Research Article

On the Existence of Central Configurations of $2k + 2p + 2l$ -Body Problems

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We prove the existence of central configurations of the $2k + 2p + 2l$ -body problems with Newtonian potentials in R^3 . In such configuration, $2k$ masses are symmetrically located on the z -axis, $2p$ masses are symmetrically located on the y -axis, and $2l$ masses are symmetrically located on the x -axis, respectively; the masses symmetrically about the origin are equal.

1. Introduction and Main Results

The Newtonian n -body problems [1–3] is concerned with the motions of n particles with masses $m_j \in R^+$ and positions $q_j \in R^3$ ($j = 1, 2, \dots, n$); the motion is governed by Newton's second law and the Universal law:

$$m_j \ddot{q}_j = \frac{\partial U(q)}{\partial q_j}, \quad (1)$$

where $q = (q_1, q_2, \dots, q_n)$ with Newtonian potential

$$U(q) = \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{|q_j - q_k|}. \quad (2)$$

Consider the space

$$X = \left\{ q = (q_1, q_2, \dots, q_n) \in R^{3n} : \sum_{j=1}^N m_j q_j = 0 \right\}; \quad (3)$$

that is, suppose that the center of mass is fixed at the origin of the coordinate axis. Because the potential is singular when two particles have same position, it is natural to assume that the configuration avoids the collision set $\Delta = \{q = (q_1, \dots, q_N) : q_j = q_k \text{ for some } k \neq j\}$. The set $X \setminus \Delta$ is called the configuration space.

Definition 1 (see [2, 3]). A configuration $q = (q_1, q_2, \dots, q_n) \in X \setminus \Delta$ is called a central configuration if there exists a constant λ such that

$$\sum_{j=1, j \neq k}^n \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad 1 \leq k \leq n. \quad (4)$$

The value of constant λ in (4) is uniquely determined by

$$\lambda = \frac{U}{I}, \quad (5)$$

where

$$I = \sum_{k=1}^n m_k |q_k|^2. \quad (6)$$

Since the general solution of the n -body problem cannot be given, great importance has been attached to search for particular solutions from the very beginning. A homographic solution is a configuration which is self-similar for all time. Central configurations and homographic solutions are linked by the Laplace theorem [3]. Collapse orbits and parabolic orbits have relations with the central configurations [2, 4–6]. So finding central configurations becomes very important.

Long and Sun [7] studied four-body central configurations with some equal masses; Albouy et al. [8] took an alternate approach to the symmetry of planar four-body convex

FIGURE 1

central configurations; Llibre and Mello [9] considered triple and quadruple nested central configurations; Corbera and Llibre [10] analyzed the existence of central configurations of p nested regular polyhedra. Based on the above works, we study central configuration for Newtonian $2k + 2p + 2l$ -body problems. In the $2k + 2p + 2l$ -body problems, $2k$ masses are symmetrically located on the z -axis, $2p$ masses are symmetrically located on the y -axis, and $2l$ masses are symmetrically located on the x -axis, the masses located on the same axis and symmetrical about the origin are equal (see Figure 1 for $k = 1$, $p = 2$, and $l = 3$).

In this paper, we extend the result of Corbera-Llibre by the following.

Theorem 2. For $2k + 2p + 2l$ -body problem in R^3 where $k \geq 1$, $p \geq 1$, and $l \geq 1$, there is at least one central configuration such that $2k$ points are located along the first axis symmetric with respect to the origin, $2p$ points are located on the second axis symmetric with respect to the origin, and $2l$ points are located on the third axis, also symmetric with respect to the origin. Along each axis, the masses symmetric with respect to the origin are equal.

2. The Proof of Theorem 2

2.1. Equations for the Central Configurations of $2+2+2l$ -Body Problems. To begin, we take a coordinate system which simplifies the analysis. The particles have positions given by

$$\begin{aligned} q_1 &= (0, 0, 1), & q_2 &= (0, 0, -1), \\ q_3 &= (0, y_1, 0), & q_4 &= (0, -y_1, 0), \\ & & y_1 &> 0, \\ q_{4+i} &= (x_i, 0, 0), & q_{4+i} &= (x_i, 0, 0), \dots, \\ q_{4+l} &= (x_l, 0, 0), & q_{4+l+i} &= (-x_i, 0, 0), \end{aligned} \quad (7)$$

where $i = 1, \dots, l$, $0 < x_1 < x_2 < \dots < x_l$.

