Research Article

Bifurcation Problems for Generalized Beam Equations

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We investigate a class of bifurcation problems for generalized beam equations and prove that the one-parameter family of problems have exactly two bifurcation points via a unified, elementary approach. The proof of the main results relies heavily on calculus facts rather than such complicated arguments as Lyapunov-Schmidt reduction technique or Morse index theory from nonlinear functional analysis.

1. Introduction and Main Results

In physics, the vibration of an elastic beam, with length 1 and one endpoint hinged at $x = 0$, which is compressed at the free edge ($x = 1$) by a force of intensity proportional to $\lambda > 0$, is governed by the so-called beam equation

$$\varphi'' + \lambda \sin \varphi = 0 \quad \text{in } (0, 1); \quad (1)$$

see [1]. The beam maintains its shape when the “force” λ is sufficiently small, but it will buckle once λ exceeds a certain value. In mathematics, the set of such values can be studied by exploiting the homogeneous Neumann boundary value problem:

$$\begin{aligned} \varphi'' + \lambda \sin \varphi &= 0 \quad \text{in } (0, 1), \\ \varphi'(0) &= \varphi'(1) = 0. \end{aligned} \quad (2)$$

Before stating precisely the properties which we will explore in BVP (2), we embed this problem into a family of such boundary value problems; that is, we introduce the family of problems

$$\begin{aligned} \varphi'' + \lambda h \circ \varphi &= 0 \quad \text{in } (a, b), \\ \varphi'(a) &= \varphi'(b) = 0, \end{aligned} \quad (3)$$

where $a, b \in \mathbb{R}$ with $a < b$, λ belongs to a certain nonempty subset of $\mathbb{R}$, and $\varphi = \varphi(x)$ is the unknown; the function $h \in$

$C^\infty(\mathbb{R}; \mathbb{R})$ satisfies the following: there exists an $l > 0$ such that

$$(H1) \quad h(x + l) = -h(x), \quad \text{for all } x \in \mathbb{R},$$

$$(H2) \quad x_0 \in \mathbb{R} \text{ for which}$$

$$h'(x_0) > 0, \quad (4)$$

$$h(x_0 + x) = -h(x_0 - x) > 0, \quad \forall x \in [0, l], \quad (5)$$

$$K : [0, l] \ni x \mapsto \sqrt{\int_{x_0}^{x_0+x} h(t) dt} \quad (6)$$

$\in [0, \infty)$ is a concave function.

Remark 1. We call the equation occurring in BVP (3) “generalized” beam equation; such equations are widely used to describe various physical phenomena.

Remark 2. It follows immediately from the hypothesis (5) that $h(x_0) = h(x_0 + l) = 0$ and from (H1) that h is $2l$ -periodic.

Remark 3. It is easy to see that the function $h(x) = \sin x$, $x \in \mathbb{R}$, satisfies the hypotheses (H1)-(H2) with $l = \pi$, $x_0 = 0$.

Trivially, BVP (3) admits the trivial solution $\varphi = 0$ for any $\lambda \in \mathbb{R}$. Here we are focused on the bifurcation theory for BVP (3). The bifurcation points are determined by eigenvalues associated with the differential operator $\varphi'' + \lambda h \circ \varphi$. At such

points, the number of solutions to (3) may change. However, very little further work has been done to determine whether the number of solutions changes at these points. In this paper, we give such a criterion for a class of nonlinear problems.

Theorem 4. *Let $a, b, \lambda \in \mathbb{R}$ with $a < b$. Assume (H1)-(H2). Then*

$$\frac{\pm\pi^2}{((b-a)^2 h'(x_0))} \quad (7)$$

are two bifurcation points for BVP (3). Besides, (3) has non-constant solutions if and only if

$$\pi^2 < |\lambda| (b-a)^2 h'(x_0). \quad (8)$$

The proof of a bifurcation assertion of a nonlinear equation often has as ingredients such topological arguments as Krasnoselskii's and Rabinowitz's theorems on bifurcation. These arguments usually have the assumption that the algebraic multiplicity of the associated linear eigenvalue problem is odd; see [1–3] and the references therein. Since then, several authors have also attempted to remove such oddness assumption; see [1, 2, 4]. In particular, Ma and Wang [2] developed an elaborate algorithm to prove steady state bifurcation assertions concerning nonlinear equations; this algorithm does not assume the oddness of the algebraic multiplicity. See [5–13] for more studies on bifurcation problems. Our approach to prove Theorem 4 does *not* assume such parity condition on the algebraic multiplicity.

