Research Article

Dimension Spectrum for Sofic Systems

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We study the dimension spectrum of sofic system with the potential functions being matrix valued. For finite-coordinate dependent positive matrix potential, we set up the entropy spectrum by constructing the quasi-Bernoulli measure and the cut-off method is applied to deal with the infinite-coordinate dependent case. We extend this method to nonnegative matrix and give a series of examples of the sofic-affine set on which we can compute the spectrum concretely.

1. Introduction

Let (Σ_A, T) be a subshift of finite type (SFT) with A being the incidence matrix and T being its shift map. Motivated by the study of the iterated function systems (IFS) and generalized Sierpiński carpets (GSC, cf. [1–5]), one considers a special type of potential functions $M : \Sigma_A \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ which take values on the set of $d \times d$ matrices. For $q \in \mathbb{R}$, define the topological pressure as follows:

$$P_M(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{I \in \Sigma_{A,n}} \sup_{x \in [I]} \left\| \prod_{l=0}^{n-1} M(T^l(x)) \right\|^q, \quad (1)$$

whenever the limit exists and $\Sigma_{A,n}$ denotes the collection of n -cylinders in Σ_A . (Here $\|A\|$ is the matrix norm; that is, $\|A\| = \mathbf{1}_{1 \times d} A \mathbf{1}_{d \times 1}$, where $\mathbf{1}_{d \times 1}$ is the $d \times 1$ column vector with entries being 1's). In [4], if M is positive, that is, $M_{i,j} > 0$, and Hölder potential with Σ_A is topologically mixing, the authors prove that the Gibbs measure for M exists uniquely and the system admits the multifractal analysis. More precisely, let

$$E_M(\alpha) = \left\{ x \in \Sigma_A : \lim_{n \rightarrow \infty} \frac{\log \left\| \prod_{i=0}^{n-1} M(T^i(x)) \right\|}{n} = \alpha \right\} \quad (2)$$

be the level set for the upper Lyapunov exponent. Then the Hausdorff dimension of the level set is obtained as follows.

Theorem 1 (see [4, Theorem 1.3]). *Let (Σ_A, T) be a SFT and let $M : \Sigma_A \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ be Hölder continuous. Then $P_M(q)$ is differentiable and for any $\alpha = P'_M(q)$, $q \neq 0$,*

$$\dim_H E_M(\alpha) = \frac{1}{\log m} \{-\alpha q + P_M(q)\}, \quad (3)$$

where $\dim_H$ denotes the Hausdorff dimension.

The study of the thermodynamic properties with these potentials relates deeply to the fractal properties of the given IFS or GSC. We emphasize that the formula (3) set up the fine structure in the Hausdorff dimension point of view for (Σ_A, T) . The authors extend this result to the case that M is nonnegative with some additional irreducible conditions' the reader may refer to [3] for the detail. When the underlying space S is a sofic shift and $d = 1$, that is, the potential function is finitary real valued, there raises a natural equilibrium measure called *semigroup measure* proposed by Kitchens and Tuncel [6]. When $d \geq 2$, the thermodynamic properties relate to fractal dynamics of given sofic affine-invariant sets (cf. [7]).

Theorem 1 investigates the dimension spectrum of SFTs; it is natural to ask whether the formula is preserved by passing to their factors (sofic shifts). To be advanced, does the formula hold for those shifts beyond sofic shifts such as the case for the specification property? Recent research revealed that some properties of SFTs are preserved for the cases beyond the specification (cf. [8–10]). This study intends to show that the

formula of dimension spectrum of SFTs is passing to their factors, that is, sofic shifts, and their Hausdorff dimension can be expressed explicitly for some class of sofic shifts.

Let $\mathbf{S}$ be a sofic shift which is a subshift of the shift space and let S be the shift map on it. The well-known Curtis-Lyndon-Hedlund Theorem ([11, Theorem 6.2.9]) demonstrates that if $\phi : \Sigma_A \rightarrow \mathbf{S}$ is a function, then ϕ is a homomorphism if and only if ϕ is a *sliding block code*. The sliding block code is induced from *block map*, that is, a map $\Phi^{(L)} : \Sigma_{A,L} \rightarrow \mathcal{A}(\mathbf{S})$ for some $L \geq 1$. For all $k \geq 1$ and if $I \in \Sigma_{A,k+L-1}$, define $\Phi_k : \Sigma_{A,k+L-1} \rightarrow \mathbf{S}_k$ by $\Phi_k([I]) := \Phi^{(L)}(\pi^{(L)}(T^{k-1}([I])))$, where $\pi^{(L)} := \pi_{\mathbb{Z}^L}$ is the *restriction map* of a cylinder to $\mathbb{Z}^L$ lattice. Then $\phi : \Sigma_A \rightarrow \mathbf{S}$ is thus defined as the limit of Φ_k ; that is, $\phi = \lim_{k \rightarrow \infty} \Phi_k$ and $\mathbf{S} := \{\phi(x) : x \in \Sigma_A\}$. (We refer reader to [11, 12] for more detail). We call ϕ *right resolving* if for all $I_1 = (i_{1,l})_{l=0}^{L-2}$ and any $I_2 = (i_{2,l})_{l=0}^{L-2}$ and $I_3 = (i_{3,l})_{l=0}^{L-2} \in \Sigma_{A,L-1}$ such that $I_1 \oplus I_2$ and $I_1 \oplus I_3 \in \Sigma_{A,L}$ we have

$$\Phi^{(L)}([I_1 \oplus I_2]) \neq \Phi^{(L)}([I_1 \oplus I_3]), \quad (4)$$

where $I_1 \oplus I_2 = i_{1,0}, i_{1,1}, \dots, i_{2,L-2} \in \Sigma_{A,L}$ if and only if $i_{1,l+1} = i_{2,l} \forall l = 0, \dots, L-2$. If $I_1, I_2 \in \mathcal{A}(\Sigma_A)$, then we define $I_1 \oplus I_2 := I_1 I_2$. Throughout this paper we assume $\phi : \Sigma_A \rightarrow \mathbf{S}$ is right resolving.

In this paper, we study that the dimension spectrum with $N : \mathbf{S} \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ is a matrix-valued potential on $\mathbf{S}$ taking values on the set of $d \times d$ matrices. To be precise, let $\Gamma_{(+)}^*(\mathbf{S})$ be the collection of $d \times d$ nonnegative (positive) matrices which are $\star$ continuous on $\mathbf{S}$; notation $\star$ stands for the H of Hölder continuous and C for continuous, the same for $\Gamma_{(+)}^*(\Sigma_A)$. For $q \in \mathbb{R}$, let $P_N(q)$ be defined similarly as in (1) and the *level set for the upper Lyapunov exponent* for N is also defined similarly as (2):

$$E_N(\alpha) = \left\{ y \in \mathbf{S} : \lim_{n \rightarrow \infty} \frac{\log \left\| \prod_{l=0}^{n-1} N(S^l([y])) \right\|}{n} = \alpha \right\}. \quad (5)$$

The main results of the present paper are the following. We want to mention here that our results were independently investigated by Feng and Huang [13, Theorem 1.4] via different approach. Our method, except for providing another point of view for the mathematical demonstration, can be applied for evaluating the topological pressure rigorously.

