Research Article

Weak Convergence for a Class of Stochastic Fractional Equations Driven by Fractional Noise

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We consider a class of stochastic fractional equations driven by fractional noise on $(t, x) \in [0, T] \times [0, 1]$ $\partial u / \partial t = D_\delta^\alpha u + f(t, x, u) + \partial^2 B^H(t, x) / \partial t \partial x$, with Dirichlet boundary conditions. We formally replace the random perturbation by a family of sequences based on Kac-Stroock processes in the plane, which approximate the fractional noise in some sense. Under some conditions, we show that the real-valued mild solution of the stochastic fractional heat equation perturbed by this family of noises converges in law, in the space $\mathcal{C}([0, T] \times [0, 1])$ of continuous functions, to the solution of the stochastic fractional heat equation driven by fractional noise.

1. Introduction

In recent years, there has been considerable interest in studying fractional equations due to interesting properties and applications in various scientific areas including image analysis, risk management, and statistical mechanics (see Droniou and Imbert [1] and Uchaikin and Zolotarev [2] for a survey of applications). Much effort has been devoted to apply the fractional calculus to mathematical problems in science and engineering. For example, Chen et al. [3] and Li et al. [4] studied the fractional-order networks and Li [5] investigated fractal time series. More works on the fields can be found in [6–12] and the references therein. Stochastic partial differential equation involving fractional Laplacian operator (which is an integrodifferential operator) has been studied by many authors. For example, Mueller [13] and Wu [14] proved the existence of a solution of stochastic fractional heat and Burgers equation perturbed by a stable noise, respectively. Other related references are Chang and Lee [15], Truman and Wu [16], Liu et al. [17], Wu [18], and the references therein.

On the other hand, weak convergence to Brownian motion, fractional Brownian motion, and related stochastic processes have been considered extensively since the work

of Taqqu [19] and Delgado and Jolis [20]. Recently, many researchers are interested in studying weak convergence of stochastic differential equation. Some surveys could be found in Bardina et al. [21], Boufoussi and Hajji [22], and Mellal and Ouknine [23]. Bardina et al. [21] studied the convergence in law, in the space $\mathcal{C}([0, t] \times [0, 1])$ of continuous functions, of the solution of

$$\frac{\partial X_n}{\partial t}(t, x) - \frac{\partial^2 X_n}{\partial x^2}(t, x) = \theta_n(t, x), \quad (1)$$

with vanishing initial data and Dirichlet boundary conditions, towards the solution of

$$\frac{\partial X}{\partial t}(t, x) - \frac{\partial^2 X}{\partial x^2}(t, x) = \dot{W}(t, x), \quad (2)$$

where $(t, x) \in [0, T] \times [0, 1]$ and θ_n is a noisy input which converges to white noise $\dot{W}$. Mellal and Ouknine [23] considered the quasilinear stochastic heat equation on $[0, 1]$

$$\frac{\partial u}{\partial t}(t, x) - \frac{\partial^2 u}{\partial x^2}(t, x) = b(u(t, x)) + \frac{\partial^2 B^H(t, x)}{\partial t \partial x}, \quad (3)$$

with Dirichlet boundary conditions

$$u(t, 0) = u(t, 1) = 0, \quad t \in [0, T], \quad (4)$$

and initial condition $u(0, x) = u_0(x)$, $x \in [0, 1]$, where $\partial^2 B^H(t, x)/\partial t \partial x$ is a fractional noise with Hurst parameter $H \in (1/2, 1)$.

Motivated by these works, we consider the weak convergence for the following stochastic fractional heat equation driven by fractional noise on $(t, x) \in [0, T] \times [0, 1]$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= D_\delta^\alpha u + f(t, x, u) + \frac{\partial^2 B^H(t, x)}{\partial t \partial x}, \\ u(t, 0) &= u(t, 1) = 0, \end{aligned} \quad (5)$$

where D_δ^α is the fractional Laplacian operator with respect to the spatial variable, to be defined in Section 2 which was recently introduced by Debbi [24] and Debbi and Dozzi [25], and $B^H(t, x)$ is a fractional noise on $[0, T] \times [0, 1]$ with Hurst index $H > 1/2$ defined on a complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P\}$. Actually, we understand (5) in the sense of Walsh [26], and so one can present a mild formulation of (5) as follows:

$$\begin{aligned} u(t, x) &= \int_0^1 G_{\alpha, \delta}(t, x - y) u_0(y) dy \\ &+ \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) f(s, y, u(s, y)) dy ds \\ &+ \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) B^H(ds, dy), \end{aligned} \quad (6)$$

where $G_{\alpha, \delta}(\cdot, \cdot)$ denotes the Green function associated with (5).

The rest of this paper is organized as follows. In Section 2, we begin by making some notation and by recalling some basic preliminaries which will be needed later. In Section 3, we will prove weak limit theorems for (5) in space $\mathcal{C}([0, t] \times [0, 1])$. Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C , which may not be the same in each occurrence. Sometimes we will emphasize the dependence of these constants upon parameters.

2. Preliminaries

In this section, we briefly recall some basic definitions of fractional noise and Green function.

2.1. Fractional Noise. For each $t \in [0, T]$, let $\mathcal{F}_t^H$ be the σ -field generated by the random variables $\{B^H(t, A), t \in [0, T], A \in \mathcal{B}[0, 1]\}$ and the sets of probability zero, and denote by $\mathcal{P}$ the σ -field of progressively measurable subsets of $[0, T] \times \Omega$.