As a matter of fact, BVP (2) is a special case of Sturm-Liouville problem or boundary value problems for elliptic partial differential equations. Therefore, BVP (2), possibly in disguise, has been studied extensively in the literature for the existence of solutions satisfying certain prescribed properties, for qualitative properties of solutions, and so on; see [14–17] and the profound references cited therein.

The remainder of this paper is organized as follows. In Section 2 we introduce some nonlinear functional analysis and formulate the problem in a formal way, and in Section 3 we give the proof of Theorem 4.

2. The Existing Bifurcation Results for BVP (2)

In this section, we mainly give a brief review of the existing results in the literature concerning bifurcation problems for BVP (2) which can be viewed as archetypes of bifurcation problems for BVP (3). Indeed, bifurcation problems for BVP (2) have been often provided as illustration examples to test the proposed abstract bifurcation-problems-solving method in the literature; see [1, 12, 13, 18].

In particular, Ma and Wang [18] proposed an abstract method which generalizes slightly the previous one obtained by Nirenberg [1]. In presenting their method, the authors fixed two Banach spaces X and Y for which X embeds continuously and densely into Y . The abstract problem which they were concerned with reads

$$L_\lambda u + G_\lambda(u) = 0, \quad (9)$$

where $L_\lambda : X \rightarrow Y$, $\lambda \in \mathbb{R}$, is a family of bounded linear operators and $G_\lambda : X \rightarrow Y$ is a family of continuous mappings. They assumed the following.

(H3) L_λ is in the form $L_\lambda = A + B_\lambda$ with A as a linear topological isomorphism of X onto Y and B_λ as compact linear operators; hence the spectrum of L_λ consists of the exactly countably many eigenvalues $\{\beta_k(\lambda)\}$ (listed by algebraic multiplicities) of L_λ ; there exists λ_0 for which

$$\begin{aligned} \beta_1(\lambda_0) &\neq \beta_j(\lambda_0), \quad \beta_j(\lambda_0) \neq 0, \quad \forall j > 1, \\ \beta_j(\lambda_0) &< 0, \quad \text{if } \lambda < \lambda_0, \\ \beta_1(\lambda_0) &= 0, \quad \text{if } \lambda = \lambda_0, \\ \beta_j(\lambda_0) &> 0, \quad \text{if } \lambda > \lambda_0. \end{aligned} \quad (10)$$

(H4) For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|G_\lambda(u)\|_Y < \varepsilon, \quad \forall u \in X \text{ with } \|u\|_X < \delta, \quad \forall \lambda \in \mathbb{R}. \quad (11)$$

G_λ is analytic in the sense that

$$G_\lambda = G_\lambda^0 + \sum_{n=1}^{\infty} G_\lambda^n \left(\underbrace{u, u, \dots, u}_{n \text{ copies}} \right); \quad (12)$$

G_λ^n is a continuous, symmetric n -form on X .

The precise problem with which they are concerned is whether there is a $\lambda_0 \in \mathbb{R}$ given in such a way that if $u_\lambda \neq 0$ with λ in a neighborhood of λ_0 is a collection of solutions to BVP (9), then

$$u_\lambda \longrightarrow 0 \quad \text{in } X, \quad \text{as } \lambda \longrightarrow \lambda_0. \quad (13)$$

If there exists a λ_0 which satisfies the above requirements, then λ_0 is called a bifurcation point for nonlinear problem (9); also, problem (9) is said to bifurcate from λ_0 .

Concerning (9), they proved the following.