Theorem A. *Let $\mathbf{S}$ be a sofic shift induced by Σ_A and let $N \in \Gamma_{(+)}^C(\mathbf{S})$ be a matrix-valued potential on $\mathbf{S}$ which depends on k coordinates. Then*

- (1) for all $q \in \mathbb{R} \setminus \{0\}$, $P_N(q)$ is differentiable;
- (2) if $\alpha = P'_N(q)$,

$$\dim_H E_N(\alpha) = \frac{1}{\log \lambda} \{-\alpha q + P_N(q)\}, \quad (6)$$

where λ is the maximal eigenvalue of A .

Theorem A deals with the finite-coordinate dependent matrix potentials. This method also allows us to set up

the dimension spectrum for infinite-coordinate dependent one for $\mathbf{S}$. We emphasize here that our method makes the discussion of the limiting measure on infinite-coordinate systems possible. Let

$$\eta_n := \sup \left\{ \frac{M_{i,j}(x)}{M_{i,j}(y)} : I \in \Sigma_{A,n}, x, y \in [I], 1 \leq i, j \leq d \right\}. \quad (7)$$

Since $M \in \Gamma_{(+)}^H(\Sigma_A)$, we have $|\log \eta_n| \leq C\lambda^{-\alpha n}$ for some $0 < \alpha < 1$ ([4, Lemma 2.2]). The following result deals with the dimension spectrum for infinite-coordinate N .

Theorem B. *Let $N \in \Gamma_{(+)}^H(\mathbf{S})$ be a matrix-valued potential on $\mathbf{S}$ which depends on infinite many coordinates. Then*

- (1) for all $q \in \mathbb{R} \setminus \{0\}$, $P_N(q)$ is differentiable;
- (2) if $\alpha = P'_N(q)$,

$$\dim_H E_N(\alpha) = \frac{1}{\log \lambda} \{-\alpha q + P_N(q)\}, \quad (8)$$

where λ is the maximal eigenvalue of A .

The block map $\Phi^{(L)}$ plays an important role in this method and one of the advantages of this method is that we can prove that (3) holds for $N \in \Gamma_{(+)}^H(\mathbf{S})$ by using the matrix theory argument (Perron-Frobenius Theorem [6]). We will show there are some interesting examples of sofic affine set that we can compute their rigorous formulae for L^q -spectrum and the pressure functions; then the dimension spectrum is thus derived by simple computation.

The content of the paper is following. In Section 2, we present the proof of Theorem A and the proof of Theorem B is given in Section 3. Section 4 extends Theorems A and B to nonnegative matrix-valued potential functions and investigates some examples.

2. Proof of Theorem A

This section gives a proof for Theorem A. We recall some definitions first. Denote by $\mathcal{M}(\Sigma_A)$ the set of probability measures on Σ_A and $\mathcal{M}_T(\Sigma_A)$ the subset of T -invariant measures of $\mathcal{M}(\Sigma_A)$, $\mathcal{M}(\mathbf{S})$ and $\mathcal{M}_S(\mathbf{S})$ are defined similarly.

Definition 2. (1) We say that $\mu \in \mathcal{M}_T(\Sigma_A)$ is *quasi-Bernoulli* if there exists a constant $C > 1$ such that

$$\begin{aligned} & C^{-1} \mu([I_1]) \mu([I_2]) \\ & \leq \mu([I_1 I_2]) \\ & \leq C \mu([I_1]) \mu([I_2]) \quad \forall I_1, I_2 \in \bigcup_{n \in \mathbb{N}} \Sigma_{A,n}. \end{aligned} \quad (9)$$

- (2) For $q \in \mathbb{R}$, the L^q -spectrum of μ is defined by

$$\tau_\mu(q) = \frac{1}{\log \lambda} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \sum_{I \in \Sigma_{A,n}} \mu([I])^q, \quad (10)$$

where λ denotes the maximal eigenvalue of A .

Our method is motivated by the idea which is proposed in [5] and the intrinsic property of the sliding block codes $\Phi^{(L)}$ and ϕ ; we formulate it briefly.

- (1) Since N depends on k coordinates, we construct Φ_k from $\Phi^{(L)}$ as mentioned above. Then the *pullback potential* on $\Sigma_{A,k+L-1}$ from Φ_k is also defined. We extend the idea of the proof of Lemma 4.3 of [5] to construct an invariant, ergodic probability measure on $\Sigma_{A,k+L-1}$ and extend this measure to some limiting measure which supports the whole Σ_A .
- (2) For all $J \in \mathbf{S}_n$ we define a measure on J by measuring one of its preimages with the measure in Σ_A which is constructed in Step 1. Although the measure in Σ_A satisfies the Markov property and probability properties, the measure on $\mathbf{S}$ cannot share the same properties. However, the space $\mathcal{M}(\mathbf{S})$ is still compact and the standard argument allows us to find an invariant and ergodic measure on $\mathbf{S}$.
- (3) Combining steps 1 with 2 we are able to show that the limiting measure is *Gibbs-like* and satisfies the *quasi-Bernoulli property* (we emphasize here that this measure is not necessary a Gibbs measure) and the L^q -spectrum preserved under the factor ϕ which is induced from the limit of Φ_k ; that is, $\phi = \lim_{k \rightarrow \infty} \Phi_k$. Therefore, the differentiability and the dimension spectrum can be preserved from ϕ .

Proof of Theorem A. We divide the proof in the following 4 steps.

Step 1. Let $\Phi^{(L)} : \Sigma_{A,L} \rightarrow \mathcal{A}(\mathbf{S})$ be a sliding block code from $\Sigma_{A,L}$ to $\mathcal{A}(\mathbf{S})$. For $k \geq 1$, define $\Phi_k = \Phi^{(k+L-1)}$ from $\Sigma_{A,k+L-1}$ to $\mathbf{S}_k$ by $\Phi_k([I]) = [J] = [j_0, \dots, j_{k-1}]$:

$$j_l = \Phi^{(L)}(\pi_{[0,L-1]}(T^{l-1}([I]))) \quad \forall l = 1, \dots, k, \quad (11)$$

where $\pi_{[0,L-1]}([I]) = i_0 i_1 \dots i_{L-1}$ denotes the projection map to coordinate $[0, L-1]$ on $\mathbb{Z}^1$ for all $I \in \Sigma_A$. Define a matrix potential M on $\Sigma_{A,k+L-1}$ by, if $I \in \Sigma_{A,k+L-1}$,

$$M([I]) = N(\Phi_k([I])). \quad (12)$$

Then M is well defined for all $I \in \Sigma_{A,k+L-1}$ from the fact that N depends on k -coordinate. Write $\Sigma^{(k)} = \Sigma_{A,k+L-1}$. Define $q^{(k)} = \#\Sigma_{A,k+L-1}$ for $k \geq 1$ and $q^{(0)} = \#\mathcal{A}(\Sigma_A)$; we setup a matrix $H^{(k)} \in \mathbf{M}_{dq^{(k-1)} \times dq^{(k-1)}}$ which is indexed by the elements of $\Sigma^{(k-1)}$ as follows: $\forall I_1, I_2 \in \Sigma^{(k-1)}$

$$H_{I_1, I_2}^{(k)} = \begin{cases} M([I_1 \oplus I_2]) & \text{if } I_1 \oplus I_2 \in \Sigma^{(k)} \\ 0_{d \times d} & \text{otherwise,} \end{cases} \quad (13)$$

where $0_{d \times d}$ denotes the $d \times d$ matrix with entries which are all zeros. For all $A = (A_{i_1, i_2})_{i_1, i_2=1}^n \in \mathbf{M}_{dn \times dn}(\mathbb{R})$ with $A_{i_1, i_2} \in \mathbf{M}_{d \times d}(\mathbb{R})$, we denote by $I(A)$ the indicator matrix of A ; that is, $I(A) \in \mathbf{M}_{n \times n}(\mathbb{R})$,

$$I(A)_{i_1, i_2} = \begin{cases} 1 & \text{if } A_{i_1, i_2} \neq 0_{d \times d} \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