We denote by $\mathcal{E}$ the set of step functions on $[0, T] \times [0, 1]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbb{1}_{[0, t] \times A}, \mathbb{1}_{[0, s] \times B} \rangle_{\mathcal{H}} = |A \cap B| R_H(t, s), \quad (7)$$

where covariance kernel $R_H(t, s) = (1/2)[t^{2H} + s^{2H} - |t - s|^{2H}]$ and $|A|$ denotes the Lebesgue measure of the set A .

According to Nualart and Ouknine [27], the mapping $\mathbb{1}_{[0, t] \times A} \rightarrow B^H(t, A)$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B^H)$ associated with B^H and denoted by

$$\varphi \mapsto B^H(\varphi) := \int_{[0, t] \times A} \varphi(s, y) B^H(ds, dy). \quad (8)$$

Define the linear operator $K_H^*: \mathcal{E} \mapsto L^2([0, T])$ by

$$\begin{aligned} K_H^*(\varphi) &= K_H(T, s) \varphi(s, x) \\ &+ \int_s^T (\varphi(r, x) - \varphi(s, x)) \frac{\partial K_H}{\partial r}(r, s) dr, \end{aligned} \quad (9)$$

where K_H is the square integrable kernel given by

$$\begin{aligned} K_H(t, s) &:= \begin{cases} c_H(t-s)^{H-1/2} + C_H s^{1/2-H} \\ \quad \times \int_s^t (u-s)^{H-3/2} \left(1 - \left(\frac{s}{u}\right)^{1/2-H}\right) du, & 0 < s \leq t, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (10)$$

with $c_H = (2H\Gamma(3/2 - H)/\Gamma(H + 1/2)\Gamma(2 - 2H))^{1/2}$, and one can get

$$\frac{\partial K_H}{\partial t}(t, s) = c_H \left(\frac{1}{2} - H\right) \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-3/2}. \quad (11)$$

Moreover, the kernel K_H satisfies the following property:

$$\int_0^{s \wedge t} K_H(t, r) K_H(s, r) dr = R_H(t, s), \quad (12)$$

$R_H(t, s)$ being the covariance kernel of the fractional Brownian motion. Then, for any pair of step functions φ and ψ in $\mathcal{E}$ we have

$$\langle K_H^*(\varphi), K_H^*(\psi) \rangle_{L^2([0, T] \times [0, 1])} = \langle \varphi, \psi \rangle_{\mathcal{H}}, \quad (13)$$

because

$$(K_H^* \mathbb{1}_{[0, t] \times A})(s, x) = K_H(t, s) \mathbb{1}_{[0, t] \times A}(s, x). \quad (14)$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert space $\mathcal{H}$ and $L^2([0, T] \times [0, 1])$. Hence, the Gaussian family $\{W(t, A), t \in [0, T], A \in \mathcal{B}[0, 1]\}$ defined by

$$W(t, A) = B^H((K_H^*)^{-1}(\mathbb{1}_{[0, t] \times A})) \quad (15)$$

is a space-time white noise, and the process B^H has an integral representation of the form

$$B^H(t, x) = \int_0^t \int_0^x K_H(t, s) W(ds, dy). \quad (16)$$

Now, we can present a mild formulation of (5) as follows:

$$\begin{aligned} u(t, x) &= \int_0^1 G_{\alpha, \delta}(t, x - y) u_0(y) dy \\ &+ \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) f(s, y, u(s, y)) dy ds \\ &+ \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) W(ds, dy). \end{aligned} \quad (17)$$

That is, the last term of (6) is equal to

$$\begin{aligned} &\int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) B^H(ds, dy) \\ &= \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) W(ds, dy). \end{aligned} \quad (18)$$

2.2. Green Function. In this subsection, we will introduce the nonlocal fractional differential operator D_δ^α defined via its Fourier transform $\mathcal{F}$ by

$$\mathcal{F}(D_\delta^\alpha \varphi)(\xi) = -|\xi|^\alpha \exp\left(-i\delta \frac{\pi}{2} \operatorname{sgn}(\xi)\right) \mathcal{F}(\varphi)(\xi). \quad (19)$$

In this paper, we will assume that $|\delta| \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$, $i = 1, \dots, d$, $[\alpha]_2$, is the largest even integer less or equal to α (even part of α), and $\delta \in 2\mathbb{N} + 1$.

The operator D_δ^α is a closed, densely defined operator on $L^2(\mathbb{R})$ and it is the infinitesimal generator of a semigroup which is in general not symmetric and not a contraction. This operator is a generalization of various well-known operators, such as the Laplacian operator (when $\alpha = 2$), the inverse of the generalized Riesz-Feller potential (when $\alpha > 2$), and the Riemann-Liouville differential operator (when $|\delta| = 2 + [\alpha]_2$ or $|\delta| = \alpha - [\alpha]$). It is self-adjoint only when $\delta = 0$ and, in this case, it coincides with the fractional power of the Laplacian. We refer the readers to Debbi [24], Debbi and Dozzi [25], and Komatsu [28] for more details about this operator.