Assume (H3)-(H4). Then λ_0 is a candidate bifurcation point of the nonlinear problem (9).

The proof of the above theorem provided in [18] utilizes such complicated methods as Lyapunov-Schmidt reduction method, Morse index theory, and so forth.

Ma and Wang [18] used the above theorem to obtain the bifurcation results for BVP (2). Indeed, they wrote firstly $X = \{u \in H^2(0, 1); u'(0) = u'(1)\}$, $Y = L^2(0, 1)$, $L_\lambda u = (d^2/dx^2)u - \lambda u$, and $G_\lambda(u) = \lambda \sin u - \lambda u$, thereby recasting BVP (2) into one of the forms (9), and secondly they solved this new bifurcation problem for BVP (2) by utilizing their abstract result.

Here we are tempted to use the results obtained in Ma and Wang [18] to solve the bifurcation problem for BVP (3); it is however obvious that the nonlinear reaction $\mathbb{R} \ni u \rightarrow h(\sin u) \in \mathbb{R}$ precludes our application of such results. In the next section, we will analyze the bifurcation problem for BVP (3) in an elementary way.

3. Proof of the Main Results

In this section we propose two lemmas and then prove Theorem 4 based on them. Various calculus theorems are employed in our proofs, and the elementary equality

$$(f^{-1})'(f(t)) = \frac{1}{f'(t)} \quad (14)$$

is also used repeatedly.

For the sake of convenience, we write

$$H(x) = K^2(x) = \int_{x_0}^{x_0+x} h(t) dt \quad (0 \leq x \leq l). \quad (15)$$

The function H is strictly increasing on $[0, l]$, $H(0) = 0$, and $H'(x) = h(x_0 + x)$. H is 2π -periodic because

$$H(x + 2l) - H(x) = \int_x^{x+2l} h(t) dt = \int_{-l}^l h(x_0 + t) dt = 0, \quad (16)$$

due to Remark 2.

Lemma 5. *The function K is strictly increasing and differentiable on $[0, l]$, with derivative*

$$K'(x) = \begin{cases} \frac{H'(x)}{(2\sqrt{H(x)})} & 0 < x \leq l, \\ \sqrt{\frac{h'(x_0)}{2}} & x = 0. \end{cases} \quad (17)$$

Moreover,

$$K'(K^{-1}(y)) = \frac{1}{2y(H^{-1})'(y^2)} \quad (0 < y < \sqrt{H(l)}). \quad (18)$$

Proof. Since $K = \sqrt{H}$ and H is positive on $(0, l]$, the value $K'(x)$ with $x \in (0, l]$ can be derived directly from (15) and thus half of (17) is obtained. Consequently,

$$K'(0) = \lim_{x \downarrow 0} K'(x) = \lim_{x \downarrow 0} \frac{H'(x)}{2\sqrt{H(x)}}. \quad (19)$$

Noting that $H' \geq 0$ on $[0, l]$, L'Hospital rule shows that

$$(K'(0))^2 = \lim_{x \downarrow 0} \frac{(H'(x))^2}{4H(x)} = \lim_{x \downarrow 0} \frac{2H'(x)H''(x)}{4H'(x)} = \frac{h'(x_0)}{2}. \quad (20)$$

Thus the other half of (17) is obtained. A simple computation using (17) and (14) gives (18). $\square$

Lemma 6. *Define*

$$p(t) = \int_0^t \frac{1}{\sqrt{H(t) - H(y)}} dy \quad (0 < t \leq l). \quad (21)$$

For $0 < t < l$, the integral above converges, and the function p is strictly increasing on $(0, l]$. Besides,

$$\lim_{t \downarrow 0} p(t) = \frac{\pi}{\sqrt{2h'(x_0)}}. \quad (22)$$

Proof. Changing of variable $y = H^{-1}(uH(t))$ in the integral, we get

$$\int_0^t \frac{1}{\sqrt{H(t) - H(y)}} dy = \int_0^1 \frac{\sqrt{H(t)}}{\sqrt{1-u}} \cdot (H^{-1})'(uH(t)) du \quad (0 < t \leq l). \quad (23)$$