It is obvious that if Σ_A is mixing, then $I(A)$ is primitive. Therefore, if we assume $H = H^{(k)}$, there exists a uniform constant $m > 0$ such that for all I_1 and $I_2 \in \Sigma^{(k-1)}$ there exists a path $(I'_l)_{l=1}^R$ with $R \leq m$, $I'_l \in \Sigma^{(k-1)}$, $I'_1 = I_1$, $I'_R = I_2$, and

$$I' := I'_1 \oplus \dots \oplus I'_R \in \Sigma^{(k+R-2)}. \quad (15)$$

Combining the fact of $N \in \Gamma_+^C(\mathbf{S})$,

$$\begin{aligned} & \prod_{l=0}^{R-1} M(\pi_{[0,k+L-2]}(T^l([I']))) \\ &= \prod_{l=0}^{R-1} N(\Phi_k \circ \pi_{[0,k+L-2]}(T^l([I']))) > 0. \end{aligned} \quad (16)$$

Thus

$$H_{I_1, I_2}^R \geq \prod_{l=1}^{R-1} M(\pi_{[0,k+L-2]}(T^l([I']))) > 0, \quad (17)$$

and also $H^m > 0$. Let $A^{(k)} = \{I^{(l)} \in \Sigma^{(k-1)} : l = 1, \dots, q^{(k-1)}\}$ be an ordered set by the lexicographic ordering and we rearrange H according to this ordering. Since H is primitive, Perron-Frobenius Theorem is applied to show that there exist eigenvalues ρ_L and $\rho_R > 0$ with corresponding eigenvectors L and $R > 0$, respectively, for H . We may also assume

$$L = (L([I^{(l)}]))_{l=1}^{q^{(k-1)}}, \quad R = (R([I^{(l)}]))_{l=1}^{q^{(k-1)}}. \quad (18)$$

That is,

$$\begin{aligned} & (LH)([I^{(l)}]) \\ &= \sum_{\substack{I^{(j)} \in \Sigma^{(k-1)}: \\ I^{(j)} \oplus I^{(l)} \in \Sigma^{(k)}}} L([I^{(j)}]) M([I^{(j)} \oplus I^{(l)}]) \\ &= \rho_L L([I^{(l)}]) \\ & (HR)([I^{(l)}]) \\ &= \sum_{\substack{I^{(j)} \in \Sigma^{(k-1)}: \\ I^{(l)} \oplus I^{(j)} \in \Sigma^{(k)}}} M([I^{(l)} \oplus I^{(j)}]) R([I^{(j)}]) \\ &= \rho_R R([I^{(l)}]). \end{aligned} \quad (19)$$

For all $I \in \Sigma^{(k+j)}$ and $j \geq 0$, let $\pi^{(k)} = \pi_{[0,k+L-2]}$ and

$$\prod_{[I]} M = \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))). \quad (20)$$

We define a measure as follows:

$$\begin{aligned}\eta_L([I]) &= \rho_L^{-j} L(\pi^{(k-1)}([I])) \left[\prod_{[I]} M \right] R(\pi^{(k-1)}(T^{j-1}([I]))), \\ \eta_R([I]) &= \rho_R^{-j} L(\pi^{(k-1)}([I])) \left[\prod_{[I]} M \right] R(\pi^{(k-1)}(T^{j-1}([I]))).\end{aligned}\quad (21)$$

It follows from (19) that if $I \in \Sigma^{(k+j-1)}$,

$$\begin{aligned}\sum_{\substack{I_1 \in \Sigma^{(k-1)}: \\ I_1 \oplus I \in \Sigma^{(k+j)}}} \eta_L([I_1 \oplus I]) &= \sum_{\substack{I_1 \in \Sigma^{(k-1)}: \\ I_1 \oplus I \in \Sigma^{(k+j)}}} \rho_L^{-j} L(\pi^{(k-1)}([I_1 \oplus I])) \left[\prod_{[I_1 \oplus I]} M \right] \\ &\quad \times R(\pi^{(k-1)}(T^{j-1}([I_1 \oplus I]))) \\ &= \rho_L^{-j} \sum_{\substack{I_1 \in \Sigma^{(k-1)}: \\ I_1 \oplus I \in \Sigma^{(k+j)}}} L(\pi^{(k-1)}([I_1])) M(\pi^{(k)}([I_1 \oplus I])) \\ &\quad \times \left[\prod_{[I]} M \right] R(\pi^{(k-1)}(T^{j-1}([I_1 \oplus I]))) \quad (22) \\ &= \rho_L^{-(j-1)} L(\pi^{(k-1)}([I])) \left[\prod_{[I]} M \right] \\ &\quad \times R(\pi^{(k-1)}(T^{j-1}([I_1 \oplus I]))) \\ &= \rho_L^{-(j-1)} L(\pi^{(k-1)}([I])) \left[\prod_{[I]} M \right] \\ &\quad \times R(\pi^{(k-1)}(T^{j-2}([I]))) \\ &= \eta_L([I]).\end{aligned}$$

That is,

$$\eta_L([I]) = \sum_{I_1 \in \Sigma^{(k-1)}: I_1 \oplus I \in \Sigma^{(k+j)}} \eta_L([I_1 \oplus I]). \quad (23)$$

It follows from the same computation we also have that

$$\eta_R([I]) = \sum_{I_1 \in \Sigma^{(k-1)}: I \oplus I_1 \in \Sigma^{(k+j)}} \eta_R([I \oplus I_1]). \quad (24)$$

This implies that $\forall n \geq 0$,

$$\begin{aligned}\sum_{I \in \Sigma^{(k+n)}} \eta_L([I]) &= \sum_{I \in \Sigma^{(k)}} \eta_L([I]), \\ \sum_{I \in \Sigma^{(k+n)}} \eta_R([I]) &= \sum_{I \in \Sigma^{(k)}} \eta_R([I]).\end{aligned}\quad (25)$$

It can be easily checked that $\rho_L = \rho_R$ and then we define

$$\eta([I]) = \eta_L \eta_{(k)}^{-1}, \quad \text{where } \eta_{(k)} = \sum_{I \in \Sigma^{(k)}} \eta_L([I]). \quad (26)$$

The Kolmogorov consistence theorem is applied to show that there exists a measure μ on Σ_A such that

$$\mu([I]) = \eta([I]) \quad \forall I \in \cup_{n \geq 0} \Sigma^{(k+n)}. \quad (27)$$

Step 2. In this step, we will define a measure on $\mathcal{M}_S(\mathbf{S})$. Since

$$\Phi_{k+j}(\Sigma^{(k+j)}) = \mathbf{S}_{k+j} \quad (28)$$

is onto, for all $J \in \mathbf{S}_{k+j}$, we define an ordered set as

$$P_J := \{I^{(l)} \in \Sigma^{(k+j)} : \Phi_{k+j}([I^{(l)}]) = J\} \quad (29)$$

and set a measure on $\mathbf{S}$:

$$\xi_1([J]) = \eta_L([I^{(1)}]), \quad \forall J \in \mathbf{S}_{k+j}, I^{(1)} \in P_J. \quad (30)$$

