Research Article

On Generalized Jordan Prederivations and Generalized Prederivations of Lie Superalgebras

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The concepts of (generalized) (θ, φ) -prederivations and (generalized) Jordan (θ, φ) -prederivations on a Lie superalgebra are introduced. It is proved that Jordan (θ, φ) -prederivations (resp., generalized Jordan (θ, φ) -prederivations) are (θ, φ) -prederivations (resp., generalized (θ, φ) -prederivations) on a Lie superalgebra under some conditions. In particular, Jordan θ -prederivations are θ -prederivations on a Lie superalgebra.

1. Introduction

Derivations and generalized derivations are interesting subjects both in mathematics and physics, and there has been a great deal of work concerning them. Leger and Luks investigated the structure of the generalized derivations of Lie algebras systematically (cf. [1]). Generalized derivations on rings were studied in [2, 3]. Generalized derivations also play a key role in Benoist's study of Levi factors in derivation algebras of nilpotent Lie algebras (cf. [4]). In 1969, Herstein showed that a Jordan derivation of a prime ring of characteristic not 2 must be a derivation (cf. [5]). Brešar together with Vukman generalized Herstein's result to Jordan (Θ, φ) -derivations (cf. [6]). In [7], the authors proved that, in a 2-torsion free noncommutative prime ring R , a generalized Jordan (θ, φ) -derivation is a generalized (θ, φ) -derivation when θ is an automorphism of R . We gave some results on generalized derivations of Lie color algebras in [8]. Moreover, Jordan θ -derivations and generalized Jordan derivations of Lie triple systems were studied in [3, 9, 10].

Prederivations (or Lie triple derivations) of Lie algebras were first introduced by Müller to study bi-invariant semi-Riemannian metrics on Lie groups. Let G be a Lie group with a bi-invariant semi-Riemannian metric and $\mathfrak{g}$ its Lie algebra. Then the Lie algebra of the group of isometries of G

fixing the identity element is a subalgebra of the prederivation algebra of $\mathfrak{g}$ (cf. [11]). Bajo proved that a real or complex Lie algebra admitting a nonsingular prederivation is necessarily nilpotent (cf. [12]). Moens generated this result to Lie algebras over any field of characteristic zero in [13]. Burde showed that the existence of nonsingular prederivations is useful for the construction of the affine structure on Lie algebras (cf. [14]). In recent years, there has been an increasing interest in investigating prederivations (cf. [2, 15–20]).

Lie superalgebras are the natural generalization of Lie algebras and have important applications both in mathematics and in physics. Lie superalgebras are also interesting from a purely mathematical point of view. So it is reasonable to extend the notion of prederivations to Lie superalgebras, which may do the same work in the structure of Lie superalgebras as the prederivations of Lie algebras did. In this paper, we introduce the concepts of (generalized) (θ, φ) -prederivations and (generalized) Jordan (θ, φ) -prederivations for a Lie superalgebra and obtain some results concerning Jordan (θ, φ) -prederivations (resp., generalized Jordan (θ, φ) -prederivations) and (θ, φ) -prederivations (resp., generalized (θ, φ) -prederivations) on a Lie superalgebra.

Throughout this paper, the base field $\mathbf{F}$ is assumed to be of characteristic not equal to 3. We now recall some elementary definitions.

Definition 1 (see [12, 19]). A prederivation (Lie triple derivation) of a Lie algebra L is a linear mapping $D : L \rightarrow L$ such that

$$D([x, y], z) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y], D(z)], \quad \forall x, y, z \in L. \quad (1)$$

Definition 2 (see [21]). A Jordan prederivation (Jordan triple derivation) of a Lie algebra L is a linear mapping $D' : L \rightarrow L$ such that

$$D'([x, y], x) = [[D'(x), y], x] + [[x, D'(y)], x] + [[x, y], D'(x)], \quad \forall x, y \in L. \quad (2)$$

2. Main Results

Definition 3. Let L be a Lie superalgebra and let $D, \theta, \varphi : L \rightarrow L$ be homogeneous linear mappings, where $d(\theta) = d(\varphi) = 0$. One denotes by $d(D)$ the $\mathbb{Z}_2$ -graded degree of a homogeneous linear mapping D of L .

(1) D is called a $(\theta, \varphi)_1$ -prederivation if

$$\begin{aligned} D([x, y], z) &= [[D(x), \theta(y)], \varphi(z)] \\ &+ (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] \\ &+ (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\theta(x), \varphi(y)], D(z)], \quad \forall x, y, z \in L. \end{aligned} \quad (3)$$

(2) D is called a $(\theta, \varphi)_2$ -prederivation if

$$\begin{aligned} D([x, y], z) &= [[D(x), \theta(y)], \theta(z)] \\ &+ (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] \\ &+ (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D(z)], \quad \forall x, y, z \in L. \end{aligned} \quad (4)$$

(3) D is called a $(\theta, \varphi)_3$ -prederivation if

$$\begin{aligned} D([x, y], z) &= [[D(x), \theta(y)], \theta(z)] \\ &+ (-1)^{d(D)d(x)} [[\varphi(x), D(y)], \theta(z)] \\ &+ (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D(z)], \quad \forall x, y, z \in L. \end{aligned} \quad (5)$$

In particular, $\forall i = 1, 2, 3$, a $(\theta, \varphi)_i$ -prederivation D is called a θ -prederivation if $\theta = \varphi$. It is clear that a $(\theta, \varphi)_i$ -prederivation is a prederivation when $\theta = \varphi = 1_T$.