By the dominated convergence theorem and (18), the integrand of the right side above equals

$$\frac{1}{2\sqrt{u(1-u)} \cdot K' \left(K^{-1} \left(\sqrt{u(H(t))} \right) \right)}, \quad (24)$$

and thus

$$p(t) = \int_0^1 \frac{1}{2\sqrt{u(1-u)} \cdot K' \left(K^{-1} \left(\sqrt{u(H(t))} \right) \right)} du. \quad (25)$$

The function $1/(K' \circ K^{-1})$ is strictly increasing on $(0, \sqrt{H(l)})$ because K^{-1} is strictly increasing and K' is strictly decreasing by our assumption (6). Hence value (24) increases as t does. Note that $\sqrt{u(H(t))} \leq \sqrt{H(t)}$ for $u \in [0, 1]$. If $t \in (0, l)$, then

$$\begin{aligned} p(t) &= \int_0^1 \frac{1}{2\sqrt{u(1-u)} \cdot K' \left(K^{-1} \left(\sqrt{u(H(t))} \right) \right)} du \\ &\leq \int_0^1 \frac{1}{2\sqrt{u(1-u)} \cdot K' \left(K^{-1} \left(\sqrt{H(t)} \right) \right)} du \\ &= \frac{\pi}{2K' \left(K^{-1} \left(\sqrt{H(t)} \right) \right)} < \infty. \end{aligned} \quad (26)$$

Hence p is real valued and (25) implies that p is strictly increasing on $(0, l]$. Finally, equalities (25), (17) and the dominated convergence theorem show that

$$\begin{aligned} \lim_{t \downarrow 0} p(t) &= \lim_{t \downarrow 0} \int_0^1 \frac{1}{2\sqrt{u(1-u)} \cdot K' \left(K^{-1} \left(\sqrt{u(H(t))} \right) \right)} du \\ &= \int_0^1 \frac{1}{2\sqrt{u(1-u)} \cdot K' \left(K^{-1}(0) \right)} du \\ &= \frac{\pi}{2\sqrt{h'(x_0)}}. \end{aligned} \quad (27)$$

The proof is complete. $\square$

Proof of Theorem 4. Assume φ is a nonconstant solution of (3). The set $\{x \in [a, b] : \varphi'(x) \neq 0\}$ is open and nonempty in $[a, b]$ and thus is a disjoint union of open intervals (provided $[a, x)$ and $(x, b]$ for $x \in (a, b)$ are viewed as "open intervals"

in $[a, b]$. Let (a_0, b_0) be such an open interval. Then φ is nonconstant on $[a_0, b_0]$ and $\varphi'(a_0) = \varphi'(b_0)$. We will show that $b_0 - a_0 > \pi/\sqrt{|\lambda|h'(x_0)}$, which is equivalent to (8).

Since differentiable functions have intermediate value property and $\varphi'(x) \neq 0$ on (a_0, b_0) , we may assume $\varphi' > 0$ on (a_0, b_0) without loss of generality. So φ is strictly increasing on $[a_0, b_0]$ with inverse function $g = \varphi^{-1}$ defined on $[\varphi(a_0), \varphi(b_0)]$. A simple computation using (14) and the chain rule for differentiation shows that

$$\begin{aligned} & \frac{d\left((1/g'(y))^2\right)}{dy} \\ &= \frac{d\left(\left((g^{-1})'(g(y))\right)^2\right)}{dy} \\ &= 2(g^{-1})'(g(y)) \cdot (g^{-1})''(g(y)) \cdot g'(y) \\ &= 2(g^{-1})''(g(y)). \end{aligned} \quad (28)$$

Condition (3) means

$$(g^{-1})''(g(y)) + \lambda h(y) = 0 \quad (\varphi(a_0) \leq y \leq \varphi(b_0)), \quad (29)$$

$$\lim_{y \downarrow \varphi(a_0)} g'(y) = \lim_{y \uparrow \varphi(b_0)} g'(y) = +\infty. \quad (30)$$

