Research Article

Exact Solutions of the Space Time Fractional Symmetric Regularized Long Wave Equation Using Different Methods

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We apply the functional variable method, exp-function method, and (G'/G) -expansion method to establish the exact solutions of the nonlinear fractional partial differential equation (NLFPDE) in the sense of the modified Riemann-Liouville derivative. As a result, some new exact solutions for them are obtained. The results show that these methods are very effective and powerful mathematical tools for solving nonlinear fractional equations arising in mathematical physics. As a result, these methods can also be applied to other nonlinear fractional differential equations.

1. Introduction

Fractional calculus is a field of mathematics that grows out of the traditional definitions of calculus. Fractional calculus has gained importance during the last decades mainly due to its applications in various areas of physics, biology, mathematics, and engineering. Some of the current application fields of fractional calculus include fluid flow, dynamical process in self-similar and porous structures, electrical networks, probability and statistics, control theory of dynamical systems, systems identification, acoustics, viscoelasticity, control theory, electrochemistry of corrosion, chemical physics, finance, optics, and signal processing [1–3].

There are several definitions of the fractional derivative which are generally not equivalent to each other. Some of these definitions are Sun and Chen's fractal derivative [4, 5], Cresson's derivative [6, 7], Grünwald-Letnikov's fractional derivative [8], Riemann-Liouville's derivative [8], and Caputo's fractional derivative [9]. But the Riemann-Liouville derivative and the Caputo derivative are the most used ones.

Lately, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear fractional differential equations. Many powerful methods have been presented. Among them are the fractional (G'/G) -expansion method [10–13], the fractional

exp-function method [14–16], the fractional first integral method [17, 18], the fractional subequation method [19–22], the fractional functional variable method [23], the fractional modified trial equation method [24, 25], and the fractional simplest equation method [26].

The paper suggests the functional variable method, the exp-function method, the (G'/G) -expansion method, and fractional complex transform to find the exact solutions of nonlinear fractional partial differential equation with the modified Riemann-Liouville derivative.

This paper is organized as follows. In Section 2, basic definitions of Jumarie's Riemann-Liouville derivative are given; in Section 3, description of the methods for FDEs is given. Then, in Section 4, these methods have been applied to establish exact solutions for the space-time fractional symmetric regularized long wave (SRLW) equation. Conclusion is given in Section 5.

2. Jumarie's Modified Riemann-Liouville Derivative

Recently, a new modified Riemann-Liouville derivative is proposed by Jumarie [27, 28]. This new definition of fractional derivative has two main advantages: firstly, comparing

with the Caputo derivative, the function to be differentiated is not necessarily differentiable; secondly, different from the Riemann-Liouville derivative, Jumarie's modified Riemann-Liouville derivative of a constant is defined to be zero. Jumarie's modified Riemann-Liouville derivative of order α is defined by

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \times \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1 \\ (f^{(n)}(x))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1, \end{cases} \quad (1)$$

where $f: R \rightarrow R$, $x \rightarrow f(x)$ denotes a continuous (but not necessarily first-order-differentiable) function. Some useful formulas and results of Jumarie's modified Riemann-Liouville derivative can be found in [28, 29]

$$D_x^\alpha x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} x^{r-\alpha}, \quad (2)$$

$$D_x^\alpha (u(x) v(x)) = v(x) D_x^\alpha u(x) + u(x) D_x^\alpha v(x), \quad (3)$$

$$D_x^\alpha f[u(x)] = f'_u(u) D_x^\alpha u(x), \quad (4)$$

$$D_x^\alpha f[u(x)] = D_u^\alpha f(u) (u'(x))^\alpha, \quad (5)$$

which are direct consequences of the equality

$$\Gamma(1+\alpha) dx = d^\alpha x. \quad (6)$$

In the above formulas (3)–(5), $u(x)$ is nondifferentiable function in (3) and (4) and differentiable in (5). The function $v(x)$ is nondifferentiable, and $f(u)$ is differentiable in (4) and nondifferentiable in (5). Because of these, the formulas (3)–(5) should be used carefully. The above equations play an important role in fractional calculus in Sections 3 and 4.