Definition 4. Let L be a Lie superalgebra and let $D, \theta, \varphi : L \rightarrow L$ be homogeneous linear mappings, where $d(\theta) = d(\varphi) = 0$.

(1) D is called a Jordan $(\theta, \varphi)_1$ -prederivation if

$$\begin{aligned} D([x, y], x) &= [[D(x), \theta(y)], \varphi(x)] \\ &+ (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(x)] \\ &+ (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\theta(x), \varphi(y)], D(x)], \quad \forall x, y \in L. \end{aligned} \quad (6)$$

(2) D is called a Jordan $(\theta, \varphi)_2$ -prederivation if

$$\begin{aligned} D([x, y], x) &= [[D(x), \theta(y)], \theta(x)] \\ &+ (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(x)] \\ &+ (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D(x)], \quad \forall x, y \in L. \end{aligned} \quad (7)$$

(3) D is called a Jordan $(\theta, \varphi)_3$ -prederivation if

$$\begin{aligned} D([x, y], x) &= [[D(x), \theta(y)], \theta(x)] \\ &+ (-1)^{d(D)d(x)} [[\varphi(x), D(y)], \theta(x)] \\ &+ (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D(x)], \quad \forall x, y \in L. \end{aligned} \quad (8)$$

In particular, $\forall i = 1, 2, 3$, a Jordan $(\theta, \varphi)_i$ -prederivation D is called a Jordan θ -prederivation if $\theta = \varphi$. It is clear that a Jordan $(\theta, \varphi)_i$ -prederivation is a Jordan prederivation when $\theta = \varphi = 1_T$.

It is clear that if D_i is a $(\theta, \varphi)_i$ -prederivation of L , then D_i is a Jordan $(\theta, \varphi)_i$ -prederivation of L , where $i = 1, 2, 3$.

In this section, L is a Lie superalgebra and θ, φ are defined to be homogeneous linear mappings of L satisfying $d(\theta) = d(\varphi) = 0$.

Theorem 5. D is a $(\theta, \varphi)_1$ -prederivation of L if and only if D is a Jordan $(\theta, \varphi)_1$ -prederivation of L such that

$$\begin{aligned} \text{(i)} \quad & [[\theta(x), \varphi(y)], D(z)] = [[\varphi(x), \theta(y)], D(z)], \\ \text{(ii)} \quad & (-1)^{d(x)d(z)} A(x, y, z) + (-1)^{d(x)d(y)} A(y, z, x) + \\ & (-1)^{d(y)d(z)} A(z, x, y) = 0, \end{aligned}$$

where $x, y, z \in L$ and $A(x, y, z) = [[D(x), \theta(y)], \varphi(z)] + (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] + (-1)^{d(D)(d(x)+d(y))} [[\theta(x), \varphi(y)], D(z)]$.

Proof. Assume that D is a $(\theta, \varphi)_1$ -prederivation of L . Clearly, D is a Jordan $(\theta, \varphi)_1$ -prederivation of L and $D([x, y], z) = -(-1)^{d(x)d(y)} D([y, x], z)$; note that

$$\begin{aligned}
 & -(-1)^{d(x)d(y)} D([y, x], z) \\
 & = -(-1)^{d(x)d(y)} [[D(y), \theta(x)], \varphi(z)] \\
 & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)d(y)} \\
 & \quad \times [[\theta(y), D(x)], \varphi(z)] \\
 & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\
 & \quad \times [[\theta(y), \varphi(x)], D(z)] \\
 & = [D(x), \theta(y), \varphi(z)] + (-1)^{d(D)d(x)} \\
 & \quad \times [[\theta(x), D(y)], \varphi(z)] \\
 & \quad + (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \theta(y)], D(z)]; \\
 & D([x, y], z) \\
 & = [[D(x), \theta(y)], \varphi(z)] \\
 & \quad + (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] \\
 & \quad + (-1)^{d(D)(d(x)+d(y))} \\
 & \quad \times [[\theta(x), \varphi(y)], D(z)],
 \end{aligned} \tag{9}$$

then (i) follows. Since D is a $(\theta, \varphi)_1$ -prederivation of L , we have $D([x, y], z) = A(x, y, z)$; hence

$$\begin{aligned}
 & (-1)^{d(x)d(z)} A(x, y, z) + (-1)^{d(x)d(y)} A(y, z, x) \\
 & \quad + (-1)^{d(y)d(z)} A(z, x, y) \\
 & = (-1)^{d(x)d(z)} D([x, y], z) \\
 & \quad + (-1)^{d(x)d(y)} D([y, z], x) \\
 & \quad + (-1)^{d(y)d(z)} D([z, x], y) \\
 & = D((-1)^{d(x)d(z)} [[x, y], z] \\
 & \quad + (-1)^{d(x)d(y)} [[y, z], x] \\
 & \quad + (-1)^{d(y)d(z)} [[z, x], y]) = 0.
 \end{aligned} \tag{10}$$