Integrating both sides of (29) and using (28), we get that

$$\frac{1}{2(g'(y))^2} + \lambda H(y) = C_1 \quad (\varphi(a_0) < y < \varphi(b_0)), \quad (31)$$

with constant C_1 . We assume $\lambda > 0$ without loss of generality. Then

$$g'(y) = \frac{1}{\sqrt{2\lambda}} \cdot \frac{1}{\sqrt{C_2 - H(y)}} \quad (\varphi(a_0) < y < \varphi(b_0)) \quad (32)$$

with $C_2 = C_1/\lambda$. Consequently,

$$H(y) < C_2 \quad (\varphi(a_0) < y < \varphi(b_0)). \quad (33)$$

As the properties of H stated before, the function $y \mapsto H(y)$ is with period $2l$, decreases on $[-l, 0]$, and increases on $[0, l]$. Equalities (30) and (32) yield that

$$H(y) \rightarrow C_2 \quad (y \downarrow \varphi(a_0) \text{ or } y \uparrow \varphi(b_0)). \quad (34)$$

This together with (33) and the properties of H shows that there exists $t \in (0, l]$ and $z \in \mathbb{Z}$ such that $C_2 = H(x_0 + t)$, $\varphi(a_0) = x_0 + 2lz - t$, and $\varphi(b_0) = x_0 + 2lz + t$. Consequently,

$$\begin{aligned} b_0 - a_0 &= g(\varphi(b_0)) - g(\varphi(a_0)) \\ &= g(\varphi(b_0) - 2lz) - g(\varphi(a_0) - 2lz) \\ &= g(x_0 + t) - g(x_0 - t) \\ &= \int_{-t}^t g'(x_0 + y) dy. \end{aligned} \quad (35)$$

Since H is an even function, it follows from (32) that

$$\begin{aligned} b_0 - a_0 &= \frac{1}{\sqrt{2\lambda}} \int_{-t}^t \frac{1}{\sqrt{H(t) - H(y)}} dy \\ &= \sqrt{\frac{2}{\lambda}} \int_0^t \frac{1}{\sqrt{H(t) - H(y)}} dy \\ &= \sqrt{\frac{2}{\lambda}} p(t). \end{aligned} \quad (36)$$

Note that p is strictly increasing on $(0, l]$ (see Lemma 6). Hence

$$b - a \geq b_0 - a_0 = \sqrt{\frac{2}{\lambda}} p(t) > \lim_{t \downarrow 0} \sqrt{\frac{2}{\lambda}} p(t) = \frac{\pi}{\sqrt{\lambda h'(x_0)}}, \quad (37)$$

which implies (8), and thus half of Theorem 4 is proved.

For the other half, we assume without loss of generality that $\lambda > 0$, and assume (8) holds; that is,

$$b - a \geq \frac{\pi}{\sqrt{\lambda h'(x_0)}}. \quad (38)$$

And we show that (2) has a nonconstant solution.

First, it follows from (27) that there exists $t_0 > 0$ such that

$$\sqrt{\frac{2}{\lambda}} p(t_0) < b - a. \quad (39)$$

Let

$$b_1 = a + \sqrt{\frac{2}{\lambda}} p(t_0) < b. \quad (40)$$

Define a continuous function g_0 on $[-t_0, t_0]$ by $g(-t_0) = a$ and

$$g'_0(y) = \frac{1}{\sqrt{2\lambda}} \cdot \frac{1}{\sqrt{H(t_0) - H(y)}} \quad (-t_0 < y < t_0). \quad (41)$$

Definition (21) yields

$$\begin{aligned} g(t_0) &= a + \int_{-t_0}^{t_0} g'_0(t) dt \\ &= a + 2 \frac{1}{\sqrt{2\lambda}} \cdot \int_0^{t_0} \frac{1}{\sqrt{H(t_0) - H(y)}} dy \\ &= a + \sqrt{\frac{2}{\lambda}} p(t_0) \\ &= b_1. \end{aligned} \quad (42)$$

