Research Article

Nonlinear Langevin Equation of Hadamard-Caputo Type Fractional Derivatives with Nonlocal Fractional Integral Conditions

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We study existence and uniqueness of solutions for a problem consisting of nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions. A variety of fixed point theorems are used, such as Banach's fixed point theorem, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative, and Leray-Schauder's degree theory. Enlightening examples illustrating the obtained results are also presented.

1. Introduction

In this paper, we study the existence and uniqueness of solutions for nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions of the form

$$\begin{aligned} D^\alpha (D^\beta + \lambda) x(t) &= f(t, x(t)), \quad 1 < t < e, \\ \sum_{i=1}^m \theta_i I^{\mu_i} x(\eta_i) &= \sum_{j=1}^n \phi_j I^{\nu_j} x(\omega_j), \\ \sum_{k=1}^p \varepsilon_k I^{\sigma_k} x(\psi_k) &= \sum_{l=1}^q \nu_l I^{\tau_l} x(\varphi_l), \end{aligned} \quad (1)$$

where D^ρ denotes the Caputo-type Hadamard fractional derivative of order ρ , $\rho = \{\alpha, \beta\}$ with $0 < \alpha, \beta < 1$, $1 < \alpha + \beta < 2$, λ is a given constant, $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, I^κ is the Hadamard fractional integral of order $\kappa > 0$, $\kappa = \{\mu_i, \nu_j, \sigma_k, \tau_l\}$, the constants $\eta_i, \omega_j, \psi_k,$

$\varphi_l \in (1, e)$, and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$, for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, p$, and $l = 1, 2, \dots, q$.

The significance of investigating nonlocal problem (1) is that the nonlocal Hadamard fractional integral conditions do not contain boundary values of unknown function x , which is a new novel for studying nonboundary problems. In particular, if $m = n = p = q = 1$, then the conditions of (1) are reduced to

$$\begin{aligned} & \frac{\theta_1}{\Gamma(\mu_1)} \int_1^{\eta_1} \left(\log \frac{t}{s}\right)^{\mu_1-1} \frac{x(s)}{s} ds \\ &= \frac{\phi_1}{\Gamma(\nu_1)} \int_1^{\omega_1} \left(\log \frac{t}{s}\right)^{\nu_1-1} \frac{x(s)}{s} ds, \\ & \frac{\varepsilon_1}{\Gamma(\sigma_1)} \int_1^{\psi_1} \left(\log \frac{t}{s}\right)^{\sigma_1-1} \frac{x(s)}{s} ds \\ &= \frac{\nu_1}{\Gamma(\tau_1)} \int_1^{\varphi_1} \left(\log \frac{t}{s}\right)^{\tau_1-1} \frac{x(s)}{s} ds, \end{aligned} \quad (2)$$

where $\log(\cdot) = \log_e(\cdot)$.

The subject of fractional differential equations has recently evolved as an interesting and popular field of research. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. More and more researchers have found that fractional differential equations play important roles in many research areas, such as physics, chemical technology, population dynamics, biotechnology, and economics. For examples and recent development of the topic, see [1–13]. However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo-type fractional derivative. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [14]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. For background material of Hadamard fractional derivative and integral, we refer to the papers [3, 15–19].

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [20]. For some new developments on the fractional Langevin equation, see, for example, [21–29].

Differential equations with nonlocal boundary conditions are often used to take into account some peculiarities of physical, chemical, biological, or other processes, which is impossible by applying classical boundary conditions. Nonlocal problem with integral boundary condition instead of classical Dirichlet or Neumann boundary condition for one-dimensional parabolic equation first was considered by Cannon [30] to describe the process of heat conduction, which was reduced to the integral equation and investigated in the spaces of classical functions. Nonlocal integral boundary conditions can be used when it is impossible to directly determine the values of the sought quantity on the boundary and we know its total amount or integral average on space domain, for example, total energy, average temperature, and total mass of impurities. Some results on problems with nonlocal integral boundary conditions for various evolution equations were investigated; for example, see [31–34].

Several new existence and uniqueness results are proved by using a variety of fixed point theorems (such as Banach contraction principle, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative, and Leray-Schauder's degree theory).

The rest of the paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, we present our existence and uniqueness results. Examples illustrating the obtained results are presented in Section 4.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 1. For at least n -times differentiable function $g : [1, \infty) \rightarrow \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order α is defined as

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n g(s) \frac{ds}{s}, \quad (3)$$

$$n-1 < \alpha < n, \quad n = [\alpha] + 1,$$

where $\delta = t(d/dt)$, $\log(\cdot) = \log_e(\cdot)$, and $[\alpha]$ denotes the integer part of the real number α .

Definition 2. The Hadamard fractional integral of order α is defined as

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{ds}{s}, \quad \alpha > 0, \quad (4)$$

provided that the integral exists.

Lemma 3 (see [35]). Let $u \in AC_\delta^n[a, b]$ or $C_\delta^n[a, b]$ and $\alpha \in \mathbb{C}$, where $X_\delta^n[a, b] = \{g : [a, b] \rightarrow \mathbb{C} : \delta^{n-1}g(t) \in X[a, b]\}$. Then, one has

$$I^\alpha (D^\alpha) u(t) = u(t) - \sum_{k=0}^{n-1} c_k (\log t)^k, \quad (5)$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n-1$, $(n = [\alpha] + 1)$.

For convenience, we set constants as follows:

$$\begin{aligned} \Omega_1 &= \sum_{i=1}^m \theta_i \frac{(\log \eta_i)^{\mu_i}}{\Gamma(\mu_i + 1)} - \sum_{j=1}^n \phi_j \frac{(\log \omega_j)^{\gamma_j}}{\Gamma(\gamma_j + 1)}, \\ \Omega_2 &= \sum_{i=1}^m \theta_i \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta + \mu_i + 1)} - \sum_{j=1}^n \phi_j \frac{(\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta + \gamma_j + 1)}, \\ \Omega_3 &= \sum_{k=1}^p \varepsilon_k \frac{(\log \psi_k)^{\sigma_k}}{\Gamma(\sigma_k + 1)} - \sum_{l=1}^q \nu_l \frac{(\log \varphi_l)^{\tau_l}}{\Gamma(\tau_l + 1)}, \\ \Omega_4 &= \sum_{k=1}^p \varepsilon_k \frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta + \sigma_k + 1)} - \sum_{l=1}^q \nu_l \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta + \tau_l + 1)}, \\ \Omega &= \Omega_1 \Omega_4 - \Omega_2 \Omega_3. \end{aligned} \quad (6)$$