Conversely, let D be a Jordan $(\theta, \varphi)_1$ -prederivation of L for which (i) and (ii) hold. Then $D([x, y], x) = A(x, y, x)$. It follows that

$$\begin{aligned}
 & D([x+z, y], x+z) \\
 & = D([x, y], x) + D([x, y], z) \\
 & \quad + D([z, y], x) + D([z, y], z)
 \end{aligned}$$

$$\begin{aligned}
 & = A(x, y, x) + A(z, y, z) \\
 & \quad + D([x, y], z) + D([z, y], x), \\
 & D([x+z, y], x+z) \\
 & = A(x+z, y, x+z) \\
 & = A(x, y, x) + A(x, y, z) \\
 & \quad + A(z, y, x) + A(z, y, z).
 \end{aligned} \tag{11}$$

Thus we obtain

$$D([x, y], z) + D([z, y], x) = A(x, y, z) + A(z, y, x). \tag{*}$$

By (i),

$$\begin{aligned}
 & A(y, x, z) \\
 & = [[D(y), \theta(x)], \varphi(z)] + (-1)^{d(D)d(y)} \\
 & \quad \times [[\theta(y), D(x)], \varphi(z)] \\
 & \quad + (-1)^{d(D)(d(x)+d(y))} [[\theta(y), \varphi(x)], D(z)] \\
 & = -(-1)^{d(x)d(y)} [[D(x), \theta(y)], \varphi(z)] \\
 & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] \\
 & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\
 & \quad \times [[\varphi(x), \theta(y)], D(z)] \\
 & = -(-1)^{d(x)d(y)} [[D(x), \theta(y)], \varphi(z)] \\
 & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)d(x)} \\
 & \quad \times [[\theta(x), D(y)], \varphi(z)] \\
 & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\
 & \quad \times [[\theta(x), \varphi(y)], D(z)] \\
 & = -(-1)^{d(x)d(y)} A(x, y, z).
 \end{aligned} \tag{12}$$

This implies that

$$\begin{aligned}
 & D([x, y], y) = -(-1)^{d(x)d(y)} D([y, x], y) \\
 & = -(-1)^{d(x)d(y)} A(y, x, y) \\
 & = A(x, y, y).
 \end{aligned} \tag{13}$$

A similar argument proves

$$D([x, y], z) + D([x, z], y) = A(x, y, z) + A(x, z, y). \tag{**}$$

By (*) $\times (-1)^{d(x)d(z)}$ + (**) $\times (-1)^{d(x)d(y)}$, we have

$$\begin{aligned}
 & (-1)^{d(x)d(z)} (D([x, y], z) + D([z, y], x)) \\
 & \quad + (-1)^{d(x)d(y)} (D([x, y], z) + D([x, z], y))
 \end{aligned}$$

$$\begin{aligned}
&= (-1)^{d(x)d(z)} (A(x, y, z) + A(z, y, x)) \\
&\quad + (-1)^{d(x)d(y)} (A(x, y, z) + A(x, z, y)); \tag{14}
\end{aligned}$$

then

$$\begin{aligned}
&(-1)^{d(x)d(z)} D([x, y], z]) \\
&\quad + (-1)^{d(x)d(y)} D([x, y], z]) \\
&\quad - (-1)^{d(y)d(z)} D([y, x], z]) \tag{15} \\
&= (-1)^{d(x)d(z)} (A(x, y, z) + A(z, y, x)) \\
&\quad + (-1)^{d(x)d(y)} (A(x, y, z) + A(x, z, y));
\end{aligned}$$

that is,

$$\begin{aligned}
&\left((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + (-1)^{d(y)(d(x)+d(z))} \right) \\
&\quad \times D([x, y], z]) \\
&= \left((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + (-1)^{d(y)(d(x)+d(z))} \right) \\
&\quad \times A(x, y, z) \\
&\quad + (-1)^{d(x)d(z)} A(z, y, x) \\
&\quad + (-1)^{d(x)d(y)} A(x, z, y) \\
&\quad - (-1)^{d(y)(d(x)+d(z))} A(x, y, z) \tag{16} \\
&= \left((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + (-1)^{d(y)(d(x)+d(z))} \right) \\
&\quad \times A(x, y, z) \\
&\quad + (-1)^{d(x)d(z)} A(z, y, x) \\
&\quad + (-1)^{d(x)d(y)} A(x, z, y) \\
&\quad + (-1)^{d(y)d(z)} A(y, x, z) \\
&= \left((-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + (-1)^{d(y)(d(x)+d(z))} \right) \\
&\quad \times A(x, y, z),
\end{aligned}$$

where the last equality uses (ii). Since $\text{chF} \neq 3$, we have $(-1)^{d(x)d(z)} + (-1)^{d(x)d(y)} + (-1)^{d(y)(d(x)+d(z))} \neq 0$, and so $D([x, y], z) = A(x, y, z)$; that is, D is a $(\theta, \varphi)_1$ -prederivation of L . $\square$

Corollary 6. D is a θ -prederivation of L if and only if D is a Jordan θ -prederivation of L .