According to Komatsu [28], D_δ^α can be represented for $1 < \alpha < 2$ by

$$\begin{aligned} D_\delta^\alpha \varphi(x) &= \int_{\mathbb{R}} \frac{\varphi(x + y) - \varphi(x) - y\varphi'(x)}{|y|^{1+\alpha}} \\ &\times \left(\kappa_-^\delta 1_{(-\infty, 0)}(y) + \kappa_+^\delta 1_{(0, +\infty)}(y) \right) dy \end{aligned} \quad (20)$$

and for $0 < \alpha < 1$ by

$$\begin{aligned} D_\delta^\alpha \varphi(x) &= \int_{\mathbb{R}} \frac{\varphi(x + y) - \varphi(x)}{|y|^{1+\alpha}} \\ &\times \left(\kappa_-^\delta 1_{(-\infty, 0)}(y) + \kappa_+^\delta 1_{(0, +\infty)}(y) \right) dy, \end{aligned} \quad (21)$$

where κ_-^δ and κ_+^δ are two nonnegative constants satisfying $\kappa_-^\delta + \kappa_+^\delta > 0$ and φ is a smooth function for which the integral exists, and φ' is its derivative. This representation identifies

it as the infinitesimal generator for a nonsymmetric α -stable Lévy process.

Let $G_{\alpha, \delta}(t, x)$ be the fundamental solution of the following Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) &= D_\delta^\alpha(t, x), \\ u(0, x) &= \delta_0(x), \quad t > 0, \quad x \in \mathbb{R}, \end{aligned} \quad (22)$$

where $\delta_0(\cdot)$ is the Dirac distribution. By Fourier transform, we see that $G_{\alpha, \delta}(t, x)$ is given by

$$\begin{aligned} G_{\alpha, \delta}(t, x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-izx - t|z|^\alpha \exp\left(-i\delta \frac{\pi}{2} \operatorname{sgn}(z)\right)\right) dz. \end{aligned} \quad (23)$$

The relevant parameters α , called the index of stability, and δ (related to the asymmetry), improperly referred to as the skewness, are real numbers satisfying $|\delta| \leq \min\{\alpha - [\alpha]_2, 2 + [\alpha]_2 - \alpha\}$, and $\delta = 0$ when $\delta \in 2\mathbb{N} + 1$.

Let us list some known facts on $G_{\alpha, \delta}(t, x)$ which will be used later on (see, e.g., Debbi [24] and Debbi and Dozzi [25]).

Lemma 1. Let $\alpha \in (0, \infty) \setminus \{\mathbb{N}\}$; one has the following:

- (1) the function $G_{\alpha, \delta}(t, \cdot)$ is not in general symmetric relatively to x and it is not everywhere positive;
- (2) for any $s, t \in (0, \infty)$ and $x \in \mathbb{R}$,

$$\frac{\partial^n}{\partial x^n} G_{\alpha, \delta}(t, x) = (s)^{-(n+1)/\alpha} G_{\alpha, \delta}(s^{-1}t, s^{-1/\alpha}x), \quad (24)$$

or equivalently,

$$\frac{\partial^n}{\partial x^n} G_{\alpha, \delta}(t, x) = (t)^{-(n+1)/\alpha} G_{\alpha, \delta}\left(1, (t)^{-1/\alpha}x\right); \quad (25)$$

- (3) $G_{\alpha, \delta}(s, \cdot) * G_{\alpha, \delta}(t, \cdot) = G_{\alpha, \delta}(s+t, \cdot)$ for any $s, t \in (0, \infty)$;
- (4) For $n \geq 1$, there exist some constants C and $C_n > 0$ such that, for all $x \in \mathbb{R}$,

$$\begin{aligned} |G_{\alpha, \delta}(1, x)| &\leq C \frac{1}{1 + |x|^{1+\alpha}}, \\ \left| \frac{\partial^n}{\partial x^n} G_{\alpha, \delta}(1, x) \right| &\leq C_n \frac{|x|^{\alpha+n-1}}{(1 + |x|^{\alpha+n})^2}; \end{aligned} \quad (26)$$

- (5) $\int_0^T \int_{\mathbb{R}} |G_{\alpha, \delta}(t, x)|^\lambda dt dx < \infty$ if and only if $1/\alpha < \lambda < \alpha$.

3. Main Results and Its Proof

Our aim is to prove that the mild solution of (5), given by (17), can be approximated in law in the space $\mathcal{C}([0, t] \times [0, 1])$ by the processes

$$\begin{aligned} u_n(t, x) &= \int_0^1 G_{\alpha, \delta}(t, x - y) u_0(y) dy \\ &+ \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) f(s, y, u_n(s, y)) dy ds \\ &+ \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) \theta_n(s, y) dy ds, \end{aligned} \quad (27)$$

where $\{\theta_n(t, x)\}_{n \in \mathbb{N}}$ is a weak approximation of a Brownian sheet; that is, $\{\theta_n(t, x)\}_{n \in \mathbb{N}}, (t, x) \in [0, T] \times [0, 1]$ is a family of Kac-Stroock processes in the plane which is square integral a.s., defined by

$$\theta_n(t, x) = n\sqrt{tx}(-1)^{N_n(t, x)}, \quad n \in \mathbb{N}, \quad (28)$$

$N_n(t, x) = N(\sqrt{nt}, \sqrt{nx})$ and $\{N(t, x), (t, x) \in [0, T] \times [0, 1]\}$ is a standard Poisson process in plane.