Besides,

$$\frac{1}{2(g'_0(y))^2} + \lambda H(y) = \lambda H(t_0) \quad (-t_0 < y < t_0). \quad (43)$$

Differentiating both sides of (45) and using (28), we get

$$(g_0^{-1})''(g_0(y)) + \lambda h(y) = 0 \quad (-t_0 \leq y \leq t_0). \quad (44)$$

From (43), it follows that

$$\lim_{y \downarrow -t_0} g_0'(y) = \lim_{y \uparrow t_0} g_0'(y) = +\infty. \quad (45)$$

Define $\varphi_0 = g_0^{-1}$. Then (44) and (45) say that

$$\begin{aligned} \varphi_0''(x) + \lambda h(\varphi_0(x)) &= 0 \quad (a < x < b_1), \\ \varphi_0'(a) &= \varphi_0'(b_1) = 0. \end{aligned} \quad (46)$$

From (46) and assumption (H1), we see that

$$\lim_{x \uparrow b_1} \varphi_0''(x) = \lim_{x \uparrow b_1} -\lambda h(\varphi_0(x)) = \lim_{x \uparrow t_0} -\lambda h(y) = h(t_0) = 0, \quad (47)$$

and similarly $\lim_{x \downarrow a} \varphi_0''(x) = 0$. By defining $\varphi_0(x) = \varphi_0(b_1)$ when $b_1 < x \leq b$, the function φ_0 is extended and thus defined on $[a, b]$ such that

$$\begin{aligned} \varphi_0''(x) + \lambda h(\varphi_0(x)) &= 0 \quad (a \leq x \leq b), \\ \varphi_0'(a) &= \varphi_0'(b) = 0. \end{aligned} \quad (48)$$

The proof is complete. $\square$

Example 7 (BVP (2) revisited). Again we are concerned with BVP (2); namely,

$$\begin{aligned} \varphi'' + \lambda \sin \varphi &= 0 \quad \text{in } (a, b), \\ \varphi'(a) &= \varphi'(b) = 0. \end{aligned} \quad (49)$$

The problem has nonconstant solutions if and only if $(b - a)^2 |\lambda| < \pi^2$. Indeed, we see that

$$\sin(x + \pi) = -\sin x \quad (x \in \mathbb{R}) \quad (50)$$

and, for some $x_0 \in \mathbb{R}$,

$$\begin{aligned} 0 < \sin x &= -\sin(-x) \quad (0 < x < \pi), \\ h'(0) &> 0, \end{aligned} \quad (51)$$

$$x \mapsto \sqrt{\int_0^x \sin t \, dt} \text{ is a concave function on } [0, \pi]. \quad (52)$$

Note that (52) is trivial since

$$\sqrt{\int_0^x \sin t \, dt} = \sqrt{1 - \cos x} = \sqrt{2} \cos \frac{x}{2} \quad (0 \leq x \leq \pi). \quad (53)$$

Remark 8. In this paper, we have solved a class of bifurcation problems for Neumann boundary value problems for semilinear elliptic equations; namely,

$$\begin{aligned} \varphi'' + \lambda h \circ \varphi &= 0 \quad \text{in } (a, b), \\ \varphi'(a) &= \varphi'(b) = 0; \end{aligned} \quad (54)$$

the governing equation occurring in this boundary value problem generalizes the classical beam equation in the sense that the nonlinear interaction assumes the form $\mathbb{R} \ni u \mapsto h(\sin u) \in \mathbb{R}$ instead of $\mathbb{R} \ni u \mapsto \sin u \in \mathbb{R}$. What is more important is that the bifurcation problem for the classical beam equation can be solved using abstract bifurcation theorems in nonlinear analysis, while the generalized beam equations can *not* be. We provided a unified approach to understand this class of problems. Indeed, our method is quite general and very elementary.

It is worthwhile to mention that bifurcation problems associated with beam equations other than the type (3) have been extensively studied; see [19–22] and the profound references cited therein. The approaches frequently used in the literature are quite different from ours and have as foundations much advanced, complicated knowledge in functional analysis.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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