Proof. If D is a Jordan θ -prederivation of L , then (i) follows immediately. (ii) holds because

$$\begin{aligned}
&(-1)^{d(x)d(z)} A(x, y, z) + (-1)^{d(x)d(y)} A(y, z, x) \\
&\quad + (-1)^{d(y)d(z)} A(z, x, y) \\
&= (-1)^{d(x)d(z)} [[D(x), \theta(y)], \theta(z)]
\end{aligned}$$

$$\begin{aligned}
&+ (-1)^{d(x)d(z)} (-1)^{d(D)d(x)} [[\theta(x), D(y)], \theta(z)] \\
&+ (-1)^{d(x)d(z)} (-1)^{d(D)(d(x)+d(y))} \\
&\quad \times [[\theta(x), \theta(y)], D(z)] \\
&+ (-1)^{d(x)d(y)} [[D(y), \theta(z)], \theta(x)] \\
&+ (-1)^{d(x)d(y)} (-1)^{d(D)d(y)} \\
&\quad \times [[\theta(y), D(z)], \theta(x)] \\
&+ (-1)^{d(x)d(y)} (-1)^{d(D)(d(y)+d(z))} \\
&\quad \times [[\theta(y), \theta(z)], D(x)] \\
&+ (-1)^{d(y)d(z)} [[D(z), \theta(x)], \theta(y)] \\
&+ (-1)^{d(y)d(z)} (-1)^{d(D)d(z)} [[\theta(z), D(x)], \theta(y)] \\
&+ (-1)^{d(y)d(z)} (-1)^{d(D)(d(x)+d(z))} \\
&\quad \times [[\theta(z), \theta(x)], D(y)] \\
&= (-1)^{d(D)d(z)} \\
&\quad \times \left((-1)^{d(z)(d(D)+d(x))} [[D(x), \theta(y)], \theta(z)] \right. \\
&\quad \quad + (-1)^{d(y)(d(D)+d(x))} [[\theta(y), \theta(z)], D(x)] \\
&\quad \quad + (-1)^{d(y)d(z)} [[\theta(z), D(x)], \theta(y)] \\
&\quad \quad + (-1)^{d(D)d(x)} \\
&\quad \quad \times \left((-1)^{d(x)(d(D)+d(y))} [[D(y), \theta(z)], \theta(x)] \right. \\
&\quad \quad \quad + (-1)^{d(z)(d(D)+d(y))} [[\theta(z), \theta(x)], D(y)] \\
&\quad \quad \quad + (-1)^{d(x)d(z)} [[\theta(x), D(y)], \theta(z)] \\
&\quad \quad \quad + (-1)^{d(D)d(y)} \\
&\quad \quad \quad \times \left((-1)^{d(y)(d(D)+d(z))} [[D(z), \theta(x)], \theta(y)] \right. \\
&\quad \quad \quad \quad + (-1)^{d(x)(d(D)+d(z))} [[\theta(x), \theta(y)], D(z)] \\
&\quad \quad \quad \quad + (-1)^{d(x)d(y)} [[\theta(y), D(z)], \theta(x)] \left. \right) = 0. \tag{17}
\end{aligned}$$

Therefore, D is a θ -prederivation of L by Theorem 5. $\square$

Theorem 7. D is a $(\theta, \varphi)_2$ -prederivation of L if and only if D is a Jordan $(\theta, \varphi)_2$ -prederivation of L such that

$$\begin{aligned}
&\text{(i)} \quad [[D(x), \theta(y)], (\varphi - \theta)(z)] = (-1)^{d(D)d(x)} [[\theta(x), D(y)], \\
&\quad \quad (\varphi - \theta)(z)], \\
&\text{(ii)} \quad (-1)^{d(x)d(z)} A'(x, y, z) + (-1)^{d(x)d(y)} A'(y, z, x) + \\
&\quad \quad (-1)^{d(y)d(z)} A'(z, x, y) = 0,
\end{aligned}$$

where $x, y, z \in L$ and $A'(x, y, z) = [[D(x), \theta(y)], \theta(z)] + (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] + (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \varphi(y)], D(z)]$.

Proof. Let D be a $(\theta, \varphi)_2$ -prederivation of L . Use the fact that

$$D([x, y], z) = -(-1)^{d(x)d(y)} D([y, x], z), \quad (18)$$

as well as the fact that

$$\begin{aligned} & -(-1)^{d(x)d(y)} D([y, x], z) \\ &= -(-1)^{d(x)d(y)} [[D(y), \theta(x)], \theta(z)] \\ & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)d(y)} [[\theta(y), D(x)], \varphi(z)] \\ & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\ & \quad \times [[\varphi(y), \varphi(x)], D(z)] \\ &= [[D(x), \theta(y)], \varphi(z)] + (-1)^{d(D)d(x)} \\ & \quad \times [[\theta(x), D(y)], \theta(z)] \\ & \quad + (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \varphi(y)], D(z)]; \end{aligned} \quad (19)$$

then we have

$$\begin{aligned} & [[D(x), \theta(y)], \theta(z)] \\ & \quad + (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] \\ &= [[D(x), \theta(y)], \varphi(z)] \\ & \quad + (-1)^{d(D)d(x)} [[\theta(x), D(y)], \theta(z)]; \end{aligned} \quad (20)$$

that is, $[[D(x), \theta(y)], (\varphi - \theta)(z)] = (-1)^{d(D)d(x)} [[\theta(x), D(y)], (\varphi - \theta)(z)]$. It is routine to prove (ii).