Theorem 2. Let $\{\theta_n(t, x), (t, x) \in [0, T] \times [0, 1]\}, n \in \mathbb{N}$, be the Kac-Stroock processes in the plane. Assume that $u_0 : [0, 1] \rightarrow \mathbb{R}$ is a continuous function and f satisfies the following linear growth conditions:

$$|f(t, x, u)| \leq C(1 + |u|) \quad (29)$$

and uniformly Lipschitz conditions

$$|f(t, x, u) - f(t, x, v)| \leq C|u - v|. \quad (30)$$

Then, the family of stochastic processes $\{u_n, n \in \mathbb{N}\}$ defined by (27) converges in law, as n tends to infinity, in the space $\mathcal{C}([0, T] \times [0, 1])$, to the mild solution u of (5), given by (17).

In order to prove Theorem 2, we will focus on the linear problem, which is amount to establish the convergence in law, in the space $\mathcal{C}([0, T] \times [0, 1])$, of the solutions of

$$\frac{\partial X_n}{\partial t} - \frac{\partial^2 X_n}{\partial x^2} = \frac{\partial^2 B_n^H}{\partial t \partial x}, \quad (31)$$

with vanishing initial data and Dirichlet boundary conditions, toward the solution of

$$\frac{\partial X}{\partial t} - \frac{\partial^2 X}{\partial x^2} = \frac{\partial^2 B^H}{\partial t \partial x}, \quad (32)$$

where the solutions of (31) and (32) are, respectively, given by

$$X_n(t, x) = \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) \theta_n(s, y) dy ds, \quad (33)$$

$$X(t, x) = \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) W(ds, dy). \quad (34)$$

In the following, we need two results which can be found in Bardina et al. [21]. The first one leads to the tightness in $\mathcal{C}([0, T] \times [0, 1])$ of a family.

Lemma 3. Let $\{X_n, n \in \mathbb{N}\}$ be a family of random variables taking values in $\mathcal{C}([0, T] \times [0, 1])$. The family of the laws of $\{X_n, n \in \mathbb{N}\}$ is tight, if there exist $p', p > 0, \delta > 2$, and a constant C such that

$$\sup_{n \geq 1} E|X_n(0, 0)|^{p'} < \infty, \quad (35)$$

and, for every $t, t' \in [0, T]$ and $x, x' \in [0, 1]$,

$$\sup_{n \geq 1} E|X_n(t', x') - X_n(t, x)|^p < C(|x' - x| + |t' - t|)^\delta. \quad (36)$$

The second one is a technical lemma.

Lemma 4. Denote by $\theta_n(t, x)$ the Kac-Stroock kernels; for any even $n \in \mathbb{N}$, there exists a constant C_n , such that, for any $t_1, t_2 \in [0, T]$ and $x_1, x_2 \in [0, 1]$ satisfying $0 < t_1 < t_2 < 2t_1$ and $0 < x_1 < x_2 < 2x_1$, one can get

$$\begin{aligned} \sup_{n \geq 1} E \left[\left(\int_{t_1}^{t_2} \int_{x_1}^{x_2} f(s, x) \theta_n(s, x) dx ds \right)^n \right] \\ \leq C_n \left(\int_{t_1}^{t_2} \int_{x_1}^{x_2} f^2(s, x) dx ds \right)^{n/2}, \end{aligned} \quad (37)$$

for any $f(s, x) \in L^2([0, T] \times [0, 1])$.

Proposition 5. The family of processes $\{X_n, n \in \mathbb{N}\}$ is tight in $\mathcal{C}([0, T] \times [0, 1])$.

Proof. We first estimate the moment of order m of the quantity

$$\begin{aligned} X_n(t', x') - X_n(t, x) \\ = \int_0^{t'} \int_0^1 K_H^* G_{\alpha, \delta}(t' - s, x' - y) \theta_n(s, y) dy ds \\ - \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) \theta_n(s, y) dy ds, \end{aligned} \quad (38)$$

which is equal to

$$\begin{aligned} \int_0^t \int_0^1 K_H^* (G_{\alpha, \delta}(t' - s, x' - y) \\ - G_{\alpha, \delta}(t' - s, x - y)) \theta_n(s, y) dy ds \\ + \int_0^t \int_0^1 K_H^* (G_{\alpha, \delta}(t' - s, x - y) \\ - G_{\alpha, \delta}(t - s, x - y)) \theta_n(s, y) dy ds \\ + \int_t^{t'} \int_0^1 K_H^* G_{\alpha, \delta}(t' - s, x' - y) \theta_n(s, y) dy ds. \end{aligned} \quad (39)$$

By Lemma 4, one can get

$$\begin{aligned}
& \sup_{n \geq 1} E |X_n(t', x') - X_n(t, x)|^m \\
& \leq C_m \left[\int_0^t \int_0^1 \left[K_H^* (G_{\alpha, \delta}(t' - s, x' - y) \right. \right. \\
& \quad \left. \left. - G_{\alpha, \delta}(t' - s, x - y)) \right]^2 dy ds \right]^{m/2} \\
& \quad + C_m \left[\int_0^t \int_0^1 \left[K_H^* (G_{\alpha, \delta}(t' - s, x - y) \right. \right. \\
& \quad \left. \left. - G_{\alpha, \delta}(t - s, x - y)) \right]^2 dy ds \right]^{m/2} \\
& \quad + C_m \left[\int_t^{t'} \int_0^1 (K_H^* G_{\alpha, \delta}(t' - s, x' - y))^2 dy ds \right]^{m/2} \\
& = C_m (I + II + III).
\end{aligned} \tag{40}$$