The masses are given by $m_1 = m_2 = M_1 = 1$, $m_3 = m_4 = M_2$, and $m_{4+i} = m_{4+l+i} = M_{2+i}$, where $i = 1, \dots, l$.

By the symmetries of the system, (4) is equivalent to the following equations:

$$\begin{aligned} -\lambda(0, 0, 1) &= \left(0, 0, \frac{-1}{4}\right) + \left(0, 0, \frac{-2}{|1 + y_1^2|^{3/2}}\right) M_2 \\ &\quad + \sum_{j=1}^l \left(0, 0, \frac{-2}{|1 + x_j^2|^{3/2}}\right) M_{2+j}, \\ -\lambda(0, y_1, 0) &= \left(0, \frac{-2y_1}{|1 + y_1^2|^{3/2}}, 0\right) + \left(0, \frac{-1}{4y_1^2}, 0\right) M_2 \\ &\quad + \sum_{j=1}^l \left(0, \frac{-2y_1}{|x_j^2 + y_1^2|^{3/2}}, 0\right) M_{2+j}, \\ -\lambda(x_1, 0, 0) &= \left(\frac{-2x_1}{|1 + x_1^2|^{3/2}}, 0, 0\right) \\ &\quad + \left(\frac{-2x_1}{|y_1^2 + x_1^2|^{3/2}}, 0, 0\right) M_2 \\ &\quad + \left(\frac{-1}{4x_1^2}, 0, 0\right) M_{2+1} \\ &\quad + \sum_{j=2}^l \left(\frac{1}{|x_j - x_1|^2} - \frac{1}{|x_j + x_1|^2}, 0, 0\right) M_{2+j}, \\ -\lambda(x_i, 0, 0) &= \left(\frac{-2x_i}{|1 + x_i^2|^{3/2}}, 0, 0\right) \\ &\quad + \left(\frac{-2x_i}{|y_1^2 + x_i^2|^{3/2}}, 0, 0\right) M_2 \\ &\quad + \sum_{j < i} \left(-\frac{1}{|x_j - x_i|^2} - \frac{1}{|x_j + x_i|^2}, 0, 0\right) M_{2+j} \\ &\quad + \left(\frac{-1}{4x_i^2}, 0, 0\right) M_{2+i} \\ &\quad + \sum_{i < j} \left(\frac{1}{|x_j - x_i|^2} - \frac{1}{|x_j + x_i|^2}, 0, 0\right) M_{2+j}, \end{aligned} \quad (8)$$

for $i = 2, \dots, l - 1$,

$$\begin{aligned} -\lambda(x_l, 0, 0) &= \left(\frac{-2x_l}{|1 + x_l^2|^{3/2}}, 0, 0\right) \\ &\quad + \left(\frac{-2x_l}{|y_1^2 + x_l^2|^{3/2}}, 0, 0\right) M_2 \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^{l-1} \left(-\frac{1}{|x_j - x_l|^2} - \frac{1}{|x_j + x_l|^2}, 0, 0 \right) M_{2+j} \\
& + \left(\frac{-1}{4x_l^2}, 0, 0 \right) M_{2+l}.
\end{aligned} \tag{9}$$

In order to simplify the equations, we defined the following:

$a_{i,1} = 1$, where $i = 1, \dots, 2 + l$; $a_{1,2} = -2/|1 + y_1^2|^{3/2}$; $a_{1,2+j} = -2/|1 + x_j^2|^{3/2}$, where $j = 1, \dots, l$; $a_{2,2} = -1/4y_1^3$; $a_{2,2+j} = -2/|x_j^2 + y_1^2|^{3/2}$, where $j = 1, \dots, l$; $a_{j+2,j+2} = -1/4x_j^3$, where $j = 1, \dots, l$; $a_{i+2,2} = -2/|y_1^2 + x_i^2|^{3/2}$, where $i = 1, \dots, l$; $a_{i+2,j+2} = -(1/|x_j - x_i|^2 x_i) - (1/|x_j + x_i|^2 x_i)$, where $1 \leq j < i \leq l$; $a_{i+2,j+2} = 1/|x_j - x_i|^2 x_i - 1/|x_j + x_i|^2 x_i$, where $1 \leq i < j \leq l$; $b_1 = 1/4$, $b_2 = 2/|1 + y_1^2|^{3/2}$, $b_{i+2} = 2/|1 + x_i^2|^{3/2}$, where $i = 1, \dots, l$.

