Research Article

Symmetries, Traveling Wave Solutions, and Conservation Laws of a $(3 + 1)$ -Dimensional Boussinesq Equation

Letlhogonolo Daddy Moleleki and Chaudry Masood Khalique

International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa

Correspondence should be addressed to Chaudry Masood Khalique; masood.khalique@nwu.ac.za

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We analyze the $(3 + 1)$ -dimensional Boussinesq equation, which has applications in fluid mechanics. We find exact solutions of the $(3 + 1)$ -dimensional Boussinesq equation by utilizing the Lie symmetry method along with the simplest equation method. The solutions obtained are traveling wave solutions. Moreover, we construct the conservation laws of the $(3 + 1)$ -dimensional Boussinesq equation using the new conservation theorem, which is due to Ibragimov.

1. Introduction

It is well known that the $(1 + 1)$ -dimensional Boussinesq equation [1],

$$u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} = 0, \quad (1)$$

describes the propagation of long waves on the surface of water with a small amplitude and plays a vital part in fluid mechanics [2]. It is completely integrable and admits multiple soliton solutions.

The $(2 + 1)$ -dimensional Boussinesq equation

$$u_{tt} - u_{xx} - u_{yy} - \alpha(u^2)_{xx} - u_{xxxx} = 0, \quad (2)$$

which describes the propagation of gravity waves on the surface of water, has been extensively studied by several authors (see, e.g., [3–7]).

The $(3 + 1)$ -dimensional Boussinesq equation is given by

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} - \alpha(u^2)_{xx} - u_{xxxx} = 0. \quad (3)$$

In [8], the author obtained one-periodic wave solution, two-periodic wave solution, and soliton solution for (3) by means of Hirota's bilinear method and the Riemann theta function. Wazwaz [5] employed a combination of Hirota's

method and Hereman's method to formally study (3) and derived two soliton solutions of (3). Some other work concerning symmetries and exact solutions of some Boussinesq equations can be seen in [9–12].

In the last few decades several methods have appeared in the literature, which can be used to find exact solutions of nonlinear evolution equations (NLEEs). Some of these methods are the inverse scattering transform method [13], the Darboux transformation method [14], the sine-cosine method [15], Hirota's bilinear method [16], Jacobi elliptic function expansion method [17], Lie group analysis [18–20], and the exp-function expansion method [21].

In this paper we use Lie group method along with the simplest equation method [22, 23] to construct some exact solutions of (3). Furthermore, we employ the new conservation theorem due to Ibragimov [24] to derive conservation laws for (3).

Lie group method, which was developed by Sophus Lie (1842–1899) in the nineteenth century, is a systematic method that can be used to find solutions of nonlinear partial differential equations (PDEs). It is based upon the study of the invariance under one-parameter Lie group of point transformations [18, 19].

Conservation laws play a very important role in the solution process and the reduction of PDEs [25–27]. They

have been used in investigating the existence, uniqueness, and stability of solutions of certain nonlinear PDEs [28–30] and also in the development of numerical methods [31, 32].

2. Traveling Wave Solutions of (3)

We obtain exact solutions of (3) using Lie group method along with the simplest equation method.

2.1. Non-Topological Soliton Solutions Using Lie Point Symmetries. The vector field

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \quad (4)$$

where ξ^i , $i = 1, 2, 3, 4$ and η depend on t, x, y, z , and u , is a generator of Lie point symmetries of the (3 + 1)-dimensional Boussinesq equation (3) if and only if

$$\begin{aligned} \text{pr}^{(4)}X \left(u_{tt} - u_{xx} - u_{yy} - u_{zz} - \alpha(u^2)_{xx} - u_{xxxx} \right) \Big|_{(3)} \\ = 0. \end{aligned} \quad (5)$$

Here $\text{pr}^{(4)}X$ is the fourth prolongation of the vector field X . The invariance condition (5) yields the determining equations, which are a system of linear partial differential equations. Solving this system we obtain the following eight Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \frac{\partial}{\partial y}, \\ X_4 &= \frac{\partial}{\partial z}, \\ X_5 &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \\ X_6 &= z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}, \\ X_7 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \\ X_8 &= -2\alpha t \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} \\ &\quad + (1 + 2\alpha u) \frac{\partial}{\partial u}. \end{aligned} \quad (6)$$