3. Description of the Methods for FDEs

We consider the following general nonlinear FDEs of the type

$$P(u, D_t^\alpha u, D_x^\beta u, D_y^\psi u, D_t^\alpha D_t^\alpha u, D_t^\alpha D_x^\beta u, D_x^\beta D_x^\beta u, D_x^\beta D_y^\psi u, D_y^\psi D_y^\psi u, \dots) = 0, \quad (7)$$

$$0 < \alpha, \beta, \psi < 1,$$

where u is an unknown function. P is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved.

The fractional complex transform [30–32] is the simplest approach to convert the fractional differential equations

into ordinary differential equations. This makes the solution procedure extremely simple. The traveling wave variable is

$$u(x, y, t) = U(\xi), \quad (8)$$

where

$$\xi = \frac{\tau x^\beta}{\Gamma(1+\beta)} + \frac{\delta y^\psi}{\Gamma(1+\psi)} + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \quad (9)$$

where τ , δ , and λ are nonzero arbitrary constants. We can rewrite (7) in the following nonlinear ODE:

$$Q(U, U', U'', U''', \dots) = 0, \quad (10)$$

where the prime denotes the derivation with respect to ξ . Now we consider three different methods.

3.1. Basic Idea of Functional Variable Method. The features of this method are presented in [33]. We describe functional variable method to find exact solutions of nonlinear space-time fractional differential equations as follows.

Let us make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_\xi = F(U) \quad (11)$$

and some successive derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \frac{1}{2} (F^2)', \\ U_{\xi\xi\xi} &= \frac{1}{2} (F^2)'' \sqrt{F^2}, \\ U_{\xi\xi\xi\xi} &= \frac{1}{2} [(F^2)''' F^2 + (F^2)'' (F^2)'], \\ &\vdots \end{aligned} \quad (12)$$

where “ $'$ ” stands for d/dU . The ODE (10) can be reduced in terms of U , F , and its derivatives by using the expressions of (12) into (10) as

$$R(U, F, F', F'', F''', F^{(4)}, \dots) = 0. \quad (13)$$

The key idea of this particular form (13) is of special interest since it admits analytical solutions for a large class of nonlinear wave type equations. Integrating (13) gives the expression of F . This and (11) give the appropriate solutions to the original problem.

3.2. Basic Idea of Exp-Function Method. According to exp-function method, developed by He and Abdou [34], we assume that the wave solution can be expressed in the following form:

$$U(\xi) = \frac{\sum_{n=-c}^d a_n \exp[n\xi]}{\sum_{m=-p}^q b_m \exp[m\xi]}, \quad (14)$$

where p, q, c , and d are positive integers which are known to be further determined and a_n and b_m are unknown constants. We can rewrite (14) in the following equivalent form:

$$U(\xi) = \frac{a_{-c} \exp[-c\xi] + \cdots + a_d \exp[d\xi]}{b_{-p} \exp[-p\xi] + \cdots + b_q \exp[q\xi]}. \quad (15)$$

This equivalent formulation plays an important and fundamental part in finding the analytic solution of problems. To determine the value of c and p , we balance the linear term of highest order of (10) with the highest order nonlinear term. Similarly, to determine the value of d and q , we balance the linear term of lowest order of (10) with lowest order nonlinear term [35–40].

3.3. Basic Idea of (G'/G) -Expansion Method. According to (G'/G) -expansion method, developed by Wang et al. [41], the solution of (10) can be expressed by a polynomial in (G'/G) as

$$U(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G} \right)^i, \quad a_m \neq 0, \quad (16)$$

where a_i ($i = 0, 1, 2, \dots, m$) are constants, while $G(\xi)$ satisfies the following second-order linear ordinary differential equation:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (17)$$

where λ and μ are constants. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in (10). By substituting (16) into (10) and using (17) we collect all terms with the same order of (G'/G) . Then by equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i ($i = 0, 1, 2, \dots, m$), λ , μ , τ , δ and λ . Finally solving the system of equations and substituting a_i ($i = 0, 1, 2, \dots, m$), λ , μ , τ , δ , λ , and the general solutions of (17) into (16), we can get a variety of exact solutions of (7) [42, 43].