Lemma 4. Let $\Omega \neq 0$, $0 < \alpha, \beta \leq 1$, and $1 < \alpha + \beta \leq 2$; let λ be a given constant, $\mu_i, \gamma_j, \sigma_k, \tau_l > 0$, constants $\eta_i, \omega_j, \psi_k, \varphi_l \in (1, e)$, and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$, for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, p$, $l = 1, 2, \dots, q$, and $h \in C([1, e], \mathbb{R})$. Then, the problem

$$D^\alpha (D^\beta + \lambda) x(t) = h(t), \quad t \in (1, e), \quad (7)$$

$$\begin{aligned} \sum_{i=1}^m \theta_i I^{\mu_i} x(\eta_i) &= \sum_{j=1}^n \phi_j I^{\gamma_j} x(\omega_j), \\ \sum_{k=1}^p \varepsilon_k I^{\sigma_k} x(\psi_k) &= \sum_{l=1}^q \nu_l I^{\tau_l} x(\varphi_l) \end{aligned} \quad (8)$$

has a unique solution given by

$$\begin{aligned}
 x(t) = & \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) \right. \\
 & \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} h(\omega_j) - \lambda I^{\beta+\gamma_j} x(\omega_j) \right] \right. \\
 & \left. \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} h(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i) \right] \right) \right. \\
 & + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \\
 & \times \left(\sum_{l=1}^q \gamma_l \left[I^{\alpha+\beta+\tau_l} h(\phi_l) - \lambda I^{\beta+\tau_l} x(\phi_l) \right] \right. \\
 & \left. \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h(\psi_k) - \lambda I^{\beta+\sigma_k} x(\psi_k) \right] \right) \right] \\
 & + I^{\alpha+\beta} h(t) - \lambda I^\beta x(t). \quad (9)
 \end{aligned}$$

Proof. Using Lemma 3, (7) can be expressed as an equivalent integral equation:

$$(D^\beta + \lambda)x(t) = I^\alpha h(t) + c_0. \quad (10)$$

It follows that

$$x(t) = I^{\alpha+\beta} h(t) - \lambda I^\beta x(t) + c_0 \frac{(\log t)^\beta}{\Gamma(\beta+1)} + c_1, \quad (11)$$

for some $c_0, c_1 \in \mathbb{R}$.

Taking the Hadamard fractional integral of order $\kappa > 0$ for (11), we have

$$\begin{aligned}
 I^\kappa x(t) = & I^{\alpha+\beta+\kappa} h(t) - \lambda I^{\beta+\kappa} x(t) + c_0 \frac{(\log t)^{\beta+\kappa}}{\Gamma(\beta+\kappa+1)} \\
 & + c_1 \frac{(\log t)^\kappa}{\Gamma(\kappa+1)}. \quad (12)
 \end{aligned}$$

Substituting $\kappa = \mu_i, \gamma_j, \sigma_k, \tau_l$ and putting $t = \eta_i, \omega_j, \psi_k, \phi_l$ in (12), respectively, and using conditions (8), we get the system of linear equations:

$$\begin{aligned}
 \Omega_1 c_1 + \Omega_2 c_0 = & \sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} h(\omega_j) - \lambda I^{\beta+\gamma_j} x(\omega_j) \right] \\
 & - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} h(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i) \right],
 \end{aligned}$$

$$\begin{aligned}
 \Omega_3 c_1 + \Omega_4 c_0 = & \sum_{l=1}^p \gamma_l \left[I^{\alpha+\beta+\tau_l} h(\phi_l) - \lambda I^{\beta+\tau_l} x(\phi_l) \right] \\
 & - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h(\psi_k) - \lambda I^{\beta+\sigma_k} x(\psi_k) \right]. \quad (13)
 \end{aligned}$$

Solving the system of linear equations for constants c_0, c_1 , we have

$$\begin{aligned}
 c_0 = & \frac{1}{\Omega} \left[\Omega_1 \left(\sum_{l=1}^p \gamma_l \left[I^{\alpha+\beta+\tau_l} h(\phi_l) - \lambda I^{\beta+\tau_l} x(\phi_l) \right] \right. \right. \\
 & \left. \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h(\psi_k) - \lambda I^{\beta+\sigma_k} x(\psi_k) \right] \right) \right. \\
 & - \Omega_3 \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} h(\omega_j) - \lambda I^{\beta+\gamma_j} x(\omega_j) \right] \right. \\
 & \left. \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} h(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i) \right] \right) \right] \\
 c_1 = & \frac{1}{\Omega} \left[\Omega_4 \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} h(\omega_j) - \lambda I^{\beta+\gamma_j} x(\omega_j) \right] \right. \right. \\
 & \left. \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} h(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i) \right] \right) \right. \\
 & - \Omega_2 \left(\sum_{l=1}^p \gamma_l \left[I^{\alpha+\beta+\tau_l} h(\phi_l) - \lambda I^{\beta+\tau_l} x(\phi_l) \right] \right. \\
 & \left. \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h(\psi_k) - \lambda I^{\beta+\sigma_k} x(\psi_k) \right] \right) \right]. \quad (14)
 \end{aligned}$$

Substituting constants c_0 and c_1 into (11), we obtain (9) as required. $\square$

3. Main Results

Let $\mathcal{C} = C([1, e], \mathbb{R})$ denote the Banach space of all continuous functions from $[1, e]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t \in [1, e]} |x(t)|$. Throughout this paper, for convenience, the expressions $I^z f(s, x(s))(y)$ and $I^z x(s)(y)$ mean

$$I^z f(s, x(s))(y) = \frac{1}{\Gamma(z)} \int_1^y \left(\log \frac{y}{s} \right)^{z-1} \frac{f(s, x(s))}{s} ds, \quad (15)$$

$$I^z x(s)(y) = \frac{1}{\Gamma(z)} \int_1^y \left(\log \frac{y}{s} \right)^{z-1} \frac{x(s)}{s} ds, \quad (16)$$

respectively, where $z > 0$ and $y \in \{t, \eta_i, \omega_j, \psi_k, \phi_l\}$.

In view of Lemma 4, we consider the operator $\mathcal{Q} : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\begin{aligned}
 (\mathcal{Q}x)(t) &= \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) \right. \\
 &\quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s))(\omega_j) \right. \right. \\
 &\quad \left. \left. - \lambda I^{\beta+\gamma_j} x(s)(\omega_j) \right] \right. \\
 &\quad \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s))(\eta_i) \right. \right. \\
 &\quad \left. \left. - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \right) \\
 &\quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \\
 &\quad \times \left(\sum_{l=1}^q \nu_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s))(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \right. \\
 &\quad \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s))(\psi_k) \right. \right. \\
 &\quad \left. \left. - \lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \right) \Bigg] \\
 &\quad + I^{\alpha+\beta} f(s, x(s))(t) - \lambda I^\beta x(s)(t).
 \end{aligned} \tag{17}$$

It should be noticed that the nonlocal problem (1) has solutions if and only if the operator $\mathcal{Q}$ has fixed points.

In the following subsections, we prove existence, as well as existence and uniqueness results, for the nonlocal problem (1) by using a variety of fixed point theorems.

We set

$$\begin{aligned}
 \Lambda(u) &= \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \right. \\
 &\quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{u+\beta+\gamma_j}}{\Gamma(u+\beta+\gamma_j+1)} \right. \\
 &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{u+\beta+\mu_i}}{\Gamma(u+\beta+\mu_i+1)} \right) \\
 &\quad \left. + \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 &\times \left(\sum_{l=1}^q |\nu_l| \frac{(\log \varphi_l)^{u+\beta+\tau_l}}{\Gamma(u+\beta+\tau_l+1)} \right. \\
 &\quad \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{u+\beta+\sigma_k}}{\Gamma(u+\beta+\sigma_k+1)} \right) \Bigg] \\
 &\quad + \frac{1}{\Gamma(u+\beta+1)},
 \end{aligned} \tag{18}$$

where $u \in \{0, \alpha\}$.

3.1. Existence and Uniqueness Result via Banach's Fixed Point Theorem

Theorem 5. Let $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume the following:

(H₁) there exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$, for each $t \in [1, e]$ and $x, y \in \mathbb{R}$.