To obtain the nontopological soliton solution of (3), we use the combination of the four translation symmetries, namely, $X = X_1 + X_2 + X_3 + \mu X_4$, where μ is a constant. Solving

the associated Lagrange system for X , we obtain the four invariants

$$\begin{aligned} g &= t - x, & f &= t - y, \\ h &= \mu t - z, & \theta &= u. \end{aligned} \quad (7)$$

Now considering θ as the new dependent variable and g, f , and h as new independent variables, (3) transforms to a nonlinear PDE in three independent variables, namely,

$$\begin{aligned} 2\mu\theta_{fh} + 2\mu\theta_{gh} + 2\theta_{fg} + (\mu^2 - 1)\theta_{hh} \\ - 2\alpha\theta_g^2 - 2\alpha\theta\theta_{gg} - \theta_{ggg} = 0. \end{aligned} \quad (8)$$

The Lie point symmetries of (8) are

$$\begin{aligned} \Gamma_1 &= \frac{\partial}{\partial g}, \\ \Gamma_2 &= \frac{\partial}{\partial f}, \\ \Gamma_3 &= \frac{\partial}{\partial h}, \\ \Gamma_4 &= (2\alpha f\mu^3 - 4\alpha h\mu^2 - 2\alpha f\mu) \frac{\partial}{\partial h} \\ &\quad + (2\mu^2\alpha f - \alpha\mu^2g - \alpha f - 3h\alpha\mu) \frac{\partial}{\partial g} \\ &\quad + (2\alpha\mu^2\theta + \mu^2 + 1) \frac{\partial}{\partial \theta}, \\ \Gamma_5 &= (\alpha f\mu^3 - \alpha f\mu - 3\alpha h\mu^2) \frac{\partial}{\partial h} \\ &\quad + (\mu^2\alpha f - \alpha\mu^2g - \alpha f - 2h\alpha\mu) \frac{\partial}{\partial g} \\ &\quad - (\alpha f\mu^2) \frac{\partial}{\partial f} + (2\alpha\mu^2\theta + \mu^2 + 1) \frac{\partial}{\partial \theta}. \end{aligned} \quad (9)$$

The use of the combination $\Gamma = \Gamma_1 + \Gamma_2 + \beta\Gamma_3$, (β is a constant) of the three translation symmetries, gives us the three invariants

$$r = f - g, \quad w = \beta f - h, \quad \theta = \phi. \quad (10)$$

Treating ϕ as the new dependent variable and r and w as new independent variables, (8) transforms to

$$\begin{aligned} (\mu^2 - 2\mu\beta - 1)\phi_{ww} - 2\beta\phi_{rw} - 2\phi_{rr} \\ - 2\alpha\phi_r^2 - 2\alpha\phi\phi_{rr} - \phi_{rrr} = 0, \end{aligned} \quad (11)$$

which is a nonlinear PDE in two independent variables. Equation (11) has three Lie point symmetries, namely,

$$\begin{aligned}\Sigma_1 &= \frac{\partial}{\partial w}, \\ \Sigma_2 &= \frac{\partial}{\partial r}, \\ \Sigma_3 &= (4w\mu\alpha\beta + 2w\alpha - 2w\mu^2\alpha) \frac{\partial}{\partial w} \\ &\quad + (w\alpha\beta + 2\mu\alpha\beta + r\alpha - \mu^2r\alpha) \frac{\partial}{\partial r} \\ &\quad + (\beta^2 - 4\mu\beta - 2 + 2\mu^2 - 4\alpha\phi\mu\beta - 2\alpha\phi + 2\alpha\mu^2\phi) \frac{\partial}{\partial \phi},\end{aligned}\quad (12)$$

and the symmetry $\Sigma = \Sigma_1 + \delta\Sigma_2$ (δ is a constant) provides the two invariants

$$\xi = \delta w - r, \quad \phi = \psi, \quad (13)$$

which gives rise to a group invariant solution $\psi = \psi(\xi)$. Using these invariants, the PDE (11) transforms to

$$\begin{aligned}(\mu^2\delta^2 - 2\mu\beta\delta^2 - \delta^2 + 2\beta\delta - 2)\psi'' \\ - 2\alpha\psi'^2 - 2\alpha\psi\psi'' - \psi'''' = 0,\end{aligned}\quad (14)$$

which is a fourth-order nonlinear ODE. This ODE can be integrated easily. Integrating it four times while choosing the constants of integration to be zero (because we are looking for soliton solutions) and then reverting back to our original variables t, x, y, z, u , we obtain the following group-invariant (nontopological soliton) solutions of the Boussinesq equation (3):

$$u(x, y, t, z) = \frac{A_1}{A_2} \operatorname{sech}^2 \left[\frac{\sqrt{A_1}}{2} (B \pm \xi) \right], \quad (15)$$

where B is a constant of integration and

$$\begin{aligned}A_1 &= \mu^2\delta^2 - 2\mu\beta\delta^2 - \delta^2 + 2\beta\delta - 2, \\ A_2 &= \frac{2\alpha}{3}, \\ \xi &= \delta z + (1 - \beta\delta)y - x + (\delta\beta - \delta\mu)t.\end{aligned}\quad (16)$$