We note here that ξ_1 is not invariant. Let

$$\xi([J]) = \xi_1([J]) \xi_{(k)}^{-1} \quad \text{where } \xi_{(k)} = \sum_{J \in \mathbf{S}_k} \xi_1([J]) \quad (31)$$

for all $J \in \mathbf{S}_{k+j}$ and assume $I_1, I_2 \in P_J$,

$$\begin{aligned}\Phi_k(\pi^{(k)}(T^l([I_1]))) &= \Phi_k(\pi^{(k)}(T^l([I_2]))) \\ \forall l = 0, \dots, j-1.\end{aligned}\quad (32)$$

Hence

$$\begin{aligned}\prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I_1]))) &= \prod_{l=0}^{j-1} N(\Phi_k(\pi^{(k)}(T^l([I_1]))) \\ &= \prod_{l=0}^{j-1} N(\Phi_k(\pi^{(k)}(T^l([I_2]))) \\ &= \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I_2]))).\end{aligned}\quad (33)$$

Set

$$\begin{aligned}U_L &= \max_{i=1, \dots, d} \max_{I \in \Sigma^{(k-1)}} L_i([I]), \\ U_R &= \max_{i=1, \dots, d} \max_{I \in \Sigma^{(k-1)}} R_i([I]), \\ V_L &= \min_{i=1, \dots, d} \min_{I \in \Sigma^{(k-1)}} L_i([I]), \\ V_R &= \min_{i=1, \dots, d} \min_{I \in \Sigma^{(k-1)}} R_i([I]),\end{aligned}\quad (34)$$

where $B_i([I])$ denotes the i -coordinate of vector of $B([I])$ for $B = L$ or R . Since $\phi = \lim_{k \rightarrow \infty} \Phi_k$ is right resolving, for all

$J \in \mathbf{S}_n$ there is at least one and at most K preimages of $I \in \Sigma^{(n)}$ such that $\Phi_n([I]) = J$. Therefore, if $j \geq 0$ and $J \in \mathbf{S}_{k+j}$, it follows from (33) that

$$\begin{aligned} & \rho^{-j} V_L V_R \left\| \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))) \right\| \\ & \leq \eta_L([I]) \leq \eta_L(\Phi_{k+j}^{-1}([J])) \\ & = \eta_L\left(\bigcup_{I \in P_j} [I]\right) \leq \sum_{I \in P_j} \eta_L([I]) \\ & \leq \rho^{-j} K U_L U_R \left\| \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))) \right\|. \end{aligned} \quad (35)$$

By the positivity of $L = (L([I^{(l)}]))_{l=1}^{q^{(k-1)}}$, $R = (R([I^{(l)}]))_{l=1}^{q^{(k-1)}}$, and $q^{(k-1)}$ is finite, we can conclude that there exist two constants P and $Q > 0$ such that $1 \leq U_L/V_L \leq P$ and $1 \leq U_R/V_R \leq Q$; then

$$\begin{aligned} & \eta_L(\Phi_{k+j}^{-1}([J])) \\ & \leq \rho^{-j} K U_L U_R \left\| \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))) \right\| \\ & \leq \rho^{-j} K P Q V_L V_R \left\| \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))) \right\| \\ & \leq \rho^{-j} K P Q L(\pi^{(k-1)}([I])) \\ & \quad \times \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))) R(\pi^{(k-1)}(T^{j-1}([I]))) \\ & = K P Q \xi_1([J]), \\ & \eta_L(\Phi_{k+j}^{-1}([J])) \geq \rho^{-j} V_L V_R \left\| \prod_{l=0}^{j-1} M(\pi^{(k)}(T^l([I]))) \right\| \\ & \geq P^{-1} Q^{-1} \xi_1([J]). \end{aligned} \quad (36)$$

This means that for all $J \in \mathbf{S}_k$ there exists a constant $C_1 > 0$ such that

$$C_1^{-1} \xi_1([J]) \leq \eta_L(\Phi_k^{-1}([J])) \leq C_1 \xi_1([J]). \quad (37)$$

And thus

$$C_2^{-1} \xi([J]) \leq \eta(\Phi_k^{-1}([J])) \leq C_2 \xi([J]), \quad (38)$$

for some $C_2 > 0$.

Step 3. Since ξ is not invariant, we follow the proof of [4] to construct an invariant and ergodic measure satisfying the

property of (21) in this step. For all $J \in \mathbf{S}_k$, define a sequence $\{\sum_{l=0}^{n-1} \xi \circ S^{-l}([J])\}_{n \geq 1}$. It follows from (37) that if $l \geq 0$,

$$\begin{aligned} & \xi_1 \circ S^{-l}([J]) \\ & = \sum_{J_1: J_1 J \in \mathbf{S}_{l+k}} \xi_1([J_1 J]) \\ & = \sum_{I_1: I_1 I \in P_{l+1}} \eta_L([I_1 I]) \\ & \leq \sum_{J_1: J_1 J \in \mathbf{S}_{l+k}} \eta_L(\Phi_{l+k}^{-1}([J_1 J])) = \eta_L(\Phi_k^{-1}([J])). \end{aligned} \quad (39)$$

Hence there exists a constant $C_3 > 0$ such that

$$C_3^{-1} \eta_L(\Phi_k^{-1}([J])) \leq \xi_1 \circ S^{-l}([J]) \leq C_3 \eta_L(\Phi_k^{-1}([J])). \quad (40)$$

Thus there exists a $C_4 > 0$ such that

$$C_4^{-1} \eta(\Phi_k^{-1}([J])) \leq \xi \circ S^{-l}([J]) \leq C_4 \eta(\Phi_k^{-1}([J])). \quad (41)$$

Since $\mathbf{S}$ is compact, then let $\nu \in \mathcal{M}(\mathbf{S})$ be the limiting measure of

$$\left\{ \sum_{l=0}^{n-1} \xi \circ S^{-l}([J]) \right\}_{n \geq 1}. \quad (42)$$

Combining the fact that $\lim_{n \rightarrow \infty} \Phi_{k+n} = \phi$ with the above computations it yields $\nu \ll \mu$ and $\mu \ll \nu$. Up to a small modification of the proof in Theorem 1.1 of [4] we also have that ν is ergodic. The Radon-Nikodym theorem applies to show that there is a constant $C > 0$ such that $\nu([J]) = C \mu(\Phi_l^{-1}([J]))$ for ν -a.e. $J \in \mathbf{S}_l$ and $l \geq k$. It follows from that ν and μ are both invariant probability measures. We obtain $C = 1$ and for all $J \in \mathbf{S}$

$$\nu([J]) = \lim_{l \rightarrow \infty} \mu(\Phi_l^{-1}([J])) = \mu(\phi^{-1}([J])). \quad (43)$$

Step 4. From the above computation we obtain that if $J \in \mathbf{S}_l$ with $l \geq k$ and $I^{(1)} \in P_j$, then $\nu([J]) = \mu([I^{(1)}]) = \eta([I^{(1)}])$. Moreover, there exists $Q_1 > 0$ such that

$$Q_1^{-1} \eta_L([I^{(1)}]) \leq \eta([I^{(1)}]) \leq Q_1 \eta_L([I^{(1)}]). \quad (44)$$