Equations (8) and (9) can be written as a linear system of the form $AX = b$ given by

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,2+1} & \cdots & a_{1,2+l} \\ a_{2,1} & a_{2,2} & a_{2,2+1} & \cdots & a_{2,2+l} \\ a_{2+1,1} & a_{2+1,2} & a_{2+1,2+1} & \cdots & a_{2+1,2+l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2+l,1} & a_{2+l,2} & a_{2+l,2+1} & \cdots & a_{2+l,2+l} \end{pmatrix} \begin{pmatrix} \lambda \\ M_2 \\ M_{2+1} \\ \vdots \\ M_{2+l} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_{2+1} \\ \vdots \\ b_{2+l} \end{pmatrix}. \tag{10}$$

The column vector is given by the variables $X = (\lambda, M_2, M_{2+1}, \dots, M_{2+l})^T$. Since the a_{ij} are function of $y_1, x_1, \dots, x_l$, we write the coefficient matrix as $A_{2+l}(y_1, x_1, \dots, x_l)$.

2.2. For $k = 1$, $p = 1$, and $l = 1$

Lemma 3. When $p = 1$, $k = 1$, and $l = 1$, there exists $(\tilde{x}_1, \tilde{y}_1) \in (0, +\infty) \times (0, +\infty)$ such that (10) has a unique solution (λ, M_2, M_3) satisfying $\lambda, M_2 > 0$ and $M_3 > 0$.

Proof. If we consider $|A_{2+1}(x_1, y_1)|$ as a function of x_1 and y_1 , then $|A_{2+1}(x_1, y_1)|$ is an analytic function and nonconstant. When $p = 1$, $k = 1$, and $l = 1$, (10) is equivalent to

$$\begin{aligned}
F_1(x_1, y_1, M_2, M_3) &= 0, \\
F_2(x_1, y_1, M_2, M_3) &= 0,
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
F_1(x_1, y_1, M_2, M_3) &= (a_{1,2} - a_{2,2}) M_2 + (a_{1,3} - a_{2,3}) M_3 \\
&+ b_2 - b_1, \\
F_2(x_1, y_1, M_2, M_3) &= (a_{1,2} - a_{3,2}) M_2 + (a_{1,3} - a_{3,3}) M_3 \\
&+ b_3 - b_1.
\end{aligned} \tag{12}$$

It is obvious that

$$\begin{aligned}
F_1(1, 1, 1, 1) &= F_2(1, 1, 1, 1) = 0, \\
\left| \frac{\partial F_1(1, 1, 1, 1)}{\partial M_2} \frac{\partial F_1(1, 1, 1, 1)}{\partial M_3} \right| &= \left(\frac{1}{4} - \frac{\sqrt{2}}{2} \right)^2 \neq 0.
\end{aligned} \tag{13}$$

By implicit function theorem, there exists $(\tilde{x}_1, \tilde{y}_1) \in (0, +\infty) \times (0, +\infty)$ such that (10) has a unique solution (λ, M_2, M_3) satisfying $\lambda, M_2 > 0$ and $M_3 > 0$. $\square$

2.3. For $k = 1$, $p = 1$, and $l \geq 1$. The proof for $k = 1$, $p = 1$, and $l \geq 1$ is done by induction. At each step, we assume we have inductively shown the existence of a central configuration symmetrically located at the x -axis, and we add two particles with zero masses. Using continuity with respect to the masses of the above defined equations, we prove the existence of a central configuration when the mass of the added particles is zero and the mass of the remaining particles are changed suitably. We then verify that the Hessian of this new system of masses has no kernel and we conclude that the two new particles can have nonzero masses which still maintain a central configuration.

For $k = 1$, $p = 1$, and $l \geq 1$, we claim that there exists $0 < y_1, 0 < x_1 < x_2 < \dots < x_l$ such that system (10) have a unique solution $\lambda = \lambda(y_1, x_1, \dots, x_l) > 0$, $M_i = M_i(y_1, x_1, \dots, x_l) > 0$ for $i = 2, \dots, l + 2$. We have seen that the claim is true for $k = 1$, $p = 1$, and $l = 1$. By induction, we assume the claim is true for $k = 1$, $p = 1$, and $l - 1$, and we will prove it for $k = 1$, $p = 1$, and l .