If

$$L\Lambda(\alpha) + |\lambda| \Lambda(0) < 1, \tag{19}$$

where constants $\Lambda(0)$ and $\Lambda(\alpha)$ are defined by (18) with $u = 0$ and α , respectively, then the nonlocal problem (1) has a unique solution on $[1, e]$.

Proof. By transforming the nonlocal problem (1) into a fixed point problem, $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (17), we have that the fixed points of the operator $\mathcal{Q}$ are solutions of problem (1). Applying the Banach contraction mapping principle, we will show that $\mathcal{Q}$ has a unique fixed point.

Setting $\sup_{t \in [1, e]} |f(t, 0)| = M < \infty$ and choosing

$$R \geq \frac{M\Lambda(\alpha)}{1 - L\Lambda(\alpha) - |\lambda| \Lambda(0)}, \tag{20}$$

we show that $\mathcal{Q}B_R \subset B_R$, where $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$. For any $x \in B_R$, we have

$$\begin{aligned}
 |(\mathcal{Q}x)(t)| &\leq \left| \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) \right. \right. \\
 &\quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s))(\omega_j) - \lambda I^{\beta+\gamma_j} x(s)(\omega_j) \right] \right. \\
 &\quad \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s))(\eta_i) \right. \right. \\
 &\quad \left. \left. - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \right) \\
 &\quad \left. + \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \right. \\
 &\quad \left. \times \left(\sum_{l=1}^q \nu_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s))(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \right. \right. \\
 &\quad \left. \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s))(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \right) \right] \Bigg|
 \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \\
& \times \left(\sum_{l=1}^q \gamma_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s)) (\varphi_l) - \lambda I^{\beta+\tau_l} x(s) (\varphi_l) \right] \right. \\
& \quad \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s)) (\psi_k) \right. \right. \\
& \quad \quad \left. \left. - \lambda I^{\beta+\sigma_k} x(s) (\psi_k) \right] \right) \\
& + I^{\alpha+\beta} f(s, x(s)) (t) - \lambda I^\beta x(s) (t) \Big| \\
& \leq \frac{1}{|\Omega|} \left[\left| \Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right| \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[I^{\alpha+\beta+\gamma_j} (|f(s, x(s)) - f(s, 0)| \right. \right. \\
& \quad \quad \left. \left. + |f(s, 0)| \right) (\omega_j) \right. \\
& \quad \quad \left. + |\lambda| I^{\beta+\gamma_j} |x(s)| (\omega_j) \right] \\
& \quad + \sum_{i=1}^m |\theta_i| \left[I^{\alpha+\beta+\mu_i} (|f(s, x(s)) - f(s, 0)| \right. \\
& \quad \quad \left. + |f(s, 0)| \right) (\eta_i) \\
& \quad \quad \left. + |\lambda| I^{\beta+\mu_i} |x(s)| (\eta_i) \right] \Big) \\
& + \left| \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right| \\
& \times \left(\sum_{l=1}^q |\gamma_l| \left[I^{\alpha+\beta+\tau_l} (|f(s, x(s)) - f(s, 0)| \right. \right. \\
& \quad \quad \left. \left. + |f(s, 0)| \right) (\varphi_l) \right. \\
& \quad \quad \left. + |\lambda| I^{\beta+\tau_l} |x(s)| (\varphi_l) \right] \\
& \quad + \sum_{k=1}^p |\varepsilon_k| \left[I^{\alpha+\beta+\sigma_k} (|f(s, x(s)) - f(s, 0)| \right. \\
& \quad \quad \left. + |f(s, 0)| \right) (\psi_k) \\
& \quad \quad \left. + |\lambda| I^{\beta+\sigma_k} |x(s)| (\psi_k) \right] \Big) \\
& + I^{\alpha+\beta} (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|) (t) \\
& + |\lambda| I^\beta |x(s)| (t)
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_3| \right) \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[(L \|x\| + M) (I^{\alpha+\beta+\gamma_j} 1) (\omega_j) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| \|x\| (I^{\beta+\gamma_j} 1) (\omega_j) \right] \right. \\
& \quad \quad \left. + \sum_{i=1}^m |\theta_i| \left[(L \|x\| + M) (I^{\alpha+\beta+\mu_i} 1) (\eta_i) \right. \right. \\
& \quad \quad \quad \left. \left. + |\lambda| \|x\| (I^{\beta+\mu_i} 1) (\eta_i) \right] \right) \\
& \quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\
& \quad \times \left(\sum_{l=1}^q |\gamma_l| \left[(L \|x\| + M) (I^{\alpha+\beta+\tau_l} 1) (\varphi_l) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| \|x\| (I^{\beta+\tau_l} 1) (\varphi_l) \right] \right. \\
& \quad \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[(L \|x\| + M) (I^{\alpha+\beta+\sigma_k} 1) (\psi_k) \right. \right. \\
& \quad \quad \quad \left. \left. + |\lambda| \|x\| (I^{\beta+\sigma_k} 1) (\psi_k) \right] \right) \\
& \quad + (L \|x\| + M) (I^{\alpha+\beta} 1) (t) + |\lambda| \|x(s)\| (I^\beta 1) (t) \\
& \leq \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_3| \right) \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[(LR + M) \frac{(\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \right. \right. \\
& \quad \quad \left. \left. + |\lambda| R \frac{(\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right] \right. \\
& \quad \quad \left. + \sum_{i=1}^m |\theta_i| \left[(LR + M) \frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \right. \right. \\
& \quad \quad \quad \left. \left. + |\lambda| R \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right] \right) \\
& \quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\
& \quad \times \left(\sum_{l=1}^q |\gamma_l| \left[(LR + M) \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} \right. \right. \\
& \quad \quad \left. \left. + |\lambda| R \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^p |\varepsilon_k| \left[(LR + M) \frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)} \right. \\
& \quad \left. + |\lambda| R \frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)} \right] \\
& + (LR + M) \frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda| R \frac{(\log t)^\beta}{\Gamma(\beta+1)} \\
& \leq (LR + M) \Lambda(\alpha) + |\lambda| R \Lambda(0) \\
& = [L\Lambda(\alpha) + |\lambda| \Lambda(0)] R + M\Lambda(\alpha) \leq R.
\end{aligned} \tag{21}$$

This implies that $\|\mathcal{Q}x\| \leq R$ for $x \in B_R$. Therefore, $\mathcal{Q}$ maps bounded subsets of B_R into bounded subsets of B_R .