Suppose, conversely, that D is a Jordan $(\theta, \varphi)_2$ -prederivation of L satisfying (i) and (ii). Note that

$$\begin{aligned} & A'(y, x, z) \\ &= [[D(y), \theta(x)], \theta(z)] \\ & \quad + (-1)^{d(D)d(y)} [[\theta(y), D(x)], \varphi(z)] \\ & \quad + (-1)^{d(D)(d(x)+d(y))} [[\varphi(y), \varphi(x)], D(z)] \\ &= -(-1)^{d(x)d(y)} [[D(x), \theta(y)], \varphi(z)] \end{aligned}$$

$$\begin{aligned} & -(-1)^{d(x)d(y)} (-1)^{d(D)d(x)} [[\theta(x), D(y)], \theta(z)] \\ & - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\ & \quad \times [[\varphi(x), \varphi(y)], D(z)] \\ &= -(-1)^{d(x)d(y)} [[D(x), \theta(y)], \theta(z)] \\ & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)d(x)} [[\theta(x), D(y)], \varphi(z)] \\ & \quad - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\ & \quad \times [[\varphi(x), \varphi(y)], D(z)] \\ &= -(-1)^{d(x)d(y)} A'(x, y, z). \end{aligned} \quad (21)$$

In the same way, we can get equalities (*) and (**). The rest of the proof is the same as the corresponding proof of Theorem 5. $\square$

A similar argument proves the following result.

Theorem 8. D is a $(\theta, \varphi)_3$ -prederivation of L if and only if D is a Jordan $(\theta, \varphi)_3$ -prederivation of L such that

$$\begin{aligned} & \text{(i) } [[D(x), (\theta - \varphi)(y)], \theta(z)] = (-1)^{d(D)d(x)} [(\theta - \varphi)(x), D(y)], \theta(z)], \\ & \text{(ii) } (-1)^{d(x)d(z)} A''(x, y, z) + (-1)^{d(x)d(y)} A''(y, z, x) + \\ & \quad (-1)^{d(y)d(z)} A''(z, x, y) = 0, \end{aligned}$$

where $x, y, z \in L$ and $A''(x, y, z) = [[D(x), \theta(y)], \theta(z)] + (-1)^{d(D)d(x)} [[\varphi(x), D(y)], \theta(z)] + (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \varphi(y)], D(z)]$.

Remark 9. Corollary 6 can also be concluded from Theorem 7 or Theorem 8 since, for any $x, y, z \in L$, $A(x, y, z) = A'(x, y, z) = A''(x, y, z)$ when D is a Jordan θ -prederivation.

Definition 10. Let L be a Lie superalgebra.

- (1) A generalized $(\theta, \varphi)_1$ -prederivation with respect to a $(\theta, \varphi)_1$ -prederivation δ_1 is a homogeneous linear mapping $D_1 : L \rightarrow L$ such that $d(D_1) = d(\delta_1)$ and

$$\begin{aligned} D_1([x, y], z) &= [[\delta_1(x), \theta(y)], \varphi(z)] \\ & \quad + (-1)^{d(D_1)d(x)} [[\theta(x), \delta_1(y)], \varphi(z)] \\ & \quad + (-1)^{d(D_1)(d(x)+d(y))} \\ & \quad \times [[\theta(x), \varphi(y)], D_1(z)], \quad \forall x, y, z \in L. \end{aligned} \quad (22)$$

- (2) A generalized $(\theta, \varphi)_2$ -prederivation with respect to a $(\theta, \varphi)_2$ -prederivation δ_2 is a homogeneous linear mapping $D_2 : L \rightarrow L$ such that $d(D_2) = d(\delta_2)$ and

$$\begin{aligned} D_2([x, y], z) &= [[\delta_2(x), \theta(y)], \theta(z)] \\ &+ (-1)^{d(D_2)d(x)} [[\theta(x), \delta_2(y)], \varphi(z)] \\ &+ (-1)^{d(D_2)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D_2(z)], \quad \forall x, y, z \in L. \end{aligned} \quad (23)$$

- (3) A generalized $(\theta, \varphi)_3$ -prederivation with respect to a $(\theta, \varphi)_3$ -prederivation δ_3 is a homogeneous linear mapping $D_3 : L \rightarrow L$ such that $d(D_3) = d(\delta_3)$ and

$$\begin{aligned} D_3([x, y], z) &= [[\delta_3(x), \theta(y)], \theta(z)] \\ &+ (-1)^{d(D_3)d(x)} [[\varphi(x), \delta_3(y)], \theta(z)] \\ &+ (-1)^{d(D_3)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D_3(z)], \quad \forall x, y, z \in L. \end{aligned} \quad (24)$$

In particular, $\forall i = 1, 2, 3$, a generalized $(\theta, \varphi)_i$ -prederivation D is called a generalized θ -prederivation with respect to a θ -prederivation δ if $\theta = \varphi$. It is clear that D is a generalized prederivation when $\theta = \varphi = 1_T$ and δ is a prederivation.