4. Exact Solutions of Space-Time Fractional Symmetric Regularized Long Wave (SRLW) Equation

We consider the space-time fractional symmetric regularized long wave (SRLW) equation [44]

$$\begin{aligned} D_t^{2\alpha} u + D_x^{2\alpha} u + u D_t^\alpha (D_x^\alpha u) \\ + D_x^\alpha u D_t^\alpha u + D_t^{2\alpha} (D_x^{2\alpha} u) = 0, \end{aligned} \quad (18)$$

$$0 < \alpha \leq 1$$

which arises in several physical applications including ion sound waves in plasma. For $\alpha = 1$, it is shown that this equation describes weakly nonlinear ion acoustic and space-charge waves, and the real-valued $u(x, t)$ corresponds

to the dimensionless fluid velocity with a decay condition [45].

We use the following transformations:

$$u(x, t) = U(\xi), \quad (19)$$

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)}, \quad (20)$$

where k and c are nonzero constants.

Substituting (20) with (1) into (18), equation (18) can be reduced into an ODE:

$$(c^2 + k^2)U'' + ckUU'' + ck(U')^2 + c^2k^2U'''' = 0, \quad (21)$$

where " U' " = $dU/d\xi$.

4.1. Exact Solutions by Functional Variable Method. Integrating (21) twice and setting the constants of integration to be zero, we obtain

$$(c^2 + k^2)U + ck\frac{U^2}{2} + c^2k^2U'' = 0 \quad (22)$$

or

$$U_{\xi\xi} = -\frac{c^2 + k^2}{c^2k^2}U - \frac{U^2}{2ck}. \quad (23)$$

Then we use the transformation (11) and (12) to convert (22) to

$$\begin{aligned} \frac{1}{2}(F^2)' &= -\frac{c^2 + k^2}{c^2k^2}U - \frac{U^2}{2ck}, \\ F(U) &= \mp U \sqrt{-\frac{c^2 + k^2}{c^2k^2} - \frac{U}{3ck}}. \end{aligned} \quad (24)$$

The solution of (21) is obtained as

$$U(\xi) = -\frac{3(c^2 + k^2)}{ck} \sec^2 \left(\frac{\sqrt{c^2 + k^2}}{2kc} \xi \right). \quad (25)$$

So we have

$$\begin{aligned} u_1(x, t) &= -\frac{3(c^2 + k^2)}{ck} \\ &\times \sec^2 \left\{ \frac{\sqrt{c^2 + k^2}}{2kc} \left(\frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \right) \right\}, \end{aligned} \quad (26)$$

which is the exact solution of space-time fractional symmetric regularized long wave (SRLW) equation. One can see that the result is different than results of Alzaidy [44].

4.2. *Exact Solutions by Exp-Function Method.* Balancing the order of U'' and U^2 in (22), we obtain

$$U'' = \frac{c_1 \exp[-(c+3p)\xi] + \dots}{c_2 \exp[-4p\xi] + \dots},$$

$$U^2 = \frac{c_3 \exp[-2c\xi] + \dots}{c_4 \exp[-2p\xi] + \dots}, \quad (27)$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in (27) we have

$$-(c+3p) = -(2c+2p), \quad (28)$$

which leads to the result:

$$p = c. \quad (29)$$

In the same way, we balance the linear term of the lowest order in (22), to determine the values of d and q

$$U'' = \frac{\dots + d_1 \exp[(d+3q)\xi]}{\dots + d_2 \exp[4q\xi]}, \quad (30)$$

$$U^2 = \frac{\dots + d_3 \exp[2d\xi]}{\dots + d_4 \exp[2q\xi]},$$

where d_i are determined coefficients only for simplicity. From (30), we have

$$3q + d = 2d + 2q, \quad (31)$$

and this gives

$$q = d. \quad (32)$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so (15) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \quad (33)$$