Next, we let $x, y \in \mathcal{C}$. Then, for $t \in [1, e]$, we have

$$\begin{aligned}
& |(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)| \\
& \leq \frac{1}{|\Omega|} \left[\left| \Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right| \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[I^{\alpha+\beta+\gamma_j} (|f(s, x(s)) - f(s, y(s))|) (\omega_j) \right. \right. \\
& \quad \left. \left. + |\lambda| I^{\beta+\gamma_j} (|x(s) - y(s)|) (\omega_j) \right] \right. \\
& \quad \left. + \sum_{i=1}^m |\theta_i| \left[I^{\alpha+\beta+\mu_i} \right. \right. \\
& \quad \times (|f(s, x(s)) - f(s, y(s))|) (\eta_i) \\
& \quad \left. \left. + |\lambda| I^{\beta+\mu_i} (|x(s) - y(s)|) (\eta_i) \right] \right) \\
& \quad + \left| \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right| \\
& \quad \times \left(\sum_{l=1}^q |\nu_l| \left[I^{\alpha+\beta+\tau_l} (|f(s, x(s)) - f(s, y(s))|) (\varphi_l) \right. \right. \\
& \quad \left. \left. + |\lambda| I^{\beta+\tau_l} (|x(s) - y(s)|) (\varphi_l) \right] \right. \\
& \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[I^{\alpha+\beta+\sigma_k} \right. \right. \\
& \quad \times (|f(s, x(s)) - f(s, y(s))|) \\
& \quad \times (\psi_k) + |\lambda| I^{\beta+\sigma_k} \\
& \quad \times (|x(s) - y(s)|) (\psi_k) \left. \right] \left. \right)
\end{aligned}$$

$$\begin{aligned}
& + I^{\alpha+\beta} (|f(s, x(s)) - f(s, y(s))|) (t) \\
& + |\lambda| I^\beta (|x(s) - y(s)|) (t) \\
& \leq \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_3| \right) \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[(L\|x - y\|) \frac{(\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \right. \right. \\
& \quad \left. \left. + |\lambda| \|x - y\| \frac{(\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right] \right. \\
& \quad \left. + \sum_{i=1}^m |\theta_i| \left[(L\|x - y\|) \frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \right. \right. \\
& \quad \left. \left. + |\lambda| \|x - y\| \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right] \right) \\
& \quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\
& \quad \times \left(\sum_{l=1}^q |\nu_l| \left[(L\|x - y\|) \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} \right. \right. \\
& \quad \left. \left. + |\lambda| \|x - y\| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right] \right. \\
& \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[(L\|x - y\|) \frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)} \right. \right. \\
& \quad \left. \left. + |\lambda| \|x - y\| \frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)} \right] \right) \\
& \quad + (L\|x - y\|) \frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda| \|x - y\| \frac{(\log t)^\beta}{\Gamma(\beta+1)} \\
& \leq L\Lambda(\alpha) \|x - y\| + |\lambda| \|x - y\| \Lambda(0) \\
& = [L\Lambda(\alpha) + |\lambda| \Lambda(0)] \|x - y\|,
\end{aligned} \tag{22}$$

which implies that $\|\mathcal{Q}x - \mathcal{Q}y\| \leq [L\Lambda(\alpha) + |\lambda|\Lambda(0)]\|x - y\|$. As $[L\Lambda(\alpha) + |\lambda|\Lambda(0)] < 1$, $\mathcal{Q}$ is a contraction. Therefore, we deduce, by the Banach contraction mapping principle, that $\mathcal{Q}$ has a fixed point which is the unique solution of nonlocal problem (1). The proof is completed. $\square$

3.2. Existence Result via Krasnoselskii's Fixed Point Theorem

Lemma 6 (Krasnoselskii's fixed point theorem) (see [36]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X . Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous;*

(c) B is a contraction mapping. Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 7. Let $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume the following:

$$(H_2) \quad |f(t, x)| \leq \rho(t), \quad \forall (t, x) \in [1, e] \times \mathbb{R} \text{ and } \rho \in C([1, e], \mathbb{R}^+).$$

If

$$|\lambda| \Lambda(0) < 1, \quad (23)$$

where $\Lambda(0)$ is defined by (18), with $u = 0$, then the nonlocal problem (1) has at least one solution on $[1, e]$.

Proof. Setting $\sup_{t \in [1, e]} |\rho(t)| = \|\rho\|$ and choosing

$$R \geq \frac{\|\rho\| \Lambda(\alpha)}{1 - \|\lambda\| \Lambda(0)}, \quad (24)$$

we consider $B_R = \{x \in \mathcal{C}([1, e], \mathbb{R}) : \|x\| \leq R\}$. We define the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_R by

$$\begin{aligned} (\mathcal{Q}_1 x)(t) &= \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) \right. \\ &\quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s))(\omega_j) \right] \right. \\ &\quad \left. \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s))(\eta_i) \right] \right) \right. \\ &\quad \left. + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \right. \\ &\quad \times \left(\sum_{l=1}^q \nu_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s))(\varphi_l) \right] \right. \\ &\quad \left. \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s))(\psi_k) \right] \right) \right] \\ &\quad + I^{\alpha+\beta} f(s, x(s))(t), \\ (\mathcal{Q}_2 x)(t) &= \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) \right. \\ &\quad \times \left(\sum_{j=1}^n -\phi_j \left[\lambda I^{\beta+\gamma_j} x(s)(\omega_j) \right] \right. \\ &\quad \left. \left. + \sum_{i=1}^m \theta_i \left[\lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \right) \right. \\ &\quad \left. + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \right. \\ &\quad \times \left(\sum_{l=1}^q -\nu_l \left[\lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \right. \\ &\quad \left. \left. + \sum_{k=1}^p \varepsilon_k \left[\lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \right) \right] \end{aligned}$$

$$\begin{aligned} &\times \left(\sum_{l=1}^q -\nu_l \left[\lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \right. \\ &\quad \left. + \sum_{k=1}^p \varepsilon_k \left[\lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \right) \Bigg] \\ &- \lambda I^\beta x(s)(t), \quad t \in [1, e]. \end{aligned} \quad (25)$$

For any $x, y \in B_R$, we have

$$\begin{aligned} &|\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)| \\ &\leq \frac{1}{|\Omega|} \left[\left| \Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right| \right. \\ &\quad \times \left(\sum_{j=1}^n |\phi_j| \left[\|\rho\| (I^{\alpha+\beta+\gamma_j} 1)(\omega_j) \right] \right. \\ &\quad \left. + |\lambda| R (I^{\beta+\gamma_j} 1)(\omega_j) \right) \\ &\quad + \sum_{i=1}^m |\theta_i| \left[\|\rho\| (I^{\alpha+\beta+\mu_i} 1)(\eta_i) \right. \\ &\quad \left. + |\lambda| R (I^{\beta+\mu_i} 1)(\eta_i) \right] \Bigg] \\ &+ \left| \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right| \\ &\times \left(\sum_{l=1}^q |\nu_l| \left[\|\rho\| (I^{\alpha+\beta+\tau_l} 1)(\varphi_l) \right] \right. \\ &\quad \left. + |\lambda| R (I^{\beta+\tau_l} 1)(\varphi_l) \right) \\ &+ \sum_{k=1}^p |\varepsilon_k| \left[\|\rho\| (I^{\alpha+\beta+\sigma_k} 1)(\psi_k) \right. \\ &\quad \left. + |\lambda| R (I^{\beta+\sigma_k} 1)(\psi_k) \right] \Bigg] \\ &+ \|\rho\| (I^{\alpha+\beta} 1)(t) + |\lambda| R (I^\beta 1)(t) \end{aligned} \quad (26)$$

$$\leq \|\rho\| \Lambda(\alpha) + R \|\lambda\| \Lambda(0) \leq R,$$

which implies that $\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\| \leq R$. It follows that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_R$.

For $x, y \in C$ and for each $t \in [1, e]$, we have

$$\|\mathcal{Q}_2 x - \mathcal{Q}_2 y\| \leq |\lambda| \Lambda(0) \|x - y\|. \quad (27)$$

Hence, by (23), $\mathcal{Q}_2$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_R as

$$\|\mathcal{Q}_1 x\| \leq \|\rho\| \Lambda(\alpha). \quad (28)$$

Now, we prove the compactness of the operator $\mathcal{Q}_1$.