With the positivity of M implements there exists a constant $Q_2 > 0$ such that for any $x \in \Sigma_A$, $n, l \in \mathbb{N}$ we have

$$\begin{aligned} & Q_2^{-1} \|M(x) \cdots M(T^{n-1}(x))\| \\ & \quad \cdot \|M(T^n(x)) \cdots M(T^{n+l-1}(x))\| \\ & \leq \|M(x) \cdots M(T^{n+l-1}(x))\| \quad (\text{Lemma 2.1 of [4]}) \\ & \leq Q_2 \|M(x) \cdots M(T^{n-1}(x))\| \\ & \quad \cdot \|M(T^n(x)) \cdots M(T^{n+l-1}(x))\|. \end{aligned} \quad (45)$$

This demonstrates η_L is quasi-Bernoulli and so are η and μ . Hence $\nu \in \mathcal{M}_S(\mathbf{S})$ is a quasi-Bernoulli measure. According to the fact that right-resolving factor ϕ cannot increase the topological entropy, we can assert that

$$\tau_\nu(q) = \tau_{\phi_*\mu}(q) = \tau_\mu(q) \quad \text{for any } q \in \mathbb{R}, \quad (46)$$

where $\phi_*\mu = \mu(\phi^{-1})$. Theorem 1.3 of [4] is applied to show that for all $q \in \mathbb{R} \setminus \{0\}$, τ_ν is differentiable and if $\alpha = P'_N(q)$,

$$\dim_H E_N(\alpha) = \frac{1}{\log \lambda} (-\alpha q + P_N(q)), \quad (47)$$

where λ denotes the maximal eigenvalue of A . Finally, the differentiability for $P_N(q)$ with $q \neq 0$ comes from the fact $P_N(q) = P_M(q)$ since ϕ is right resolving and M is the pullback potential of N . This completes the proof. $\square$

Remark 3. We remark that in the proof of Theorem A, $\nu \in \mathcal{M}_S(\mathbf{S})$ is not a Gibbs measure, and in the following, we will show that this method allow us to approximate the potential depending on infinite coordinate for $N \in \Gamma_+^H(\mathbf{S})$.

3. Proof of Theorem B

In this section, we extend our result to the matrix-valued potentials that are infinite-coordinate dependent.

Proof of Theorem B. The first statement is an immediate consequence of Theorem A since $N^{(k)}$ depends on k -coordinate. It is still remaining to prove the second statement.

For $k \geq 1, I \in \Sigma^{(k)}$ with $I \in P_j$, let $H^{(k)} \in \mathbf{M}_{dq^{(k-1)} \times dq^{(k-1)}}(\mathbb{R})$ and let $\rho^{(k)}$ be its maximal eigenvalue as in Theorem A. Since $I(H^{(k)})$ is primitive and $M^{(k)}$ is positive, $H^{(k)}$ is also primitive for all $k \in \mathbb{N}$. We claim that $\eta_k^{-1} \rho^{(k)} \leq \rho^{(k+1)} \leq \eta_k \rho^{(k)}$ for $k \geq 1$. Indeed, let $H^{(k)}$ and $H^{(k+1)}$ be indexed by $\Sigma^{(k-1)}$ and $\Sigma^{(k)}$, respectively. For all $I \in \Sigma^{(k+1)}$, $[J] = [j_0 \cdots j_k] := \Phi_{k+1}([I])$,

$$\begin{aligned} M^{(k+1)}([I]) &:= N^{(k+1)}(\Phi_{k+1}([I])) = \max_{y \in [j_0 \cdots j_k]} N(y) \\ &\leq \eta_k \max_{y \in [j_0 j_1 \cdots j_{k-1}]} N(y) = \eta_k N^{(k)}([J^*]) = \eta_k M^{(k)}([I^*]), \end{aligned} \quad (48)$$

where $J^* := j_0 \cdots j_{k-1}$ and $I^* := i_0 \cdots i_{k+L-1}$. Therefore, for $m \geq 1$,

$$\begin{aligned} \|H^{(k+1)m}\| &= \sum_{I \in \Sigma^{(k+m)}} \left\| \prod_{l=0}^{m-1} M^{(k+1)}(\pi^{(k+1)}(T^l([I]))) \right\| \\ &\leq (\eta_k)^m \sum_{I \in \Sigma^{(k+m)}} \left\| \prod_{l=0}^{m-1} M^{(k)}(\pi^{(k)}(T^l([I]))) \right\| \end{aligned}$$

$$\begin{aligned} &\leq C(\eta_k)^m \sum_{I \in \Sigma^{(k+m-1)}} \left\| \prod_{l=0}^{m-1} M^{(k)}(\pi^{(k)}(T^l([I]))) \right\| \\ &= C(\eta_k)^m \|H^{(k)m}\|, \end{aligned} \quad (49)$$

for some $C > 1$. Hence, $\|H^{(k+1)m}\|^{1/m} \leq C^{1/m}(\eta_k) \|H^{(k)m}\|^{1/m}$. Taking $m \rightarrow \infty$ we have

$$\rho^{(k+1)} \leq \eta_k \rho^{(k)}, \quad \text{for } k \geq 1. \quad (50)$$

Using the same argument, we also have

$$\rho^{(k+1)} \geq \eta_k^{-1} \rho^{(k)}, \quad \text{for } k \geq 1. \quad (51)$$

On the other hand, for $I \in \Sigma^{(k+j)}$ with $j \geq 0$ being fixed,

$$\begin{aligned} \eta_L^{(k+1)}([I]) &= (\rho^{(k+1)})^{-j} L(\pi^{(k)}([I])) \\ &\quad \times \left[\prod_{[I]} M^{(k+1)} \right] R(\pi^{(k)}(T^{j-1}([I]))) \\ &\leq D \times (\rho^{(k+1)})^{-j} \left\| \prod_{[I]} M^{(k+1)} \right\| \\ &\leq D \times (\rho^{(k)})^{-j} \eta_k^j \left\| \prod_{[I]} M^{(k+1)} \right\| \quad (\text{By (51)}) \\ &\leq D' \times (\rho^{(k)})^{-j} \eta_k^j \eta_k^j \left\| \prod_{[I]} M^{(k)} \right\| \\ &\leq D'' \eta_k^{2j} \eta_L^{(k)}([I]), \end{aligned} \quad (52)$$

for some $D, D', D'' > 0$. Similarly we have

$$\eta_L^{(k+1)}([I]) \geq (D'' \eta_k^{2j})^{-1} \eta_L^{(k)}([I]) \quad (53)$$

and there exists $D_1 > 0$ such that

$$(D_1 \eta_k^{2j})^{-1} \mu^{(k)}([I]) \leq \mu^{(k+1)}([I]) \leq D_1 \eta_k^{2j} \mu^{(k)}([I]). \quad (54)$$

The fact that $\lim_{k \rightarrow \infty} \eta_k = 1$ asserts and there exists $D_2 > 0, n \in \mathbb{N}$ such that for $k \geq n$ we have

$$D_2^{-1} \mu^{(k)}([I]) \leq \mu^{(k+1)}([I]) \leq D_2 \mu^{(k)}([I]). \quad (55)$$

This demonstrates $\mu^{(k)} \rightarrow \tilde{\mu}$ as $k \rightarrow \infty$ for some $\tilde{\mu}$. Define $\tilde{\nu} = \phi_* \tilde{\mu}$. $\nu^{(k)} = \phi_* \mu^{(k)}$ implies

$$\lim_{k \rightarrow \infty} \nu^{(k)} = \lim_{k \rightarrow \infty} \phi_* \mu^{(k)} = \phi_* \tilde{\mu} = \tilde{\nu}. \quad (56)$$

It can also be checked that $\tilde{\mu}$ satisfies the quasi-Bernoulli property and for all $q \in \mathbb{R} \setminus \{0\}$,

$$\tau_{\tilde{\mu}}(q) = \lim_{k \rightarrow \infty} \tau_{\mu^{(k)}}(q). \quad (57)$$

Using the same proof of Theorem A, we have

$$\begin{aligned}\tau_{\bar{\mu}}(q) &= \lim_{k \rightarrow \infty} \tau_{\mu^{(k)}}(q) = \lim_{k \rightarrow \infty} \tau_{\phi_* \mu^{(k)}}(q) \\ &= \lim_{k \rightarrow \infty} \tau_{\nu^{(k)}}(q) = \tau_{\bar{\nu}}(q).\end{aligned}\quad (58)$$

Combining Theorems 1 with A, we conclude that $P_N(q)$ is thus differentiable and the desired equality (8) follows. This completes the proof. $\square$

4. Examples

This section illustrates several examples that help for the understanding of our results.