Assume by induction hypothesis that there exists $0 < \tilde{y}_1, 0 < \tilde{x}_1 < \tilde{x}_2 < \dots < \tilde{x}_{l-1}$ such that system (10) with $l - 1$ instead of l has a unique solution $(\tilde{\lambda}, \tilde{M}_2, \tilde{M}_{2+1}, \dots, \tilde{M}_{2+l})$, where $\tilde{\lambda} = \lambda(\tilde{y}_1, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{l-1}) > 0$ and $\tilde{M}_i = M_i(\tilde{y}_1, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{l-1}) > 0$.

We need the next Lemma.

Lemma 4. There exists $\tilde{x}_l > \tilde{x}_{l-1}$ such that $\tilde{\lambda} = \lambda(\tilde{y}_1, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{l-1})$, $\tilde{M}_i = M_i(\tilde{y}_1, \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{l-1})$ for $i = 2, \dots, l + 1$, and $\tilde{M}_{l+2} = 0$ is a solution of (10).

Proof. Since $\tilde{M}_{l+2} = 0$, we have that the first $l + 1$ equations of (10) are satisfied when $\lambda = \tilde{\lambda}$ and $M_i = \tilde{M}_i$ for $i = 2, \dots, l + 1$ and $M_{l+2} = \tilde{M}_{l+2} = 0$. Substituting this solution into the last equations of (10), we get the equation

$$\begin{aligned}
f(x_l) &= \tilde{\lambda} - \frac{2}{|y_1^2 + x_l^2|^{3/2}} \tilde{M}_2 \\
&- \sum_{i=1}^{l-1} \left(\frac{1}{|x_l - x_i|^2 x_l} + \frac{1}{|x_l + x_i|^2 x_l} \right) M_{i+2} \\
&- \frac{1}{4x_l^3} \tilde{M}_{l+2} - \frac{2}{|1 + x_l^2|^{3/2}}.
\end{aligned} \tag{14}$$

We have that

$$\lim_{x_l \rightarrow +\infty} f(x_l) = \tilde{\lambda} > 0, \quad \lim_{x_l \rightarrow \tilde{x}_{l-1}^+} f(x_l) = -\infty. \quad (15)$$

Therefore, there exists at least a value $x_l = \tilde{x}_l > \tilde{x}_{l-1}$ satisfying equation $f(x_l) = 0$. This completes the proof of Lemma 4. $\square$

By using the implicit function theorem we will prove that the solution of (10) given in Lemma 4 can be continued to a solution with $M_{l+2} > 0$.

Let $s = (\lambda, M_2, \dots, M_{l+1}, M_{l+2}, y_1, x_1, \dots, x_l)$, we define

$$\begin{aligned} g_1(s) &= a_{1,1}\lambda + a_{1,2}M_2 + \sum_{j=1}^l a_{1,2+j}M_{2+j} - b_1, \\ g_2(s) &= a_{2,1}\lambda + a_{2,2}M_2 + \sum_{j=1}^l a_{2,2+j}M_{2+j} - b_2, \\ g_{2+i}(s) &= a_{2+i,1}\lambda + a_{2+i,2}M_2 + \sum_{j=1}^l a_{2+i,2+j}M_{2+j} - b_{2+i}, \\ &\vdots \\ g_{2+l}(s) &= a_{2+l,1}\lambda + a_{2+l,2}M_2 + \sum_{j=1}^l a_{2+l,2+j}M_{2+j} - b_{2+l}. \end{aligned} \quad (16)$$

It is not difficult to see that the system (10) is equivalent to $g_i(s) = 0$ for $i = 1, \dots, l+2$. Let $\tilde{s} = (\tilde{\lambda}, \tilde{M}_2, \tilde{M}_{2+1}, \dots, \tilde{M}_{2+l}, \tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_l)$ be the solution of system (10) given in Lemma 4. The differential of (16) with respect to the variables $(\lambda, M_2, M_{2+1}, \dots, M_{2+l-1}, x_l)$ is