We define $\sup_{(t,x) \in [(1,e) \times B_R]} |f(t, x)| = \bar{f} < \infty$. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $x \in B_R$. Then, we have

$$\begin{aligned}
& |(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)| \\
& \leq \left| \frac{1}{\Omega} \left[\Omega_3 \left(\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta + 1)} \right) \right. \right. \\
& \quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s)) (\omega_j) \right] \right. \\
& \quad \left. \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s)) (\eta_i) \right] \right) \right. \\
& \quad + \Omega_1 \left(\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta + 1)} \right) \\
& \quad \times \left(\sum_{l=1}^q \gamma_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s)) (\varphi_l) \right] \right. \\
& \quad \left. \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s)) (\psi_k) \right] \right) \right] \right| \\
& + \left| I^{\alpha+\beta} f(s, x(s))(t_2) - I^{\alpha+\beta} f(s, x(s))(t_1) \right| \\
& \leq \frac{\bar{f}}{|\Omega|} \left[|\Omega_3| \left| \frac{(\log t_1)^\beta - (\log t_2)^\beta}{\Gamma(\beta + 1)} \right| \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha + \beta + \gamma_j + 1)} \right. \\
& \quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha + \beta + \mu_i + 1)} \right) \\
& \quad + |\Omega_1| \left| \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta + 1)} \right| \\
& \quad \times \left(\sum_{l=1}^q |\gamma_l| \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha + \beta + \tau_l + 1)} \right. \\
& \quad \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha + \beta + \sigma_k + 1)} \right) \Bigg] \\
& + \frac{\bar{f}}{\Gamma(\alpha + \beta + 1)} \left| (\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right|,
\end{aligned} \tag{29}$$

which is independent of x and tends to zero as $t_2 - t_1 \rightarrow 0$. Thus, $\mathcal{Q}_1$ is equicontinuous. So $\mathcal{Q}_1$ is relatively compact on B_R . Hence, by the Arzelà-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_R . Thus, all the assumptions of Lemma 6 are satisfied. So the conclusion of Lemma 6 implies that the nonlocal problem (1) has at least one solution on $[1, e]$. $\square$

3.3. Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 8 (nonlinear alternative for single valued maps) (see [37]). Let E be a Banach space; let C be a closed, convex subset of E ; let U be an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \rightarrow C$ is a continuous, compact (i.e., $F(\bar{U})$ is a relatively compact subset of C) map. Then, either

- (i) F has a fixed point in $\bar{U}$, or
- (ii) there is a $u \in \partial U$ (the boundary of U in C) and $\xi \in (0, 1)$ with $u = \xi F(u)$.

Theorem 9. Let $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume the following:

- (H₃) there exist a continuous nondecreasing function $Y : [0, \infty) \rightarrow (0, \infty)$ and a function $\rho \in C([1, e], \mathbb{R}^+)$ such that

$$|f(t, u)| \leq \rho(t) Y(\|u\|) \quad \text{for each } (t, u) \in [1, e] \times \mathbb{R}; \tag{30}$$

- (H₄) there exists a constant $M > 0$ such that

$$\frac{M}{\|\rho\| Y(M) \Lambda(\alpha) + |\lambda| M \Lambda(0)} > 1, \tag{31}$$

where $\Lambda(0)$ and $\Lambda(\alpha)$ are defined by (18) with $u = 0$ and α , respectively.

Then, the nonlocal problem (1) has at least one solution on $[1, e]$.

Proof. Let the operator $\mathcal{Q}$ be defined by (17). Firstly, we will show that $\mathcal{Q}$ maps bounded sets (balls) into bounded sets in $C([1, e], \mathbb{R})$. For a number $R > 0$, let $B_R = \{x \in C([1, e], \mathbb{R}) : \|x\| \leq R\}$ be a bounded ball in $C([1, e], \mathbb{R})$. Then, for $t \in [1, e]$, we have

$$\begin{aligned}
& |(\mathcal{Q}x)(t)| \\
& \leq \left| \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta + 1)} \Omega_3 \right) \right. \right. \\
& \quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s)) (\omega_j) - \lambda I^{\beta+\gamma_j} x(s) (\omega_j) \right] \right. \\
& \quad \left. \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s)) (\eta_i) \right. \right. \right. \\
& \quad \left. \left. \left. - \lambda I^{\beta+\mu_i} x(s) (\eta_i) \right] \right) \right] \right|
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \\
& \times \left(\sum_{l=1}^q \gamma_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s)) (\varphi_l) - \lambda I^{\beta+\tau_l} x(s) (\varphi_l) \right] \right. \\
& \quad \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s)) (\psi_k) \right. \right. \\
& \quad \quad \left. \left. - \lambda I^{\beta+\sigma_k} x(s) (\psi_k) \right] \right) \\
& + I^{\alpha+\beta} f(s, x(s)) (t) - \lambda I^\beta x(s) (t) \Bigg| \\
& \leq \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_3| \right) \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[I^{\alpha+\beta+\gamma_j} \|\rho\| \Upsilon(\|x\|) (\omega_j) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| I^{\beta+\gamma_j} \|x\| (\omega_j) \right] \right. \\
& \quad \left. + \sum_{i=1}^m |\theta_i| \left[I^{\alpha+\beta+\mu_i} \|\rho\| \Upsilon(\|x\|) (\eta_i) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| I^{\beta+\mu_i} \|x\| (\eta_i) \right] \right) \\
& \quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\
& \quad \times \left(\sum_{l=1}^q |\gamma_l| \left[I^{\alpha+\beta+\tau_l} \|\rho\| \Upsilon(\|x\|) (\varphi_l) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| I^{\beta+\tau_l} \|x\| (\varphi_l) \right] \right. \\
& \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[I^{\alpha+\beta+\sigma_k} \|\rho\| \Upsilon(\|x\|) (\psi_k) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| I^{\beta+\sigma_k} \|x\| (\psi_k) \right] \right) \\
& \quad + I^{\alpha+\beta} \|\rho\| \Upsilon(\|x\|) (t) + |\lambda| I^\beta \|x\| (t) \\
& \leq \|\rho\| \Upsilon(\|x\|) \Lambda(\alpha) + |\lambda| R \Lambda(0) \\
& \leq \|\rho\| \Upsilon(R) \Lambda(\alpha) + |\lambda| R \Lambda(0), \tag{32}
\end{aligned}$$

and, consequently,

$$\|\mathcal{Q}x\| \leq \|\rho\| \Upsilon(R) \Lambda(\alpha) + |\lambda| R \Lambda(0). \tag{33}$$