Substituting (33) into (22) and using Maple, we obtain

$$\frac{1}{A} [R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) + R_{-3} \exp(-3\xi)] = 0, \quad (34)$$

where

$$A = (b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi))^3,$$

$$R_3 = k^2 a_1 b_1^2 + c^2 a_1 b_1^2 + \frac{1}{2} c k a_1^2 b_1,$$

$$R_2 = k^2 a_0 b_1^2 + c^2 a_0 b_1^2 - c^2 k^2 a_1 b_1 b_0 + c k a_1 a_0 b_1 + 2c^2 a_1 b_1 b_0 + \frac{1}{2} c k a_1^2 b_0$$

$$+ c^2 k^2 a_0 b_1^2 + 2k^2 a_1 b_1 b_0,$$

$$R_1 = 2k^2 a_0 b_1 b_0 + c^2 a_1 b_0^2 + c^2 a_{-1} b_1^2 + k^2 a_{-1} b_1^2 - c^2 k^2 a_0 b_1 b_0 + c k a_1 a_{-1} b_1 - 4c^2 k^2 a_1 b_1 b_{-1}$$

$$+ c k a_1 a_0 b_0 + k^2 a_1 b_0^2 + 2c^2 a_0 b_1 b_0 + \frac{1}{2} c k a_0^2 b_1$$

$$+ 2k^2 a_1 b_1 b_{-1} + c^2 k^2 a_1 b_0^2 + 4c^2 k^2 a_{-1} b_1^2$$

$$+ 2c^2 a_1 b_1 b_{-1} + \frac{1}{2} c k a_1^2 b_{-1},$$

$$R_0 = 2k^2 a_0 b_1 b_{-1} + 2c^2 a_{-1} b_1 b_0 + 2k^2 a_1 b_0 b_{-1}$$

$$+ 2k^2 a_{-1} b_1 b_0 + 2c^2 a_1 b_0 b_{-1} + 2c^2 a_0 b_1 b_{-1}$$

$$+ 3c^2 k^2 a_{-1} b_0 b_1 + c k a_1 a_{-1} b_0 + c k a_0 a_{-1} b_1$$

$$+ 3c^2 k^2 a_1 b_0 b_{-1} + \frac{1}{2} c k a_0^2 b_0 - 6c^2 k^2 a_0 b_1 b_{-1}$$

$$+ k^2 a_0 b_0^2 + c^2 a_0 b_0^2 + c k a_1 a_0 b_{-1},$$

$$R_{-1} = k^2 a_1 b_{-1}^2 + c^2 a_1 b_{-1}^2 + c^2 a_{-1} b_0^2 + k^2 a_{-1} b_0^2$$

$$- 4c^2 k^2 a_{-1} b_1 b_{-1} - c^2 k^2 a_0 b_{-1} b_0 + c k a_1 a_{-1} b_{-1}$$

$$+ c k a_0 a_{-1} b_0 + \frac{1}{2} c k a_{-1}^2 b_1 + 2k^2 a_{-1} b_1 b_{-1}$$

$$+ 2k^2 a_0 b_0 b_{-1} + c^2 k^2 a_{-1} b_0^2 + \frac{1}{2} c k a_0^2 b_{-1}$$

$$+ 2c^2 a_0 b_0 b_{-1} + 2c^2 a_{-1} b_1 b_{-1} + 4c^2 k^2 a_1 b_{-1}^2,$$

$$R_{-2} = k^2 a_0 b_{-1}^2 + c^2 a_0 b_{-1}^2 - c^2 k^2 a_{-1} b_0 b_{-1} + c k a_0 a_{-1} b_{-1}$$

$$+ \frac{1}{2} c k a_{-1}^2 b_0 + c^2 k^2 a_0 b_{-1}^2 + 2c^2 a_{-1} b_0 b_{-1}$$

$$+ 2k^2 a_{-1} b_0 b_{-1},$$

$$R_{-3} = c^2 a_{-1} b_{-1}^2 + k^2 a_{-1} b_{-1}^2 + \frac{1}{2} c k a_{-1}^2 b_{-1}.$$

Solving this system of algebraic equations by using Maple, we get the following results:

$$\begin{aligned} a_1 &= 0, & a_0 &= \mp \frac{6k^2 b_0}{\sqrt{-k^2 - 1}}, & a_{-1} &= 0, \\ b_1 &= \frac{b_0^2}{4b_{-1}}, & b_0 &= b_0, & b_{-1} &= b_{-1}, \\ c &= \mp \sqrt{\frac{k^2}{-k^2 - 1}}, & k &= k, \end{aligned} \quad (36)$$

where b_0 and b_{-1} are arbitrary parameters. Substituting these results into (33), we get the following exact solution:

$$U(\xi) = \mp \frac{6k^2 b_0 / \sqrt{-k^2 - 1}}{(b_0^2 / 4b_{-1}) \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}, \quad (37)$$

where b_0 and b_1 are arbitrary parameters and $\xi = (kx^\alpha / \Gamma(1 + \alpha)) \mp \sqrt{k^2 / (-k^2 - 1)} (t^\alpha / \Gamma(1 + \alpha))$.