Definition 11. Let L be a Lie superalgebra.

- (1) A generalized Jordan $(\theta, \varphi)_1$ -prederivation with respect to a Jordan $(\theta, \varphi)_1$ -prederivation δ_1 is a homogeneous linear mapping $D_1 : L \rightarrow L$ such that $d(D_1) = d(\delta_1)$ and

$$\begin{aligned} D_1([x, y], x) &= [[\delta_1(x), \theta(y)], \varphi(x)] \\ &+ (-1)^{d(D_1)d(x)} [[\theta(x), \delta_1(y)], \varphi(x)] \\ &+ (-1)^{d(D_1)(d(x)+d(y))} \\ &\times [[\theta(x), \varphi(y)], D_1(x)], \quad \forall x, y \in L. \end{aligned} \quad (25)$$

- (2) A generalized Jordan $(\theta, \varphi)_2$ -prederivation with respect to a Jordan $(\theta, \varphi)_2$ -prederivation δ_2 is a homogeneous linear mapping $D_2 : L \rightarrow L$ such that $d(D_2) = d(\delta_2)$ and

$$\begin{aligned} D_2([x, y], x) &= [[\delta_2(x), \theta(y)], \theta(x)] \\ &+ (-1)^{d(D_2)d(x)} [[\theta(x), \delta_2(y)], \varphi(x)] \\ &+ (-1)^{d(D_2)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D_2(x)], \quad \forall x, y \in L. \end{aligned} \quad (26)$$

- (3) A generalized Jordan $(\theta, \varphi)_3$ -prederivation with respect to a Jordan $(\theta, \varphi)_3$ -prederivation δ_3 is a homogeneous linear mapping $D_3 : L \rightarrow L$ such that $d(D_3) = d(\delta_3)$ and

$$\begin{aligned} D_3([x, y], x) &= [[\delta_3(x), \theta(y)], \theta(x)] \\ &+ (-1)^{d(D_3)d(x)} [[\varphi(x), \delta_3(y)], \theta(x)] \\ &+ (-1)^{d(D_3)(d(x)+d(y))} \\ &\times [[\varphi(x), \varphi(y)], D_3(x)], \quad \forall x, y \in L. \end{aligned} \quad (27)$$

In particular, $\forall i = 1, 2, 3$, a generalized Jordan $(\theta, \varphi)_i$ -prederivation D is called a generalized Jordan θ -prederivation with respect to a Jordan θ -prederivation δ if $\theta = \varphi$. It is clear that D is a generalized Jordan prederivation when $\theta = \varphi = 1_T$ and δ is a Jordan prederivation.

Theorem 12. D is a generalized $(\theta, \varphi)_1$ -prederivation of L with respect to a $(\theta, \varphi)_1$ -prederivation δ if and only if D is a generalized Jordan $(\theta, \varphi)_1$ -prederivation of L with respect to a $(\theta, \varphi)_1$ -prederivation δ such that

$$(i) [[\theta(x), \varphi(y)], D(z)] = [[\varphi(x), \theta(y)], D(z)],$$

$$(ii) (-1)^{d(x)d(z)} B(x, y, z) + (-1)^{d(x)d(y)} B(y, z, x) + (-1)^{d(y)d(z)} B(z, x, y) = 0,$$

where $x, y, z \in L$ and $B(x, y, z) = [[\delta(x), \theta(y)], \varphi(z)] + (-1)^{d(D)d(x)} [[\theta(x), \delta(y)], \varphi(z)] + (-1)^{d(D)(d(x)+d(y))} [[\theta(x), \varphi(y)], D(z)]$.

Proof. Suppose that D is a generalized $(\theta, \varphi)_1$ -prederivation of L with respect to a $(\theta, \varphi)_1$ -prederivation δ . Clearly, D is a generalized Jordan $(\theta, \varphi)_1$ -prederivation of L and $D([x, y, z]) = -(-1)^{d(x)d(y)} D([y, x, z])$. (i) follows from the fact that

$$\begin{aligned} &-(-1)^{d(x)d(y)} D([y, x], z) \\ &= -(-1)^{d(x)d(y)} [[\delta(y), \theta(x)], \varphi(z)] \\ &-(-1)^{d(x)d(y)} (-1)^{d(D)d(y)} [[\theta(y), \delta(x)], \varphi(z)] \\ &-(-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\ &\times [[\theta(y), \varphi(x)], D(z)] \\ &= [[\delta(x), \theta(y)], \varphi(z)] + (-1)^{d(D)d(x)} \\ &\times [[\theta(x), \delta(y)], \varphi(z)] \\ &+ (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \theta(y)], D(z)]. \end{aligned} \quad (28)$$