Next, we will show that $\mathcal{Q}$ maps bounded sets into equicontinuous sets of $C([1, e], \mathbb{R})$. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $x \in B_R$. Then, we have

$$\begin{aligned}
& |(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)| \\
& \leq \left| \frac{1}{\Omega} \left[\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)} \Omega_3 \right. \right. \\
& \quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s)) (\omega_j) \right. \right. \\
& \quad \quad \left. \left. - \lambda I^{\beta+\gamma_j} x(s) (\omega_j) \right] \right. \\
& \quad \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s)) (\eta_i) \right. \right. \\
& \quad \quad \left. \left. - \lambda I^{\beta+\mu_i} x(s) (\eta_i) \right] \right) \\
& \quad \left. + \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)} \Omega_1 \right. \\
& \quad \times \left(\sum_{l=1}^q \gamma_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s)) (\varphi_l) \right. \right. \\
& \quad \quad \left. \left. - \lambda I^{\beta+\tau_l} x(s) (\varphi_l) \right] \right. \\
& \quad \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s)) (\psi_k) \right. \right. \\
& \quad \quad \left. \left. - \lambda I^{\beta+\sigma_k} x(s) (\psi_k) \right] \right) \Bigg| \\
& \quad + |I^{\alpha+\beta} f(s, x(s)) (t_2) - \lambda I^\beta x(s) (t_2) \\
& \quad \quad - I^{\alpha+\beta} f(s, x(s)) (t_1) + \lambda I^\beta x(s) (t_1)| \\
& \leq \frac{1}{|\Omega|} \left[\left| \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)} \right| |\Omega_3| \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[I^{\alpha+\beta+\gamma_j} \|\rho\| \Upsilon(\|x\|) (\omega_j) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| I^{\beta+\gamma_j} \|x\| (\omega_j) \right] \right. \\
& \quad \left. + \sum_{i=1}^m |\theta_i| \left[I^{\alpha+\beta+\mu_i} \|\rho\| \Upsilon(\|x\|) (\eta_i) \right. \right. \\
& \quad \quad \left. \left. + |\lambda| I^{\beta+\mu_i} \|x\| (\eta_i) \right] \right) \\
& \quad \left. + \left| \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)} \Omega_1 \right| \right]
\end{aligned}$$

$$\begin{aligned}
& + \left| \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta + 1)} \right| |\Omega_1| \\
& \times \left(\sum_{l=1}^q |\gamma_l| \left[I^{\alpha+\beta+\tau_l} \|\rho\| \Upsilon(\|x\|)(\varphi_l) \right. \right. \\
& \quad \left. \left. + |\lambda| I^{\beta+\tau_l} \|x\|(\varphi_l) \right] \right. \\
& \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[I^{\alpha+\beta+\sigma_k} \|\rho\| \Upsilon(\|x\|)(\psi_k) \right. \right. \\
& \quad \left. \left. + |\lambda| I^{\beta+\sigma_k} \|x\|(\psi_k) \right] \right) \\
& + \frac{\|\rho\| \Upsilon(\|x\|)}{\Gamma(\alpha + \beta + 1)} \left| (\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right| \\
& + \frac{|\lambda| \|x\|}{\Gamma(\beta + 1)} \left| (\log t_1)^\beta - (\log t_2)^\beta + 2 \left(\log \frac{t_2}{t_1} \right)^\beta \right| \\
& \leq \frac{1}{|\Omega|} \left[\left| \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta + 1)} \right| |\Omega_3| \right. \\
& \quad \times \left(\sum_{j=1}^n |\phi_j| \left[\frac{\|\rho\| \Upsilon(R) (\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha + \beta + \gamma_j + 1)} \right. \right. \\
& \quad \left. \left. + \frac{|\lambda| R (\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta + \gamma_j + 1)} \right] \right. \\
& \quad \left. + \sum_{i=1}^m |\theta_i| \left[\frac{\|\rho\| \Upsilon(R) (\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha + \beta + \mu_i + 1)} \right. \right. \\
& \quad \left. \left. + \frac{|\lambda| R (\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta + \mu_i + 1)} \right] \right) \\
& \quad + \left| \frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta + 1)} \right| |\Omega_1| \\
& \quad \times \left(\sum_{l=1}^q |\gamma_l| \left[\frac{\|\rho\| \Upsilon(R) (\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha + \beta + \tau_l + 1)} \right. \right. \\
& \quad \left. \left. + \frac{|\lambda| R (\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta + \tau_l + 1)} \right] \right. \\
& \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[\frac{\|\rho\| \Upsilon(R) (\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha + \beta + \sigma_k + 1)} \right. \right. \\
& \quad \left. \left. + \frac{|\lambda| R (\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta + \sigma_k + 1)} \right] \right) \Bigg]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\|\rho\| \Upsilon(R)}{\Gamma(\alpha + \beta + 1)} \left| (\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta} \right| \\
& + \frac{|\lambda| R}{\Gamma(\beta + 1)} \left| (\log t_1)^\beta - (\log t_2)^\beta + 2 \left(\log \frac{t_2}{t_1} \right)^\beta \right|.
\end{aligned} \tag{34}$$

As $t_2 - t_1 \rightarrow 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_R$. Therefore, by the Arzelà-Ascoli theorem, the operator $\mathcal{Q} : C([1, e], \mathbb{R}) \rightarrow C([1, e], \mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in [1, e]$, and following similar computations to the first step, we have

$$|x(t)| \leq \|\rho\| \Upsilon(\|x\|) \Lambda(\alpha) + |\lambda| \|x\| \Lambda(0), \tag{35}$$

which leads to

$$\frac{\|x\|}{\|\rho\| \Upsilon(\|x\|) \Lambda(\alpha) + |\lambda| \|x\| \Lambda(0)} \leq 1. \tag{36}$$

In view of (H_4) , there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([1, e], \mathbb{R}) : \|x\| < M\}. \tag{37}$$

We see that the operator $\mathcal{Q} : \bar{U} \rightarrow C([1, e], \mathbb{R})$ is continuous and completely continuous. From the choice of U , there is no $x \in \partial U$ such that $x = \xi \mathcal{Q}x$ for some $\xi \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder's type, we deduce that $\mathcal{Q}$ has a fixed point $x \in \bar{U}$ which is a solution of the boundary value problem (1). This completes the proof. $\square$

3.4. Existence Result via Leray-Schauder's Degree Theory

Theorem 10. Let $f : [1, e] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. In addition, one assumes the following:

(H_5) there exist constants $0 \leq \mu < [1 - |\lambda| \Lambda(0)] [\Lambda(\alpha)]^{-1}$ and $M > 0$ such that

$$|f(t, x)| \leq \mu |x| + M \quad \forall (t, x) \in [1, e] \times \mathbb{R}, \tag{38}$$

where $\Lambda(0)$ and $\Lambda(\alpha)$ are given by (18) with $u = 0$ and α , respectively.

Then, the nonlocal problem (1) has at least one solution on $[1, e]$.

Proof. We define an operator $\mathcal{Q} : C([1, e], \mathbb{R}) \rightarrow C([1, e], \mathbb{R})$ as in (17) and consider the fixed point problem

$$x = \mathcal{Q}x. \tag{39}$$

We are going to prove that there exists a fixed point $x \in C([1, e], \mathbb{R})$ satisfying (39). It is sufficient to show that $\mathcal{Q} : \bar{B}_R \rightarrow C([1, e], \mathbb{R})$ satisfies

$$x \neq \kappa \mathcal{Q}x, \quad \forall x \in \partial B_R, \quad \forall \kappa \in [0, 1], \tag{40}$$

where $B_R = \{x \in C([1, e], \mathbb{R}) : \max_{t \in [1, e]} |x(t)| < R, R > 0\}$. We define

$$H(\kappa, x) = \kappa \mathcal{Q}x, \quad x \in C([1, e], \mathbb{R}), \quad \kappa \in [0, 1]. \quad (41)$$