Using the continuous embedding established in [27]

$$L^{1/H}([0, T] \times [0, 1]) \subset \mathcal{H}, \tag{41}$$

we obtain

$$\begin{aligned}
I &= \left[\int_0^t \int_0^1 \left[K_H^* (G_{\alpha, \delta}(t' - s, x' - y) \right. \right. \\
& \quad \left. \left. - G_{\alpha, \delta}(t' - s, x - y)) \right]^2 dy ds \right]^{m/2} \\
&\leq C_H \left[\int_0^t \int_0^1 (G_{\alpha, \delta}(t' - s, x' - y) \right. \\
& \quad \left. - G_{\alpha, \delta}(t' - s, x - y))^{1/H} dy ds \right]^{mH} \\
&= C_H \|G_{\alpha, \delta}(t' - \cdot, x' - *) \\
& \quad - G_{\alpha, \delta}(t' - \cdot, x - *)\|_{L^{1/H}([0, T] \times [0, 1])}^m \\
&= C_H \left\| G_{\alpha, \delta}(t' - \cdot, x' - *) - G_{\alpha, \delta}(t' - \cdot, x - *) \right\|^\theta \\
& \quad \cdot \|G_{\alpha, \delta}(t' - \cdot, x' - *) \\
& \quad - G_{\alpha, \delta}(t' - \cdot, x - *)\|_{L^{1/H}([0, T] \times [0, 1])}^{1-\theta} \\
&\leq C_H \left\| G_{\alpha, \delta}(t' - \cdot, x' - *) - G_{\alpha, \delta}(t' - \cdot, x - *) \right\|^\theta \\
& \quad \cdot \|G_{\alpha, \delta}(t' - \cdot, x' - *)\|_{L^{1/H}([0, T] \times [0, 1])}^{1-\theta}
\end{aligned}$$

$$\begin{aligned}
& + \left\| G_{\alpha, \delta}(t' - \cdot, x' - *) - G_{\alpha, \delta}(t' - \cdot, x - *) \right\|^\theta \\
& \quad \cdot \|G_{\alpha, \delta}(t' - \cdot, x - *)\|_{L^{1/H}([0, T] \times [0, 1])}^{1-\theta} \\
& = C_H (I_1 + I_2).
\end{aligned} \tag{42}$$

Let $\theta \in (0, \min\{1, (\alpha + 1)H - 1\})$. Thanks to the mean-value theorem, one can get

$$\begin{aligned}
I_1 &= \left\| |x' - x|^{\theta} \left| \frac{\partial}{\partial x} G_{\alpha, \delta}(t' - \cdot, \xi - *) \right|^\theta \right. \\
& \quad \times \|G_{\alpha, \delta}(t' - \cdot, x' - *)\|_{L^{1/H}([0, T] \times [0, 1])}^{1-\theta} \\
&= \left(|x' - x|^{\theta/H} \int_0^T \int_0^1 \left| \frac{\partial}{\partial x} G_{\alpha, \delta}(t' - s, \xi - y) \right|^{\theta/H} \right. \\
& \quad \times \|G_{\alpha, \delta}(t' - s, x' - y)\|^{(1-\theta)/H} dy ds \Big)^{mH}, \\
& \int_0^1 \left| \frac{\partial}{\partial x} G_{\alpha, \delta}(t' - s, \xi - y) \right|^{\theta/H} \\
& \quad \times \|G_{\alpha, \delta}(t' - s, x' - y)\|^{(1-\theta)/H} dy \\
&\leq \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_{\alpha, \delta}(t' - s, \xi - y) \right|^{\theta/H} \\
& \quad \times \|G_{\alpha, \delta}(t' - s, x' - y)\|^{(1-\theta)/H} dy \\
&= (t' - s)^{1/\alpha - (1+\theta)/\alpha H} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} G_{\alpha, \delta}(1, \xi - y) \right|^{\theta/H} \\
& \quad \times \|G_{\alpha, \delta}(1, y)\|^{(1-\theta)/H} dy \\
&\leq C_{H, \theta} (t' - s)^{1/\alpha - (1+\theta)/\alpha H}.
\end{aligned} \tag{43}$$

Therefore, if $1/\alpha - (1+\theta)/\alpha H + 1 > 0$, that is, $\theta < (\alpha + 1)H - 1$, then