4.1. Computation of Dimension Spectrum. Suppose X is an irreducible subshift of finite type and $\pi : X \rightarrow Y$ is a factor. Chazottes and Ugalde [14] indicate that if a matrix-valued push-forward potential function $\mathbf{N}$ is *row allowable* and is positive on periodic points, then there exists a unique Gibbs measure on Y . Here $\mathbf{N}$ is called row allowable if there is no zero row in $\mathbf{N}$. Before extending our results to nonnegative matrix-valued potential functions, we give the definition of a *column allowable* matrix first.

Definition 4. We call $A \in \mathbf{M}_{n \times n}(\mathbb{R})$ column allowable if for all $1 \leq j \leq n$, we have $\sum_{i=1}^n A_{ij} \geq 1$. We also denote by $\mathcal{N}_n$ the collection of column allowable matrices of size $n \times n$.

It can be easily verified that $\mathcal{N}_n$ forms a semigroup under matrix product.

Lemma 5. If A and $B \in \mathcal{N}_n$, then $AB \in \mathcal{N}_n$.

Proof. Indeed, for all $j = 1, \dots, n$,

$$\begin{aligned}\sum_{i=1}^n (AB)_{ij} &= \sum_{i=1}^n \sum_{k=1}^n A_{ik} B_{kj} \\ &= \sum_{k=1}^n B_{kj} \left(\sum_{i=1}^n A_{ik} \right) \\ &\geq \sum_{k=1}^n B_{kj} \geq 1.\end{aligned}\quad (59)$$

This completes the proof. $\square$

For nonnegative matrix-valued potential N , we have the following result.

Theorem 6. Let $N \in \Gamma^H(\mathbf{S}) \cap \mathcal{N}_d$ depend on k -coordinate and there exists a finite set $\Lambda \subset \bigcup_{n \geq k} \Sigma^{(n)} =: \Sigma^*$ such that for all I_1 and $I_2 \in \Sigma^*$ there exists $I \in \Lambda$ such that $I_1 \oplus I \oplus I_2 \in \Sigma^*$ and $M([I]) = N(\Phi_k([I])) > 0$ for all $I \in \Lambda$, and then (6) holds.

Proof. We give the proof for the case that all elements in Λ are equal length and the case for different length is in the same fashion. It follows from the proof in Theorem A that $H^{(k)}$ can be constructed which is indexed by the $\Sigma^{(k)}$. Since for any $\omega \in \Lambda$, $|\omega| = k + L - 1$ (we assume that Λ is equal length and the definition $\Lambda \subset \bigcup_{n \geq k} \Sigma^{(n)}$ allows us to define all elements which have equal length of $k + L - 1$) we also assume that Λ consist of only one element; say I^* . Without loss of generality, assume $I^* \in \Sigma^{(k-1)}$. It suffices to show that $H^{(k)}$ is primitive. Indeed, for any I_1 and $I_2 \in \Sigma^{(k-1)}$, since $N \in \mathcal{N}_d$, Lemma 5 is thus applied to show that

$$\begin{aligned}H_{I_1, I_2}^2 &= M([I_1 \oplus I^*]) M([I^* \oplus I_2]) \\ &= N(\Phi_k([I_1 \oplus I^*])) N(\Phi_k([I^* \oplus I_2])) > 0.\end{aligned}\quad (60)$$

This means that $H^2 > 0$. The other case can be done similarly. Therefore, the same proof as in Theorem A leads to (6) and the proof is completed. $\square$

In the proof of Theorem A, the L^q -spectrum plays an important role for the computing of dimension spectrum. We emphasize that for a measure $\mu \in \mathcal{M}_T(\Sigma_A)$, it is not easy to compute the rigorous formula for τ_μ . If the measure μ is given as in Theorem A, the following theorem provides a class of matrix-valued potentials for which we can compute its L^q -spectrum explicitly. Let $N \in \Gamma^H(\mathbf{S}) \cap \mathcal{N}_d$ depend on k -coordinate and $H := H^{(k)}$ as defined in Theorem A; we define a matrix $\mathbf{R}(q) \in \mathbf{M}_{q^{(k-1)} \times q^{(k-1)}}$ from H (recall that $q^{(k)} = \#\Sigma_{A, k+L-1}$) as follows:

$$\mathbf{R}(q)_{I_1, I_2} = \begin{cases} \rho(M([I_1 \oplus I_2]))^q, & \text{if } I_1 \oplus I_2 \in \Sigma^{(k)}; \\ 0, & \text{otherwise.} \end{cases}\quad (61)$$

$\rho(A) \in \mathbb{R}$ denotes the maximal eigenvalue of $A \in \mathbf{M}_{d \times d}(\mathbb{R})$.

Proposition 7. Under the same assumptions of Theorem 6, assume that

$$N = \{N([J])\}_{J \in \mathbf{S}_k} \in \Gamma^H(\mathbf{S}) \quad (62)$$

satisfies that

$$N([J_1]) N([J_2]) = N([J_2]) N([J_1]) \quad \forall J_1 \neq J_2 \in \mathbf{S}_k. \quad (63)$$

Assume that $H^{(k)}$ and $\nu \in \mathcal{M}_S(\mathbf{S})$ are as defined in Theorem A. Then

$$\tau_\nu(q) = \frac{1}{\log \lambda} (-q \log \rho + \log \Theta(q)), \quad (64)$$

where $\Theta(q)$ is the maximal root of the characteristic polynomial of $\mathbf{R}(q) \in \mathbf{M}_{q^{(k-1)} \times q^{(k-1)}}$.