As shown in Theorem 9, the operator $\mathcal{Q}$ is continuous, uniformly bounded, and equicontinuous. Then, by the Arzelà-Ascoli theorem, a continuous map h_κ defined by $h_\kappa(x) = x - H(\kappa, x) = x - \kappa \mathcal{Q}x$ is completely continuous. If (40) is true, then the following Leray-Schauder degrees are well defined and, by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\kappa, B_R, 0) &= \deg(I - \kappa \mathcal{Q}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) \\ &= 1 \neq 0, \quad 0 \in B_R, \end{aligned} \quad (42)$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder's degree, $h_1(x) = x - \mathcal{Q}x = 0$ for at least one $x \in B_R$. In order to prove (40), we assume that $x = \kappa \mathcal{Q}x$ for some $\kappa \in [0, 1]$. Then

$$\begin{aligned} & |(\mathcal{Q}x)(t)| \\ & \leq \left| \frac{1}{\Omega} \left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_3 \right) \right. \right. \\ & \quad \times \left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s)) (\omega_j) \right. \right. \\ & \quad \left. \left. - \lambda I^{\beta+\gamma_j} x(s) (\omega_j) \right] \right. \\ & \quad \left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s)) (\eta_i) \right. \right. \\ & \quad \left. \left. - \lambda I^{\beta+\mu_i} x(s) (\eta_i) \right] \right) \\ & \quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \\ & \quad \times \left(\sum_{l=1}^q \gamma_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s)) (\varphi_l) \right. \right. \\ & \quad \left. \left. - \lambda I^{\beta+\tau_l} x(s) (\varphi_l) \right] \right. \\ & \quad \left. - \sum_{k=1}^p \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s, x(s)) (\psi_k) \right. \right. \\ & \quad \left. \left. - \lambda I^{\beta+\sigma_k} x(s) (\psi_k) \right] \right) \\ & \quad \left. + I^{\alpha+\beta} f(s, x(s))(t) - \lambda I^\beta x(s)(t) \right| \\ & \leq \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_3| \right) \right. \\ & \quad \times \left(\sum_{j=1}^n |\phi_j| \left[I^{\alpha+\beta+\gamma_j} |f(s, x(s))| (\omega_j) \right. \right. \\ & \quad \left. \left. + |\lambda| I^{\beta+\gamma_j} |x(s)| (\omega_j) \right] \right. \\ & \quad \left. + \sum_{i=1}^m |\theta_i| \left[I^{\alpha+\beta+\mu_i} |f(s, x(s))| (\eta_i) \right. \right. \\ & \quad \left. \left. + |\lambda| I^{\beta+\mu_i} |x(s)| (\eta_i) \right] \right) \\ & \quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)} |\Omega_1| + |\Omega_2| \right) \\ & \quad \times \left(\sum_{l=1}^q |\gamma_l| \left[I^{\alpha+\beta+\tau_l} |f(s, x(s))| (\varphi_l) \right. \right. \\ & \quad \left. \left. + |\lambda| I^{\beta+\tau_l} |x(s)| (\varphi_l) \right] \right. \\ & \quad \left. + \sum_{k=1}^p |\varepsilon_k| \left[I^{\alpha+\beta+\sigma_k} |f(s, x(s))| (\psi_k) \right. \right. \\ & \quad \left. \left. + |\lambda| I^{\beta+\sigma_k} |x(s)| (\psi_k) \right] \right) \\ & \quad + I^{\alpha+\beta} |f(s, x(s))|(t) + |\lambda| I^\beta |x(s)|(t) \\ & \leq (\mu \|x\| + M) \Lambda(\alpha) + |\lambda| \|x\| \Lambda(0) \\ & = [\mu \Lambda(\alpha) + |\lambda| \Lambda(0)] \|x\| + M \Lambda(\alpha). \end{aligned} \quad (43)$$

Computing directly for $\|x\| = \sup_{t \in [1, e]} |x(t)|$, we have

$$\|x\| \leq \frac{M \Lambda(\alpha)}{1 - \mu \Lambda(\alpha) - |\lambda| \Lambda(0)}. \quad (44)$$

If $R = M \Lambda(\alpha) / (1 - \mu \Lambda(\alpha) - |\lambda| \Lambda(0)) + 1$, inequality (39) holds. This completes the proof. $\square$

4. Examples

In this section, we present some examples to illustrate our results.

Example 1. Consider the following nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions:

$$\begin{aligned} D^{1/2} \left(D^{2/3} + \frac{1}{9} \right) x(t) &= \frac{\log t}{3t} \cdot \frac{|x(t)|}{2 + |x(t)|} + \frac{3}{2}t, \quad 1 < t < e, \\ \frac{13}{4} I^{2/3} x \left(\frac{e+7}{7} \right) &+ \frac{13}{3} I^{3/4} x \left(\frac{2e+7}{7} \right) + \frac{13}{2} I^{4/5} x \left(\frac{3e+7}{7} \right) \\ &= \frac{1}{3} I^{1/6} x \left(\frac{e+2}{4} \right) + \frac{1}{4} I^{1/2} x \left(\frac{e+1}{2} \right), \\ \frac{2}{3} I^{1/5} x \left(\frac{3e}{4} \right) &+ \frac{3}{7} I^{1/4} x \left(\frac{4e}{5} \right) = 3 I^{1/5} x \left(\frac{2e}{5} \right). \end{aligned} \quad (45)$$

Here, $\alpha = 1/2$, $\beta = 2/3$, $\lambda = 1/9$, $m = 3$, $n = 2$, $p = 2$, $q = 1$, $\theta_1 = 13/4$, $\theta_2 = 13/3$, $\theta_3 = 13/2$, $\mu_1 = 2/3$, $\mu_2 = 3/4$, $\mu_3 = 4/5$, $\eta_1 = (e+7)/7$, $\eta_2 = (2e+7)/7$, $\eta_3 = (3e+7)/7$, $\phi_1 = 1/3$, $\phi_2 = 1/4$, $\gamma_1 = 1/6$, $\gamma_2 = 1/2$, $\omega_1 = (e+2)/4$, $\omega_2 = (e+1)/2$, $\varepsilon_1 = 2/3$, $\varepsilon_2 = 3/7$, $\sigma_1 = 1/5$, $\sigma_2 = 1/4$, $\psi_1 = 3e/4$, $\psi_2 = 4e/5$, $\nu_1 = 3$, $\tau_1 = 1/5$, $\varphi_1 = 2e/5$, and $f(t, x) = (\log t |x(t)|) / (3t(2 + |x(t)|)) + 3t/2$. Since $|f(t, x) - f(t, y)| \leq (1/6)|x - y|$, then (H_1) is satisfied with $L = 1/6$. We can show that

$$\begin{aligned} \Lambda(\alpha) &= \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \right. \\ &\quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \right) \\ &\quad + \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \\ &\quad \times \left(\sum_{l=1}^q |\nu_l| \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} \right. \\ &\quad \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)} \right) \Bigg] \\ &\quad + \frac{1}{\Gamma(\alpha+\beta+1)} \\ &\approx 2.527538367, \end{aligned}$$

$$\begin{aligned} \Lambda(0) &= \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \right. \\ &\quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right) \end{aligned}$$

$$\begin{aligned} &+ \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \\ &\times \left(\sum_{l=1}^q |\nu_l| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right. \\ &\quad \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)} \right) \Bigg] + \frac{1}{\Gamma(\beta+1)} \\ &\approx 4.083246365. \end{aligned} \quad (46)$$

Thus, $L\Lambda(\alpha) + |\lambda|\Lambda(0) \approx 0.8749504351 < 1$. Hence, by Theorem 5, the nonlocal problem (45) has a unique solution on $[1, e]$.