$$I_1 \leq C_{T, H, \theta} |x' - x|^{m\theta}. \tag{44}$$

Similarly, one can get

$$I_2 \leq C_{T, H, \theta} |x' - x|^{m\theta}. \tag{45}$$

Hence

$$I \leq C_{T, H, \theta} |x' - x|^{m\theta}. \tag{46}$$

Now we are in the position to deal with *II*. Consider

$$\begin{aligned}
II &= \left[\int_0^t \int_0^1 \left[K_H^* (G_{\alpha,\delta}(t' - s, x - y) \right. \right. \\
&\quad \left. \left. - G_{\alpha,\delta}(t - s, x - y) \right) \right]^2 dy ds \right]^{m/2} \\
&\leq C_H \left[\int_0^t \int_0^1 (G_{\alpha,\delta}(t' - s, x - y) \right. \\
&\quad \left. - G_{\alpha,\delta}(t - s, x - y))^{1/H} dy ds \right]^{mH} \\
&= C_H \|G_{\alpha,\delta}(t' - \cdot, x - *) \\
&\quad - G_{\alpha,\delta}(t - \cdot, x - *)\|_{L^{1/H}([0,T] \times [0,1])}^m \\
&= C_H \|G_{\alpha,\delta}(t' - \cdot, x - *) - G_{\alpha,\delta}(t - \cdot, x - *)\|^\mu \\
&\quad \cdot \|G_{\alpha,\delta}(t' - \cdot, x - *) \\
&\quad - G_{\alpha,\delta}(t - \cdot, x - *)\|_{L^{1/H}([0,T] \times [0,1])}^m \\
&\leq C_H \left[\|G_{\alpha,\delta}(t' - \cdot, x - *) - G_{\alpha,\delta}(t - \cdot, x - *)\|^\mu \right. \\
&\quad \cdot \|G_{\alpha,\delta}(t' - \cdot, x - *)\|_{L^{1/H}([0,T] \times [0,1])}^m \\
&\quad \left. + \|G_{\alpha,\delta}(t' - \cdot, x - *) - G_{\alpha,\delta}(t - \cdot, x - *)\|^\mu \right. \\
&\quad \left. \cdot \|G_{\alpha,\delta}(t - \cdot, x - *)\|_{L^{1/H}([0,T] \times [0,1])}^m \right] \\
&= C_H (II_1 + II_2).
\end{aligned} \tag{47}$$

By mean-value theorem, it holds that

$$\begin{aligned}
II_1 &= \left\| |t' - t|^\mu \left| \frac{\partial}{\partial t} G_{\alpha,\delta}(\eta - \cdot, x - *) \right|^\mu \right. \\
&\quad \cdot \|G_{\alpha,\delta}(t' - \cdot, x - *)\|_{L^{1/H}([0,T] \times [0,1])}^m \\
&= \left(\int_0^T \int_0^1 |t' - t|^{\mu/H} \left| \frac{\partial}{\partial t} G_{\alpha,\delta}(\eta - s, x - y) \right|^{\mu/H} \right. \\
&\quad \left. \cdot |G_{\alpha,\delta}(t' - s, x - y)|^{(1-\mu)/H} dy ds \right)^{mH}.
\end{aligned} \tag{48}$$

Noting that

$$G_{\alpha,\delta}(t, x) = t^{-1/\alpha} G_{\alpha,\delta}(1, t^{-1/\alpha} x), \tag{49}$$

one can get

$$\begin{aligned}
&\frac{\partial}{\partial t} G_{\alpha,\delta}(t, x) \\
&= \left(-\frac{1}{\alpha} \right) t^{-1-1/\alpha} G_{\alpha,\delta}(1, t^{-1/\alpha} x) \\
&\quad + \left(-\frac{1}{\alpha} \right) t^{-1-2/\alpha} \left[x \frac{\partial}{\partial z} G_{\alpha,\delta}(1, z) \Big|_{z=t^{-1/\alpha} x} \right].
\end{aligned} \tag{50}$$

Then

$$\begin{aligned}
&\int_0^1 \left| \frac{\partial}{\partial t} \mathcal{G}_{\alpha,\delta}(\eta - s, x - y) \right|^{\mu/H} \\
&\quad \cdot |\mathcal{G}_{\alpha,\delta}(t' - s, x - y)|^{(1-\mu)/H} dy \\
&\leq \int_R \left| \frac{\partial}{\partial t} \mathcal{G}_{\alpha,\delta}(\eta - s, x - y) \right|^{\mu/H} \\
&\quad \cdot |\mathcal{G}_{\alpha,\delta}(t' - s, x - y)|^{(1-\mu)/H} dy \\
&\leq \int_R \left| \left(-\frac{1}{\alpha} \right) (\eta - s)^{-1-1/\alpha} G_{\alpha,\delta}(1, (\eta - s)^{-1/\alpha} (x - y)) \right. \\
&\quad \left. + \left(\frac{1}{\alpha} \right) (\eta - 2)^{-1-2/\alpha} \right. \\
&\quad \left. \times \left[(x - y) \frac{\partial}{\partial z} G_{\alpha,\delta}(1, z) \Big|_{z=(\eta-s)^{-1/\alpha}(x-y)} \right] \right|^{\mu/H} \\
&\quad \cdot |\mathcal{G}_{\alpha,\delta}(t' - s, x - y)|^{(1-\mu)/H} dy \\
&\leq C_{\alpha,\delta,H} \left[(t - s)^{-(\alpha\mu+1)/H\alpha+1/\alpha} \right. \\
&\quad \left. + (t - s)^{-(\alpha\mu+\mu+1)/H\alpha+1/\alpha} \right].
\end{aligned} \tag{51}$$

Therefore, if $\mu \in (0, ((\alpha + 1)H - 1)/\alpha)$, we have

$$II_1 \leq C_{\alpha,\delta,H,T,\mu} |t' - t|^{m\mu}. \tag{52}$$

Similarly,

$$II_2 \leq C_{\alpha,\delta,H,T,\mu} |t' - t|^{m\mu}. \tag{53}$$

Thus, we have

$$II \leq C_{T,H,\theta} |t' - t|^{m\mu}. \tag{54}$$

Now we deal with *III*; similar to the proof of *I*, we get

$$\begin{aligned}
III &= \left[\int_t^{t'} \int_0^1 (K_H^* G_{\alpha,\delta}(t' - s, x' - y))^2 dy ds \right]^{m/2} \\
&\leq C_{\alpha,\delta,H} \left[\int_t^{t'} \int_0^1 |G_{\alpha,\delta}(t' - s, x' - y)|^{1/H} dy ds \right]^{mH} \\
&\leq C_{\alpha,\delta,H} \left[\int_t^{t'} (t' - t)^{-1/H\alpha+1/\alpha} \right. \\
&\quad \left. \times \left(\int_0^1 |G_{\alpha,\delta}(1, z)|^{1/H} dz \right) ds \right]^{mH} \\
&\leq C_{\alpha,\delta,H} |t' - t|^{m((\alpha+1)H-1)/\alpha}.
\end{aligned} \tag{55}$$