Proof. Let

$$H = [M([I_1 \oplus I_2])]_{I_1, I_2 \in \Sigma^{(k-1)}} \quad (65)$$

be constructed as in Step 1 of the proof of Theorem A. Since elements of N are mutually commuted, then the set of

matrices $\{M([I])\}_{I \in \Sigma^{(k)}}$ can be diagonalized simultaneously. That is, there exists a unique $P \in \mathbf{M}_{d \times d}(\mathbb{R})$ such that $PM([I])P^{-1} := D([I])$ is a diagonal matrix for all $I \in \Sigma^{(k)}$. Since H is primitive, there exist L and $R > 0$ such that (19) holds. We first compute the L^q -spectrum τ_μ , where $\mu \in \mathcal{M}_T(\Sigma_A)$ is defined in the proof of Theorem A with the property that there exists a constant $C' > 0$ such that for each $I \in \Sigma_{A,n}$,

$$\frac{1}{C'} \rho^{-n} \left\| \prod_{[I]} M \right\| \leq \mu([I]) \leq C' \rho^{-n} \left\| \prod_{[I]} M \right\|. \quad (66)$$

This induces

$$\begin{aligned} \tau_\mu(q) &= \frac{1}{\log \lambda} \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{I \in \Sigma_{A,n}} \mu([I])^q \\ &= \frac{1}{\log \lambda} \left(-q \log \rho + \lim_{n \rightarrow \infty} \frac{1}{n} \right. \\ &\quad \times \log \sum_{I \in \Sigma_{A,n}} \left\| \prod_{l=0}^{n-1} M(\pi^{(k)}(T^l([I]))) \right\|^q \Big) \\ &= \frac{1}{\log \lambda} \left(-q \log \rho + \lim_{n \rightarrow \infty} \frac{1}{n} \right. \\ &\quad \times \log \sum_{I \in \Sigma_{A,n}} \left\| P^{-1} \prod_{l=0}^{n-1} D(\pi^{(k)}(T^l([I]))) P \right\|^q \Big) \\ &= \frac{1}{\log \lambda} \left(-q \log \rho + \lim_{n \rightarrow \infty} \frac{1}{n} \right. \\ &\quad \times \log \sum_{I \in \Sigma_{A,n}} \prod_{l=0}^{n-1} \rho(M(\pi^{(k)}(T^l([I])))^q \Big) \\ &= \frac{1}{\log \lambda} (-q \log \rho + \log \Theta(q)). \end{aligned} \quad (67)$$

We note here that the second equality comes from the positivity of L , R and P is invertible. Since $\tau_\nu(q) = \tau_\mu(q) = \tau_{\phi_*\mu}(q)$, the proof is completed. $\square$

Here we give a concrete example for the dimension spectrum of sofic system.

Example 8. Let Σ_A be the golden mean shift with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad (68)$$

and the right-resolving sliding block code with $L = 2$:

$$\Phi^{(2)}([00]) = a, \quad \Phi^{(2)}([01]) = b, \quad \Phi^{(2)}([10]) = b. \quad (69)$$

Define a matrix potential on $\mathbf{S}_1$, that is, $k = 1$, as in Proposition 7 by

$$N_a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad N_b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (70)$$

Then $\Lambda = \{[00]\}$ and $M([00]) = N_a > 0$. A little modification of the proof of Theorem A indicates that $P_N(q)$ is differentiable. Suppose $\alpha = P'_N(q)$ with $q \neq 0$; Theorem 6 is applied to show that

$$\dim_H E_N(\alpha) = -\alpha q + P_N(q). \quad (71)$$

On the other hand, one can easily compute that

$$\mathbf{R}(q) = \begin{bmatrix} 2^q & 1^q \\ 1^q & 0 \end{bmatrix} \quad (72)$$

and Proposition 7 applies to show that

$$\begin{aligned} \tau_\nu(q) &= \frac{1}{\log g} \left(-q \log(1 + \sqrt{2}) + \log \frac{2^q + \sqrt{2^{2q} + 4}}{2} \right), \\ \text{where } g &= \frac{1 + \sqrt{5}}{2}. \end{aligned} \quad (73)$$

4.2. Computation of Pressure. Let (Σ_A, T) be a subshift of finite type and $P_M(q)$ be its pressure for $q \in \mathbb{R}$. If $P_M(q)$ is differentiable, Theorem 1.3 of [4] demonstrates that the dimension spectrum can be computed via the formula of $P_M(q)$. However, the computation of the explicit formula for $P_M(q)$ is not easy. If $(\mathbf{S}, S)$ a sofic system, we provide a wide class of matrix potential on $\mathbf{S}$ for which we can compute its $P_N(q)$ rigorously which leads to the dimension spectrum of $E_N(\alpha)$. We first give a theorem which is analogous to Theorem 1.3 of [4].

Theorem 9. Let $N \in \Gamma_+^H(\mathbf{S})$. We have for any $\alpha = P'_N(q)$ with $q \neq 0$

$$\dim_H E_N(\alpha) = \frac{1}{\log \lambda} (-\alpha q + P_N(q)). \quad (74)$$

Proof. Up to a minor modification, the proof is identical to the proof of Theorem 1.3 of [4] and we omit it here. $\square$

We prove the following class for which we can compute its $P_N(q)$ and $\dim_H E_N(\alpha)$.

Theorem 10. If $N \in \Gamma^C(\mathbf{S})$ depends on k -coordinate, then it satisfies the following properties.

- (1) Let $H = H^{(k)}$ be the matrix constructed in Theorem A which is primitive.
- (2) Let $M \in \Gamma^C(\Sigma_A^{(k)})$ be induced from N as in the proof of Theorem A. If there exists a sequence of real numbers $\chi = (\chi_I)_{I \in \Sigma_A^{(k)}}$ and $K \in \mathbf{M}_{1 \times d}(\mathbb{R})$ is a row vector such that for any $I \in \Sigma_A^{(k)}$ we have

$$KM([I]) = \chi_I K. \quad (75)$$

Then

$$P_N(q) = \log \Theta(q), \quad (76)$$

where $\Theta(q)$ denotes the maximal eigenvalue of $\mathbf{F}(q) \in \mathbf{M}_{q^{(k-1)} \times q^{(k-1)}}(\mathbb{R})$ defined in (77) and thus it is differentiable. Furthermore, (74) can be computed explicitly.

Proof. Define $\mathbf{F}(q) \in \mathbf{M}_{q^{(k-1)} \times q^{(k-1)}}(\mathbb{R})$ by

$$\mathbf{F}(q)_{I_1, I_2} = \begin{cases} (KM([I_1 \oplus I_2])Y)^q, & \text{if } I_1 \oplus I_2 \in \Sigma^{(k)}; \\ 0, & \text{otherwise,} \end{cases} \quad (77)$$

where $Y \in \mathbf{M}_{d \times 1}(\mathbb{R})$ is a column vector with $KY = 1$. Since H is primitive, then the left and right eigenvectors are positive; that is, $L, R > 0$. Combining (75) with Perron-Frobenius Theorem we have

$$\begin{aligned} P_N(q) &= \lim_{n \rightarrow \infty} \frac{1}{n} \\ &\quad \times \log \sum_{J \in \mathbf{S}_n} \sup_{y \in [J]} \|N(y) N(S(y)) \cdots N(S^{n-1}(y))\|^q \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{J \in \mathbf{S}_n} \|N(y) N(S(y)) \cdots N(S^{n-1}(y))\|^q \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \\ &\quad \times \log \sum_{I \in P_J, x \in I, \phi(x)=y} \|M(x) M(T(x)) \cdots M(T^{n-1}(x))\|^q \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{I \in \Sigma_{A,n+L-1}} \left\| \prod_{l=0}^{n-1} M(\pi^{(k)}(\Phi_k(T^l([I]))) \right\|^q \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \\ &\quad \times \log \sum_{I \in \Sigma_{A,n+L-1}} \left(\chi_{\pi^{(k)}(\Phi_k([I]))} \cdots \chi_{\pi^{(k)}(\Phi_k(T^{n-1}([I]))})} \right)^q \\ &\quad \text{(By (77))} \\ &= \log \Theta(q). \end{aligned} \quad (78)$$

The second equality exists because N is finite-coordinate dependent and 4th equality comes from the right-resolving property of ϕ . This completes the proof. $\square$

Remark 11. (1) In Theorem A, we always assume that if one is regarded as $\Sigma_A = (G, E)$ where $G = \{I : I \in \Sigma^{(k-1)}\}$ and edges,

$$E = \{(I_1, I_2) : I_1 \oplus I_2 \in \Sigma^{(k)}\}. \quad (79)$$

Then there is only one level from I_1 to $I_2 \in \Sigma^{(k)}$; that is, the number of levels of (I_1, I_2) for any I_1 and I_2 is equal to one, and thus $H^{(k)}$ can be constructed with the entry which is

FIGURE 1: The substitution rules associated with K_T .

a single smaller matrix. However, if there is more than one level from I_1 to I_2 , we only need to modify $H_{I_1, I_2}^{(k)}$ by

$$H_{I_1, I_2}^{(k)} = \begin{cases} l(I_1 \oplus I_2) M([I_1 \oplus I_2]), & \text{if } I_1 \oplus I_2 \in \Sigma^{(k)}; \\ 0_{d \times d}, & \text{otherwise,} \end{cases} \quad (80)$$

where $l(I_1 \oplus I_2)$ denotes the number of levels from I_1 to I_2 . Since ϕ is right resolving, Theorem A still follows.