Example 2. Consider the following nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions:

$$\begin{aligned} D^{2/3} \left(D^{2/5} + \frac{1}{12} \right) x(t) &= \frac{t \log t}{t + \sin \pi t} \cdot \frac{\arccot x(t)}{1 + |x(t)|} + t^2 \sin \pi t, \quad 1 < t < e, \\ 4 I^{2/3} x \left(\frac{2e}{3} \right) &+ 3 I^{3/5} x \left(\frac{2e+1}{3} \right) \\ &= \frac{2}{5} I^{2/3} x \left(\frac{e+1}{3} \right) + \frac{1}{2} I^{1/2} x \left(\frac{e+2}{3} \right) \\ &\quad + \frac{2}{5} I^{3/2} x \left(\frac{e+3}{3} \right), \\ \frac{5}{7} I^{2/5} x \left(\frac{3e}{7} \right) &+ \frac{6}{7} I^{3/4} x \left(\frac{4e}{7} \right) \\ &= \frac{1}{5} I^{1/5} x \left(\frac{3e}{5} \right) + \frac{2}{7} I^{2/5} x \left(\frac{4e}{5} \right). \end{aligned} \quad (47)$$

Here, $\alpha = 2/3$, $\beta = 2/5$, $\lambda = 1/12$, $m = 2$, $n = 3$, $p = 2$, $q = 2$, $\theta_1 = 4$, $\theta_2 = 3$, $\mu_1 = 2/3$, $\mu_2 = 3/5$, $\eta_1 = 2e/3$, $\eta_2 = (2e+1)/3$, $\phi_1 = 2/5$, $\phi_2 = 1/2$, $\phi_3 = 2/5$, $\gamma_1 = 2/3$, $\gamma_2 = 1/2$, $\gamma_3 = 3/2$, $\omega_1 = (e+1)/3$, $\omega_2 = (e+2)/3$, $\omega_3 = (e+3)/3$, $\varepsilon_1 = 5/7$, $\varepsilon_2 = 6/7$, $\sigma_1 = 2/5$, $\sigma_2 = 3/4$, $\psi_1 = 3e/7$, $\psi_2 = 4e/7$, $\nu_1 = 1/5$, $\nu_2 = 2/7$, $\tau_1 = 1/5$, $\tau_2 = 2/5$, $\varphi_1 = 3e/5$, $\varphi_2 = 4e/5$, and $f(t, x) = (t \log t \arccot x(t)) / ((t + \sin \pi t)(1 + |x(t)|)) + t^2 \sin \pi t$. Since $|f(t, x)| \leq (t \log t) / (t + \sin \pi t) + t^2 \sin \pi t$, then (H_2) is satisfied. We can find that

$$\begin{aligned} \Lambda(0) &= \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \right. \\ &\quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right. \\ &\quad \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right) \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \\
 & \times \left(\sum_{l=1}^q |\gamma_l| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right. \\
 & \quad \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)} \right) \Bigg] + \frac{1}{\Gamma(\beta+1)} \\
 & \approx 10.69222877.
 \end{aligned} \tag{48}$$

Thus, $|\lambda|\Lambda(0) \approx 0.8910190640 < 1$. Hence, by Theorem 7, the nonlocal problem (47) has at least one solution on $[1, e]$.

Example 3. Consider the following nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions:

$$\begin{aligned}
 & D^{1/2} \left(D^{4/5} + \frac{1}{7} \right) x(t) \\
 & = \frac{\sin x(t)}{5\pi^2 + \cos^2 \pi x(t)} + \frac{2 + \log t}{\pi^2}, \quad 1 < t < e, \\
 & 2I^{3/2} x \left(\frac{e+1}{3} \right) + 3I^{4/3} x \left(\frac{e+2}{3} \right) + 4I^{5/4} x \left(\frac{e+3}{3} \right) \\
 & \quad + 5I^{6/5} x \left(\frac{e+4}{3} \right) = \frac{2}{3} I^{2/3} x \left(\frac{e+3}{4} \right), \\
 & \frac{1}{3} I^{3/5} x \left(\frac{2e}{5} \right) + \frac{1}{6} I^{3/4} x \left(\frac{3e}{5} \right) \\
 & = 5I^{3/5} x \left(\frac{2e}{5} \right) + \frac{1}{2} I^{2/5} x \left(\frac{3e}{5} \right).
 \end{aligned} \tag{49}$$

Here, $\alpha = 1/2$, $\beta = 4/5$, $\lambda = 1/7$, $m = 4$, $n = 1$, $p = 2$, $q = 2$, $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 4$, $\theta_4 = 5$, $\mu_1 = 3/2$, $\mu_2 = 4/3$, $\mu_3 = 5/4$, $\mu_4 = 6/5$, $\eta_1 = (e+1)/3$, $\eta_2 = (e+2)/3$, $\eta_3 = (e+3)/3$, $\eta_4 = (e+4)/3$, $\phi_1 = 2/3$, $\gamma_1 = 2/3$, $\omega_1 = (e+3)/4$, $\varepsilon_1 = 1/3$, $\varepsilon_2 = 1/6$, $\sigma_1 = 3/5$, $\sigma_2 = 3/4$, $\psi_1 = 2e/5$, $\psi_2 = 3e/5$, $\nu_1 = 5$, $\nu_2 = 1/2$, $\tau_1 = 3/5$, $\tau_2 = 2/5$, $\varphi_1 = 2e/5$, $\varphi_2 = 3e/5$, and $f(t, x) = (\sin x(t))/(5\pi^2 + \cos^2 \pi x(t)) + (2 + \log t)/\pi^2$. Then, we get that

$$\begin{aligned}
 \Lambda(\alpha) & = \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \right. \\
 & \quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \right. \\
 & \quad \left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \right) \right. \\
 & \quad \left. + \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & \times \left(\sum_{l=1}^q |\gamma_l| \frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} \right. \\
 & \quad \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)} \right) \Bigg] \\
 & + \frac{1}{\Gamma(\alpha+\beta+1)} \\
 & \approx 2.675517413, \\
 \Lambda(0) & = \frac{1}{|\Omega|} \left[\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)} \right) \right. \\
 & \quad \times \left(\sum_{j=1}^n |\phi_j| \frac{(\log \omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)} \right. \\
 & \quad \left. \left. + \sum_{i=1}^m |\theta_i| \frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)} \right) \right. \\
 & \quad \left. + \left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2| \right) \right. \\
 & \quad \times \left(\sum_{l=1}^q |\gamma_l| \frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)} \right. \\
 & \quad \left. \left. + \sum_{k=1}^p |\varepsilon_k| \frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)} \right) \right] + \frac{1}{\Gamma(\beta+1)} \\
 & \approx 5.058796431.
 \end{aligned} \tag{50}$$

Clearly,

$$\begin{aligned}
 |f(t, x)| & = \left| \frac{\sin x(t)}{5\pi^2 + \cos^2 \pi x(t)} + \frac{2 + \log t}{\pi^2} \right| \\
 & \leq \left(\frac{2 + \log t}{5\pi^2} \right) (|x(t)| + 5).
 \end{aligned} \tag{51}$$

Choosing $\rho(t) = (2 + \log t)/(5\pi^2)$ and $Y(|x|) = |x| + 5$, we can show that

$$\frac{M}{\|\rho\| Y(M) \Lambda(\alpha) + |\lambda| M \Lambda(0)} > 1, \tag{52}$$

which implies that $M > 7.092618387$. Hence, by Theorem 9, the nonlocal problem (49) has at least one solution on $[1, e]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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