Finally, if we take $b_{-1} = 1$ and $b_0 = 2$, (37) becomes

$$\begin{aligned} u(x, t) &= \mp \frac{6k^2}{\sqrt{-k^2 - 1}} \\ &\times \frac{1}{1 + \cosh\left((kx^\alpha / \Gamma(1 + \alpha)) \mp \sqrt{k^2 / (-k^2 - 1)} (t^\alpha / \Gamma(1 + \alpha))\right)} \end{aligned} \quad (38)$$

and we obtain the hyperbolic function solution of the space-time fractional symmetric regularized long wave (SRLW) equation. Comparing our result to the results in [46], it can be seen that our solution has never been obtained.

4.3. Exact Solutions by (G'/G) -Expansion Method. Recently, Zayed et al. [47] obtained solitary wave solutions to SRLW equation by means of improved (G'/G) -expansion method. But they applied this method to (22). Namely, they took the constants of integration as zero.

In our study, we integrate (21) twice with respect to ξ and we get

$$(c^2 + k^2)U + ck \frac{U^2}{2} + c^2 k^2 U'' + \xi_0 U + \xi_1 = 0, \quad (39)$$

where ξ_0 and ξ_1 are constants of integration.

Use ansatz (39), for the linear term of highest order U'' with the highest order nonlinear term U^2 . By simple calculation, balancing U'' with U^2 in (39) gives

$$m + 2 = 2m \quad (40)$$

so that

$$m = 2. \quad (41)$$

Suppose that the solutions of (41) can be expressed by a polynomial in (G'/G) as follows:

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \quad (42)$$

By using (17) and (42) we have

$$\begin{aligned} U''(\xi) &= 6b_2 \left(\frac{G'}{G}\right)^4 + (2b_1 + 10b_2 \lambda) \left(\frac{G'}{G}\right)^3 \\ &\quad + (8b_2 \mu + 3b_1 \lambda + 4b_2 \lambda^2) \left(\frac{G'}{G}\right)^2 \\ &\quad + (6b_2 \lambda \mu + 2b_1 \mu + b_1 \lambda^2) \left(\frac{G'}{G}\right) \\ &\quad + 2b_2 \mu^2 + b_1 \lambda \mu, \end{aligned} \quad (43)$$

$$\begin{aligned} U^2(\xi) &= b_2^2 \left(\frac{G'}{G}\right)^4 + 2b_1 b_2 \left(\frac{G'}{G}\right)^3 + 2b_0 b_2 \left(\frac{G'}{G}\right)^2 \\ &\quad + b_1^2 \left(\frac{G'}{G}\right)^2 + 2b_0 b_1 \left(\frac{G'}{G}\right) + b_0^2. \end{aligned}$$

Substituting (42) and (43) into (39), collecting the coefficients of $(G'/G)^i$ ($i = 0, \dots, 4$), and setting it to zero, we obtain the following system:

$$\begin{aligned} -\frac{1}{2}cka_2^2 + 6c^2k^2a_2 &= 0, \\ 2c^2k^2a_1 - cka_1a_2 + 10c^2k^2a_2\lambda &= 0, \\ -\frac{1}{2}cka_1^2 - cka_0a_2 + 8c^2k^2a_2\mu + 3c^2k^2a_1\lambda \\ &\quad + \xi_0a_2 + 4c^2k^2a_2\lambda^2 + k^2a_2 + c^2a_2 = 0, \\ -cka_0a_1 + c^2k^2a_1\lambda^2 + k^2a_1 + \xi_0a_1 + 6c^2k^2a_2\lambda\mu \\ &\quad + c^2a_1 + 2c^2k^2a_1\mu = 0, \\ -\frac{1}{2}cka_0^2 + 2c^2k^2a_2\mu^2 + c^2a_0 + \xi_0a_0 + k^2a_0 \\ &\quad + c^2k^2a_1\lambda\mu + \xi_1 = 0. \end{aligned} \quad (44)$$