Since D is a generalized $(\theta, \varphi)_1$ -prederivation of L with respect to δ , we have $D([x, y], z) = B(x, y, z)$; hence

$$\begin{aligned}
 & (-1)^{d(x)d(z)} B(x, y, z) + (-1)^{d(x)d(y)} B(y, z, x) \\
 & + (-1)^{d(y)d(z)} B(z, x, y) \\
 & = (-1)^{d(x)d(z)} D([x, y], z) \\
 & + (-1)^{d(x)d(y)} D([y, z], x) \\
 & + (-1)^{d(y)d(z)} D([z, x], y) \\
 & = D((-1)^{d(x)d(z)} [[x, y], z] + (-1)^{d(x)d(y)} [[y, z], x] \\
 & + (-1)^{d(y)d(z)} [[z, x], y]) = 0.
 \end{aligned} \tag{29}$$

Conversely, if D is a generalized Jordan $(\theta, \varphi)_1$ -prederivation of L with respect to a $(\theta, \varphi)_1$ -prederivation δ satisfying (i) and (ii), then refer to the proof of Theorem 5; it suffices to prove $B(x, y, z) = -(-1)^{d(x)d(y)} B(y, x, z)$. In fact,

$$\begin{aligned}
 & B(y, x, z) \\
 & = [[\delta(y), \theta(x)], \varphi(z)] \\
 & + (-1)^{d(D)d(y)} [[\theta(y), \delta(x)], \varphi(z)] \\
 & + (-1)^{d(D)(d(x)+d(y))} [[\theta(y), \varphi(x)], D(z)] \\
 & = -(-1)^{d(x)d(y)} [[\delta(x), \theta(y)], \varphi(z)] \\
 & - (-1)^{d(x)d(y)} (-1)^{d(D)d(x)} [[\theta(x), \delta(y)], \varphi(z)] \\
 & - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\
 & \times [[\varphi(x), \theta(y)], D(z)] \\
 & = -(-1)^{d(x)d(y)} [[\delta(x), \theta(y)], \varphi(z)] \\
 & - (-1)^{d(x)d(y)} (-1)^{d(D)d(x)} \\
 & \times [[\theta(x), \delta(y)], \varphi(z)] \\
 & - (-1)^{d(x)d(y)} (-1)^{d(D)(d(x)+d(y))} \\
 & \times [[\theta(x), \varphi(y)], D(z)] \\
 & = -(-1)^{d(x)d(y)} B(x, y, z).
 \end{aligned} \tag{30}$$

This completes the proof. $\square$

Corollary 13. D is a generalized θ -prederivation of L with respect to a θ -prederivation δ if and only if D is a generalized

Jordan θ -prederivation of L with respect to a Jordan θ -prederivation δ such that

$$\begin{aligned}
 & (-1)^{d(z)(d(D)+d(x))} [[\theta(x), \theta(y)], (D - \delta)(z)] \\
 & + (-1)^{d(x)(d(D)+d(y))} [[\theta(y), \theta(z)], (D - \delta)(x)] \\
 & + (-1)^{d(y)(d(D)+d(z))} [[\theta(z), \theta(x)], (D - \delta)(y)] = 0.
 \end{aligned} \tag{31}$$

Proof. If D is a generalized Jordan θ -prederivation of L , then (i) follows immediately. (ii) holds because

$$\begin{aligned}
 & (-1)^{d(x)d(z)} B(x, y, z) + (-1)^{d(x)d(y)} B(y, z, x) \\
 & + (-1)^{d(y)d(z)} B(z, x, y) \\
 & = (-1)^{d(x)d(z)} [[\delta(x), \theta(y)], \theta(z)] \\
 & + (-1)^{d(x)d(z)} (-1)^{d(D)d(x)} [[\theta(x), \delta(y)], \theta(z)] \\
 & + (-1)^{d(x)d(z)} (-1)^{d(D)(d(x)+d(y))} [[\theta(x), \theta(y)], D(z)] \\
 & + (-1)^{d(x)d(y)} [[\delta(y), \theta(z)], \theta(x)] \\
 & + (-1)^{d(x)d(y)} (-1)^{d(D)d(y)} [[\theta(y), \delta(z)], \theta(x)] \\
 & + (-1)^{d(x)d(y)} (-1)^{d(D)(d(y)+d(z))} [[\theta(y), \theta(z)], D(x)] \\
 & + (-1)^{d(y)d(z)} [[\delta(z), \theta(x)], \theta(y)] \\
 & + (-1)^{d(y)d(z)} (-1)^{d(D)d(z)} [[\theta(z), \delta(x)], \theta(y)] \\
 & + (-1)^{d(y)d(z)} (-1)^{d(D)(d(x)+d(z))} [[\theta(z), \theta(x)], D(y)] \\
 & = (-1)^{d(x)d(z)} [[\delta(x), \theta(y)], \theta(z)] \\
 & + (-1)^{d(x)d(z)} (-1)^{d(D)d(x)} [[\theta(x), \delta(y)], \theta(z)] \\
 & + (-1)^{d(x)d(z)} (-1)^{d(D)(d(x)+d(y))} [[\theta(x), \theta(y)], \delta(z)] \\
 & + (-1)^{d(x)d(y)} [[\delta(y), \theta(z)], \theta(x)] \\
 & + (-1)^{d(x)d(y)} (-1)^{d(D)d(y)} [[\theta(y), \delta(z)], \theta(x)] \\
 & + (-1)^{d(x)d(y)} (-1)^{d(D)(d(y)+d(z))} [[\theta(y), \theta(z)], \delta(x)] \\
 & + (-1)^{d(y)d(z)} [[\delta(z), \theta(x)], \theta(y)] \\
 & + (-1)^{d(y)d(z)} (-1)^{d(D)d(z)} [[\theta(z), \delta(x)], \theta(y)] \\
 & + (-1)^{d(y)d(z)} (-1)^{d(D)(d(x)+d(z))} [[\theta(z), \theta(x)], \delta(y)] \\
 & = (-1)^{d(D)d(z)}
 \end{aligned}$$