$$\begin{aligned} D(\tilde{s}) &= \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,2+1} & \cdots & a_{1,2+l-1} & 0 \\ a_{2,1} & a_{2,2} & a_{2,2+1} & \cdots & a_{2,2+l-1} & 0 \\ a_{2+1,1} & a_{2+1,2} & a_{2+1,2+1} & \cdots & a_{2+1,2+l-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2+l-1,1} & a_{2+l-1,2} & a_{2+l-1,2+1} & \cdots & a_{2+l-1,2+l-1} & 0 \\ a_{2+l,1} & a_{2+l,2} & a_{2+l,2+1} & \cdots & a_{2+l,2+l-1} & \frac{\partial g_{2+l}(\tilde{s})}{\partial x_l} \end{vmatrix} \\ &= |A_{2+l-1}(\tilde{s})| \frac{\partial g_{2+l}(\tilde{s})}{\partial x_l}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} \frac{\partial g_{2+l}(\tilde{s})}{\partial x_l} &= \frac{6x_l}{|y_1^2 + x_l^2|^{5/2}} \tilde{M}_2 \\ &\quad + \sum_{i=1}^{l-1} \left(\frac{2}{|x_l - x_i|^3 x_l} + \frac{1}{|x_l - x_i|^2 x_l^2} \right) \tilde{M}_{2+i} \end{aligned}$$

$$\begin{aligned} &+ \frac{2}{|x_l - x_i|^3 x_l} + \frac{2}{|x_l - x_i|^2 x_l^2} \Big) \tilde{M}_{2+i} \\ &+ \frac{3}{4x_l^4} \tilde{M}_{2+l} + \frac{6x_l}{|1 + x_l^2|^{3/2}} > 0. \end{aligned} \quad (18)$$

We have assumed that $(\tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_{l-1})$ makes the system (10) with $l-1$ instead of l have a unique solution, so $|A_{2+l-1}(\tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_{l-1})| \neq 0$; therefore, $D(\tilde{s}) \neq 0$. Applying the implicit function theorem, there exists a neighborhood U of $(\tilde{M}_{l+2}, \tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_{l-1})$ and unique analytic functions $\lambda = \lambda(M_{l+2}, y_1, x_1, \dots, x_{l-1})$, $M_i = M_i(M_{l+2}, y_1, x_1, \dots, x_{l-1})$ for $i = 2, \dots, l+1$ and $x_l = x_l(M_{l+2}, y_1, x_1, \dots, x_{l-1})$, such that $(\lambda, M_2, M_{2+1}, \dots, M_{2+l})$ is the solution of the system (10) for all $(M_{l+2}, y_1, x_1, \dots, x_{l-1}) \in U$. The determinant is calculated as

$$\begin{aligned} |A_{2+l}(y_1, x_1, \dots, x_l)| &= a_{2+l,2+l} |A_{2+l-1}(y_1, x_1, \dots, x_{l-1})| \\ &\quad + \sum_{j=1}^{2+l-1} a_{2+l,j} B_{2+l,j} \\ &= -\frac{1}{4x_l^2} |A_{2+l-1}(y_1, x_1, \dots, x_{l-1})| \\ &\quad + \sum_{j=1}^{2+l-1} a_{2+l,j} B_{2+l,j}, \end{aligned} \quad (19)$$

where $B_{2+l,j}$ is the algebraic cofactor of $a_{2+l,j}$.

We see that $|A_{2+l-1}(y_1, x_1, \dots, x_{l-1})|$, $B_{2+l,j}$, and $a_{2+l,j}$ for $j = 1, \dots, 2+l-1$ do not contain the factor $1/x_l^3$. If we consider $|A_{2+l}(y_1, x_1, \dots, x_l)|$ as a function of x_l , then $|A_{2+l}(y_1, x_1, \dots, x_l)|$ is analytic and nonconstant. It is obvious that $\partial g_i(\tilde{s}) / \partial M_{l+2} = a_{i,l+2} \neq 0$ for $i = 1, 2, \dots, l+2$. We can find $(\tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_l)$ sufficiently close to $(\tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_l)$ such that $|A_{2+l}(\tilde{y}_1, \tilde{x}_1, \dots, \tilde{x}_l)| \neq 0$ and, therefore, a solution $(\lambda, M_2, \dots, M_{2+l})$ of system (10) satisfying $\lambda > 0$, $M_i > 0$ for $i = 2, \dots, l+2$.

2.4. For $k \geq 1$, $p \geq 1$, and $l \geq 1$. The proof for $k = 1$, $p \geq 1$, and $l \geq 1$ is also done by induction. At each step, two particles are symmetrically located at the y -axis. Using the similar arguments above we prove Theorem 2 for $k = 1$, $p \geq 1$, and $l \geq 1$. At each step, two particles are symmetrically located at the z -axis. Using the similar arguments above we prove Theorem 2 for $k \geq 1$, $p \geq 1$, and $l \geq 1$.

The proof of Theorem 2 is completed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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