Together with (40)–(55), one can get

$$\begin{aligned}
&\sup_{n \geq 1} E |X_n(t', x') - X_n(t, x)|^m \\
&\leq C \left[|t' - t|^{m\mu} + |x' - x|^{m\theta} \right].
\end{aligned} \tag{56}$$

By Lemma 3, the proof can be completed. $\square$

Proposition 6. *The family of processes $\{X_n, n \in \mathbb{N}\}$ defined by (33) converges to the process X given by (34), in the sense of finite-dimensional distributions, as n tends to infinity, in the space $\mathcal{C}([0, T] \times [0, 1])$.*

Proof. We claim that, for any $a_1, \dots, a_m \in \mathbb{R}$ and $(t_1, x_1), \dots, (t_m, x_m) \in [0, T] \times [0, 1]$, the law of linear combination

$$\sum_{j=1}^m a_j X_n(t_j, x_j) \quad (57)$$

converges weakly to the law of a random variable defined by

$$\sum_{j=1}^m a_j X(t_j, x_j). \quad (58)$$

This will be done by proving the convergence of the corresponding characteristic functions; that is,

$$\begin{aligned} & E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X_n(t_j, x_j) \right\} \right] \\ & \longrightarrow E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X(t_j, x_j) \right\} \right]. \end{aligned} \quad (59)$$

Since for any fixed $(t, x) \in [0, T] \times [0, 1]$, $K_H^* G_{\alpha, \delta}(t-s, x-y) \in L^2([0, T] \times [0, 1])$, then for any $(t_j, x_j) \in [0, T] \times [0, 1]$, $j \in \{1, 2, \dots, m\}$, there exists a sequence $(K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k$ of simple functions such that $(K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k$ converges to $K_H^* G_{\alpha, \delta}(t_j-s, x_j-y)$ in $L^2([0, T] \times [0, 1])$ as $k \rightarrow \infty$.

To simplify notation, we define

$$\begin{aligned} X_n^{j,k} &= \int_0^t \int_0^1 (K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k \theta_n(s, y) dy ds, \\ X^{j,k} &= \int_0^t \int_0^1 (K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k W(ds, dy). \end{aligned} \quad (60)$$

Then

$$\begin{aligned} & \left| E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X_n(t_j, x_j) \right\} \right] \right. \\ & \left. - E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X(t_j, x_j) \right\} \right] \right| \leq V + VI + VII, \end{aligned} \quad (61)$$

where

$$\begin{aligned} V &= \left| E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X_n(t_j, x_j) \right\} \right] \right. \\ & \left. - E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X^{j,k}(t_j, x_j) \right\} \right] \right|, \end{aligned}$$

$$\begin{aligned} VI &= \left| E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X_n^{j,k}(t_j, x_j) \right\} \right] \right. \\ & \left. - E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X^{j,k}(t_j, x_j) \right\} \right] \right|, \end{aligned}$$

$$\begin{aligned} VII &= \left| E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X^{j,k}(t_j, x_j) \right\} \right] \right. \\ & \left. - E \left[\exp \left\{ i \xi \sum_{j=1}^m a_j X(t_j, x_j) \right\} \right] \right|. \end{aligned} \quad (62)$$

We will proceed to prove (59) in three steps.

Step 1. By the mean-value theorem, there exists a constant $C > 0$ such that

$$\begin{aligned} V &\leq C \max_{1 \leq j \leq m} \{ E |X_n(t_j, x_j) - X_n^{j,k}(t_j, x_j)| \} \\ &= C \max_{1 \leq j \leq m} \left\{ E \left| \int_0^T \int_0^1 [K_H^* G_{\alpha, \delta}(t_j-s, x_j-y) \right. \right. \\ & \quad \left. \left. - (K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k \right] \right. \\ & \quad \left. \times \theta_n(s, y) dy ds \right| \}. \end{aligned} \quad (63)$$

Using the same method as the proof of Lemma 4, by Hölder inequality we can get

$$\begin{aligned} & E \left| \int_0^T \int_0^1 [K_H^* G_{\alpha, \delta}(t_j-s, x_j-y) \right. \\ & \quad \left. - (K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k \right] \theta_n(s, y) dy ds \right| \\ & \leq C \left(\int_0^T \int_0^1 [K_H^* G_{\alpha, \delta}(t_j-s, x_j-y) \right. \\ & \quad \left. - (K_H^* G_{\alpha, \delta}(t_j-s, x_j-y))_k \right]^2 dy ds \right)^{1/2}. \end{aligned} \quad (64)$$

So V uniformly converges to 0 with respect to n .

