac{[ns]}{n}, s_1, \dots, s_k\right) \right]^2 ds_1 \dots ds_k \\ &\leq C^{2k} \left| \frac{[nt]}{n} - \frac{[ns]}{n} \right|^{2H}. \end{aligned} \quad (65)$$

Notice that for any $s < t < r$,

$$\begin{aligned} E|Z_t^n - Z_s^n| |Z_r^n - Z_t^n| &\leq (E(Z_t^n - Z_s^n)^2)^{1/2} (E(Z_r^n - Z_t^n)^2)^{1/2} \\ &\leq C^{2k} \left| \frac{[nt]}{n} - \frac{[ns]}{n} \right|^H \left| \frac{[nr]}{n} - \frac{[nt]}{n} \right|^H \\ &\leq C^{2k} \left| \frac{[nr]}{n} - \frac{[ns]}{n} \right|^{2H}. \end{aligned} \quad (66)$$

We see that

$$E|Z_t^n - Z_s^n| |Z_r^n - Z_t^n| \leq C^{2k} |r - s|^{2H}, \quad (67)$$

if $r - s \geq 1/n$. On the other hand if $r - s < 1/n$, then either s and t or t and r belong to a same subinterval $[m/n, (m+1)/n]$ for some integer m . Thus the left-hand side of (67) is zero.

Therefore, (67) holds for all $s < t < r$ and $Z^n \xrightarrow{d} Z_{H,k}$ by Theorem 4. This completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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