$$\begin{aligned}
& \times (-1)^{d(z)(d(D)+d(x))} [[\delta(x), \theta(y)], \theta(z)] \\
& + (-1)^{d(y)(d(D)+d(x))} [[\theta(y), \theta(z)], \delta(x)] \\
& + (-1)^{d(y)d(z)} [[\theta(z), \delta(x)], \theta(y)] \\
& + (-1)^{d(D)d(x)} \\
& \times (-1)^{d(x)(d(D)+d(y))} [[\delta(y), \theta(z)], \theta(x)] \\
& + (-1)^{d(z)(d(D)+d(y))} [[\theta(z), \theta(x)], \delta(y)] \\
& + (-1)^{d(x)d(z)} [[\theta(x), \delta(y)], \theta(z)] \\
& + (-1)^{d(D)d(y)} \\
& \times (-1)^{d(y)(d(D)+d(z))} [[\delta(z), \theta(x)], \theta(y)] \\
& + (-1)^{d(x)(d(D)+d(z))} [[\theta(x), \theta(y)], \delta(z)] \\
& + (-1)^{d(x)d(y)} [[\theta(y), \delta(z)], \theta(x)] = 0.
\end{aligned} \tag{32}$$

Note that δ is a θ -prederivation by Corollary 6. Therefore, D is a generalized θ -prederivation of L with respect to a θ -prederivation δ by Theorem 12. $\square$

As before, one can prove the following theorems.

Theorem 14. D is a generalized $(\theta, \varphi)_2$ -prederivation of L with respect to a $(\theta, \varphi)_2$ -prederivation δ if and only if D is a generalized Jordan $(\theta, \varphi)_2$ -prederivation of L with respect to a $(\theta, \varphi)_2$ -prederivation δ such that

$$\begin{aligned}
& \text{(i)} \quad [[\delta(x), \theta(y)], (\varphi - \theta)(z)] = (-1)^{d(\delta)d(x)} [[\theta(x), \delta(y)], \\
& \quad (\varphi - \theta)(z)], \\
& \text{(ii)} \quad (-1)^{d(x)d(z)} B'(x, y, z) + (-1)^{d(x)d(y)} B'(y, z, x) + \\
& \quad (-1)^{d(y)d(z)} B'(z, x, y) = 0,
\end{aligned}$$

where $x, y, z \in L$ and $B'(x, y, z) = [[\delta(x), \theta(y)], \theta(z)] + (-1)^{d(D)d(x)} [[\theta(x), \delta(y)], \varphi(z)] + (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \varphi(y)], D(z)]$.

Theorem 15. D is a generalized $(\theta, \varphi)_3$ -prederivation of L with respect to a $(\theta, \varphi)_3$ -prederivation δ if and only if D is a generalized Jordan $(\theta, \varphi)_3$ -prederivation of L with respect to a $(\theta, \varphi)_3$ -prederivation δ such that

$$\begin{aligned}
& \text{(i)} \quad [[\delta(x), (\theta - \varphi)(y)], \theta(z)] = (-1)^{d(\delta)d(x)} [[(\theta - \varphi)(x), \\
& \quad \delta(y)], \theta(z)], \\
& \text{(ii)} \quad (-1)^{d(x)d(z)} B''(x, y, z) + (-1)^{d(x)d(y)} B''(y, z, x) + \\
& \quad (-1)^{d(y)d(z)} B''(z, x, y) = 0,
\end{aligned}$$

where $x, y, z \in L$ and $B''(x, y, z) = [[\delta(x), \theta(y)], \theta(z)] + (-1)^{d(\delta)d(x)} [[\varphi(x), \delta(y)], \theta(z)] + (-1)^{d(D)(d(x)+d(y))} [[\varphi(x), \varphi(y)], D(z)]$.

Remark 16. Corollary 13 can also be concluded from Theorem 14 or Theorem 15 since, for any $x, y, z \in L$, $B(x, y, z) = B'(x, y, z) = B''(x, y, z)$ when D is a generalized Jordan θ -prederivation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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