Step 2. We proceed to deal with VI. Using the mean-value theorem again,

$$\begin{aligned}
 VI &\leq C \max_{1 \leq j \leq m} \{E |X_n^{jk} - X^{jk}|\} \\
 &= C \max_{1 \leq j \leq m} \left\{ E \left| \int_0^T \int_0^1 (K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k \right. \right. \\
 &\quad \times \theta_n(s, y) dy ds \\
 &\quad \left. \left. - \int_0^T \int_0^1 (K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k \right. \right. \\
 &\quad \left. \left. \times W(ds, dy) \right| \right\}. \quad (65)
 \end{aligned}$$

Thanks to Donsker's theorem, as $n \rightarrow \infty$, the laws of processes

$$\int_0^T \int_0^1 (K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k \theta_n(s, y) dy ds \quad (66)$$

converge weakly to the law of

$$\int_0^T \int_0^1 (K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k W(ds, dy), \quad (67)$$

since $(K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k$ is a simple function. So we can get $VI \rightarrow 0$, as $n \rightarrow \infty$.

Step 3. Finally we deal with VII. Using the mean-value theorem again, one can get

$$\begin{aligned}
 VII &\leq C \max_{1 \leq j \leq m} \{E |X^{jk} - X(t_j, x_j)|\} \\
 &= C \max_{1 \leq j \leq m} \left\{ E \left| \int_0^T \int_0^1 (K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k \right. \right. \\
 &\quad \times W(ds, dy) \\
 &\quad \left. \left. - \int_0^T \int_0^1 K_H^* G_{\alpha, \delta}(t_j - s, x_j - y) \right. \right. \\
 &\quad \left. \left. \times W(ds, dy) \right| \right\}. \quad (68)
 \end{aligned}$$

Then by the Hölder inequality and the variance for a stochastic integral, we get

$$\begin{aligned}
 &E \left| \int_0^T \int_0^1 (K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k W(ds, dy) \right. \\
 &\quad \left. - \int_0^T \int_0^1 K_H^* G_{\alpha, \delta}(t_j - s, x_j - y) W(ds, dy) \right| \\
 &\leq \left(\int_0^T \int_0^1 [(K_H^* G_{\alpha, \delta}(t_j - s, x_j - y))_k \right. \\
 &\quad \left. - K_H^* G_{\alpha, \delta}(t_j - s, x_j - y)]^2 dy ds \right)^{1/2}. \quad (69)
 \end{aligned}$$

So, $VII \rightarrow 0$, for all $1 \leq j \leq m$, as $n \rightarrow \infty$. Our proof is completed. $\square$

As a consequence of the last two properties, we can state the following.

Theorem 7. *The family of processes $\{X_n, n \in \mathbb{N}\}$ defined by (33) converges in law, as n tends to infinity, in the space $\mathcal{C}([0, T] \times [0, 1])$, to the process X defined by (34).*

Now, we can give the proof of Theorem 2.

Proof of Theorem 2. Let us recall first the mild solution of (5), which is given by

$$\begin{aligned}
 u(t, x) &= \int_0^1 G_{\alpha, \delta}(t, x - y) u_0(y) dy \\
 &\quad + \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) f(s, y, u(s, y)) dy ds \\
 &\quad + \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) W(ds, dy), \quad (70)
 \end{aligned}$$

and the approximation sequence toward the mild solution of (5), which fulfils

$$\begin{aligned}
 u_n(t, x) &= \int_0^1 G_{\alpha, \delta}(t, x - y) u_0(y) dy \\
 &\quad + \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) f(s, y, u_n(s, y)) dy ds \\
 &\quad + \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t - s, x - y) \theta_n(s, y) dy ds, \quad (71)
 \end{aligned}$$

where $\{\theta_n(t, x), (t, x) \in [0, T] \times [0, 1]\}$, $n \in \mathbb{N}$, stand for the Kac-Stroock process which is square integrable a.s.

Moreover, the approximating sequence u_n has continuous paths a.s., for all $n \in \mathbb{N}$ which can be obtained by using the properties of the Green function, the fact that $\theta_n \in L^2([0, T] \times [0, 1])$ a.s., together with a Gronwall-type argument.

On the other hand, consider the following function:

$$\begin{aligned}
 \psi : \mathcal{C}([0, T] \times [0, 1]) &\longrightarrow \mathcal{C}([0, T] \times [0, 1]), \\
 \eta &\longrightarrow z_\eta, \quad (72)
 \end{aligned}$$

where $\eta : [0, T] \times [0, 1] \rightarrow \mathbb{R}$ is a continuous function, and

$$\begin{aligned}
 z_\eta(t, x) &= \int_0^1 G_{\alpha, \delta}(t, x - y) u_0(y) dy \\
 &\quad + \int_0^t \int_0^1 G_{\alpha, \delta}(t - s, x - y) f(s, y, u_n(s, y)) dy ds \\
 &\quad + \eta(t, x). \quad (73)
 \end{aligned}$$

Then it can be proved that this last function admits a unique continuous solution. Now, according to Theorem 3.5 in Bardina et al. [21], the function ψ is continuous. Considering

$$\begin{aligned} X_n(t, x) &= \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t-s, x-y) \theta_n(s, y) dy ds, \\ X(t, x) &= \int_0^t \int_0^1 K_H^* G_{\alpha, \delta}(t-s, x-y) W(ds, dy), \end{aligned} \quad (74)$$

one can get that X_n converges in law in $\mathcal{C}([0, T] \times [0, 1])$ to X , as n goes to infinity. On the other hand, we have that $u_n = \psi(X_n)$ and $u = \psi(X)$, and hence the continuity of ψ implies the convergence in law of u_n to u in $\mathcal{C}([0, T] \times [0, 1])$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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