Solving this system by using Maple gives

$$\begin{aligned} a_0 &= \frac{\xi_0 + c^2 + k^2 + c^2k^2\lambda^2 + 8c^2k^2\mu}{ck}, & a_1 &= 12ck\lambda, \\ a_2 &= 12ck, & c &= c, \\ k &= k, & \xi_0 &= \xi_0, \end{aligned} \quad (45)$$

$$\begin{aligned} \xi_1 &= (-2c^2k^2 - 8c^4k^4\lambda^2\mu + 16c^4k^4\mu^2 + c^4k^4\lambda^4 - k^4 \\ &\quad - c^4 - 2k^2\xi_0 - 2c^2\xi_0 - \xi_0^2)(2ck)^{-1}, \end{aligned}$$

where λ , μ , ξ_0 , and ξ_1 are arbitrary constants.

By using (42), expression (45) can be written as

$$U(\xi) = \frac{\xi_0 + c^2 + k^2 + c^2 k^2 \lambda^2 + 8c^2 k^2 \mu}{ck} + 12ck\lambda \left(\frac{G'}{G} \right) + 12ck \left(\frac{G'}{G} \right)^2. \quad (46)$$

Substituting general solutions of (17) into (46) we have three types of travelling wave solutions of space-time fractional symmetric regularized long wave (SRLW) equation. These are the following.

When $\lambda^2 - 4\mu > 0$,

$$U_1(\xi) = \frac{\xi_0 + k^2 + c^2}{ck} - 2ck(\lambda^2 - 4\mu) + 3ck(\lambda^2 - 4\mu) \times \left(\frac{C_1 \sinh(1/2) \sqrt{\lambda^2 - 4\mu\xi} + C_2 \cosh(1/2) \sqrt{\lambda^2 - 4\mu\xi}}{C_1 \cosh(1/2) \sqrt{\lambda^2 - 4\mu\xi} + C_2 \sinh(1/2) \sqrt{\lambda^2 - 4\mu\xi}} \right)^2, \quad (47)$$

where $\xi = (kx^\alpha/\Gamma(1+\alpha)) + (ct^\alpha/\Gamma(1+\alpha))$.

When $\lambda^2 - 4\mu < 0$,

$$U_2(\xi) = \frac{\xi_0 + k^2 + c^2}{ck} - 2ck(\lambda^2 - 4\mu) + 3ck(4\mu - \lambda^2) \times \left(\frac{-C_1 \sin(1/2) \sqrt{4\mu - \lambda^2\xi} + C_2 \cos(1/2) \sqrt{4\mu - \lambda^2\xi}}{C_1 \cos(1/2) \sqrt{4\mu - \lambda^2\xi} + C_2 \sin(1/2) \sqrt{4\mu - \lambda^2\xi}} \right)^2, \quad (48)$$

where $\xi = (kx^\alpha/\Gamma(1+\alpha)) + (ct^\alpha/\Gamma(1+\alpha))$.

When $\lambda^2 - 4\mu = 0$,

$$u_3(x, t) = \frac{\xi_0 + k^2 + c^2}{ck} + 12ck \times \left(\frac{C_2}{C_1 + C_2((kx^\alpha/\Gamma(1+\alpha)) + (ct^\alpha/\Gamma(1+\alpha)))} \right)^2. \quad (49)$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then U_1 and U_2 become

$$u_1(x, t) = \frac{\xi_0 + k^2 + c^2}{ck} - 2ck\lambda^2 + 3ck\lambda^2 \tanh^2 \left\{ \frac{\lambda}{2} \left(\frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \right) \right\}. \quad (50)$$

Comparing our results to Zayed's results [47], it can be seen that these results are new.

5. Conclusion

In this paper, the functional variable method, the exp-function method, and (G'/G) -expansion method have been successfully employed to obtain solution of the space-time fractional symmetric regularized long wave (SRLW) equation. These solutions include the generalized hyperbolic function solutions, generalized trigonometric function solutions, and rational function solutions, which may be very useful to understand the nonlinear FDEs and our result can turn into hyperbolic solution when suitable parameters are chosen. To the best of our knowledge, the solutions obtained in this paper have not been reported in literature. Maple has been used for programming and computations in this work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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