(2) In the assumption (75) of Theorem 10, one can easily check that the result remains if there exists a column vector K such that for any $I \in \Sigma_A^{(k)}$

$$M([I])K = \chi_I K. \quad (81)$$

(3) One can also easily check that for those classes of Theorem 10, if $\nu \in \mathcal{M}_S(S)$ is the measure in Theorem A, Then the L^q -spectrum is

$$\tau_\nu(q) = \frac{1}{\log \lambda} (-q \log \rho + \log \Theta(q)), \quad (82)$$

where ρ is the maximal eigenvalue of $H^{(k)}$ and $\Theta(q)$ is the maximal eigenvalue of (77).

In the following example, the computation of pressure helps for the computation of dimension spectrum of sofic affine-invariant set.

Example 12 (sofic affine-invariant set. See [15, Example 1]). Consider $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ which is invariant under

$$T = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}. \quad (83)$$

Let $D = \{0, \dots, 4\} \times \{0, \dots, 2\}$ and for any $\{d_k\}_{k=1}^\infty \in D^\mathbb{N}$ the base T representation is as follows (reader may refer to [7] for details):

$$R_T(\{d_k\}) = \sum_{k=1}^\infty \begin{bmatrix} 5^{-k} & 0 \\ 0 & 3^{-k} \end{bmatrix} d_k. \quad (84)$$

Let A be a matrix index by D which is incidence matrix and $K_T(A)$ is also defined as the image of R_T ; that is,

$$K_T(A) = \{R_T\{d_k\} : A(d_k, d_{k+1}) = 1 \text{ for } k \geq 1\}. \quad (85)$$

FIGURE 2: The first and second figures are fourth and seventh steps of iteration, respectively.

Let $\mathbf{S}$ be a sofic system that is induced by projecting Σ_A on y -direction, see Figure 1, and let $N = (N([s]))_{s=0}^2$ be defined as

$$\begin{aligned} N([0]) &= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, & N([1]) &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \\ N([2]) &= \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (86)$$

Let

$$K_T^{(n)}(\mathbf{S}) = \{R_T\{d_k\} : (d_k)_{k \geq 1} \in \mathbf{S}_n = \phi(\Sigma_{A,n+L-1})\}, \quad (87)$$

$$K_T(\mathbf{S}) = \{R_T\{d_k\} : (d_k)_{k \geq 1} \in \mathbf{S} = \phi(\Sigma_A)\}.$$

$K_T^{(4)}(\mathbf{S})$ and $K_T^{(7)}(\mathbf{S})$ are represented in Figure 2. Define $\Phi^{(2)}$ by

$$\begin{aligned} \Phi^{(2)}([i0]) &= [0] & \text{for } i = 0, 3, 4, \\ \Phi^{(2)}([i1]) &= [1] & \text{for } i = 0, 1, 2, \\ \Phi^{(2)}([i2]) &= [2] & \text{for } i = 1, 2, 3, 4. \end{aligned} \quad (88)$$

Then one can easily check that if $K = (1, 1)$, then $\chi = (\chi_I)_{I \in \Sigma_2}$ can be defined as

$$\chi_0 = \chi_1 = \chi_2 = 2. \quad (89)$$

$H^{(1)}$ can be defined as follows ((1) of Remark 11):

$$\begin{aligned} H^{(1)} &= \begin{bmatrix} N([0]) + 2N([1]) + 2N([2]) & N([1]) + 2N([2]) \\ N([0]) & 2N([0]) + N([1]) \end{bmatrix}. \end{aligned} \quad (90)$$

For any $q \in \mathbb{R}$, define $\mathbf{F}(q)$ from H by taking $K = (1, 1)$ and $Y = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$:

$$\mathbf{F}(q) = \begin{bmatrix} 5 \times 2^q & 3 \times 2^q \\ 1 \times 2^q & 3 \times 2^q \end{bmatrix}, \quad (91)$$

and then

$$P_N(q) = \log \Theta(q) = \log 6 + q \log 2. \quad (92)$$

Theorem 6 indicates that $\alpha = \log 2$ and

$$\dim_H E_N(\log 2) = \frac{1}{\log 6} (-q \log 2 + (\log 6 + q \log 2)) = 1 \quad (93)$$

which is constant multifractal.

If N is symmetric, we also have the following estimate.

Corollary 13. Let $N \in \Gamma^C(\mathbf{S})$ depend on k -coordinate and let $H = H^{(k)}$ be the matrix constructed in Theorem A which is primitive. Let $M \in \Gamma^C(\Sigma_A^{(k)})$ be induced from N , if for any $I \in \Sigma_A^{(k)}$, there exist $a_I, b_I \in \mathbb{R}$ such that

$$M([I]) = \begin{bmatrix} a_I & b_I \\ b_I & a_I \end{bmatrix}. \quad (94)$$

Then

$$P_N(q) = \log \Theta(q), \quad (95)$$

where $\Theta(q)$ denotes the maximal eigenvalue of $\mathbf{F}(q) \in \mathbf{M}_{q^{(k-1)} \times q^{(k-1)}}(\mathbb{R})$ which is defined by

$$\mathbf{F}(q)_{I_1, I_2} = \begin{cases} (a_I + b_I)^q & \text{if } I := I_1 \oplus I_2 \in \Sigma^{(k)} \\ 0 & \text{otherwise.} \end{cases} \quad (96)$$

Proof. Since for any $I \in \Sigma_A^{(k)}$,

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a_I & b_I \\ b_I & a_I \end{bmatrix} = (a_I + b_I) \begin{bmatrix} 1 & 1 \end{bmatrix}. \quad (97)$$

Theorem 10 is applied to show that $P_N(q) = \log \Theta(q)$, where $\Theta(q)$ is the maximal eigenvalue of (96). The proof is completed. $\square$

Example 14 (continued). Under the same substitution rule of Example 12, if the potentials on $\mathcal{A}(\mathbf{S})$ are as follows

$$\begin{aligned} N([0]) &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, & N([1]) &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \\ N([2]) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \end{aligned} \quad (98)$$

one can easily check that $H^{(1)}$ is primitive and define

$$\mathbf{F}(q) = \begin{bmatrix} 3^q + 2 \times 4^q + 2 \times 1^q & 4^q + 2 \times 1^q \\ 3^q & 2 \times 3^q + 4^q \end{bmatrix}. \quad (99)$$

Then

$$P_N(q) = \log 2 + \log(1 + 3^q + 4^q). \quad (100)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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