2.2. Exact Solutions of (3) Using Simplest Equation Method.

We now use the simplest equation method to obtain more solutions of the nonlinear ODE (14), which will then give us more exact solutions for our Boussinesq equation (3). Bernoulli and Riccati equations will be used as the simplest equations [22, 23].

2.2.1. Solutions of (3) Using the Bernoulli Equation as the Simplest Equation. In this case the balancing procedure yields $M = 2$ so the solutions of (14) have the form

$$F(k) = A_0 + A_1G + A_2G^2. \quad (17)$$

Inserting (17) into (14) and using the Bernoulli equation [23] and then equating the coefficients of powers of G^i to zero gives us the following algebraic system of six equations:

$$\begin{aligned}-20\alpha A_2^2 b^2 - 120A_2 b^4 &= 0, \\ -336A_2 ab^3 - 24A_1 b^4 - 36\alpha A_2^2 ab \\ &\quad - 24\alpha A_1 b^2 A_2 = 0, \\ -2A_1 a^2 - 2A_1 a^2 \delta^2 \beta \mu - A_1 a^4 + 2A_1 a^2 \delta \beta \\ &\quad + A_1 a^2 \delta^2 \mu^2 - \delta^2 A_1 a^2 - 2\alpha A_0 A_1 a^2 = 0, \\ -6\alpha A_1^2 b^2 + 6A_2 b^2 \delta^2 \mu^2 - 12\alpha A_0 A_2 b^2 - 6\delta^2 A_2 b^2 \\ &\quad + 12A_2 b^2 \delta \beta - 12A_2 b^2 - 16\alpha A_2^2 a^2 \\ &\quad - 12A_2 b^2 \delta^2 \beta \mu - 330A_2 a^2 b^2 \\ &\quad - 42\alpha A_1 a A_2 b - 60A_1 ab^3 = 0, \\ -6A_1 ab \delta^2 \beta \mu - 16A_2 a^4 - 6\alpha A_0 A_1 ab + 3A_1 ab \delta^2 \mu^2 \\ &\quad - 8\alpha A_0 A_2 a^2 - 8A_2 a^2 \delta^2 \beta \mu \\ &\quad - 4\alpha A_1^2 a^2 - 3\delta^2 A_1 ab + 6A_1 ab \delta \beta - 15A_1 a^3 b \\ &\quad - 8A_2 a^2 - 6A_1 ab + 4A_2 a^2 \delta^2 \mu^2 \\ &\quad - 4\delta^2 A_2 a^2 + 8A_2 a^2 \delta \beta = 0, \\ 10A_2 ab \delta^2 \mu^2 - 130A_2 a^3 b - 20A_2 ab - 4A_1 b^2 \delta^2 \beta \mu \\ &\quad + 4A_1 b^2 \delta \beta - 4A_1 b^2 \\ &\quad - 4\alpha A_0 A_1 b^2 + 20A_2 ab \delta \beta - 50A_1 a^2 b^2 \\ &\quad - 10\delta^2 A_2 ab - 10\alpha A_1^2 ab - 18\alpha A_1 a^2 A_2 \\ &\quad - 2\delta^2 A_1 b^2 + 2A_1 b^2 \delta^2 \mu^2 - 20\alpha A_0 A_2 ab \\ &\quad - 20A_2 ab \delta^2 \beta \mu = 0.\end{aligned}\quad (18)$$

These equations can be solved with the aid of Mathematica and one possible solution for A_0 , A_1 , and A_2 is

$$\begin{aligned}A_0 &= \frac{-2 - 2\delta^2 \beta \mu - a^2 + 2\delta \beta + \delta^2 \mu^2 - \delta^2}{2\alpha}, \\ A_1 &= \frac{-6ab}{\alpha}, \\ A_2 &= \frac{-6b^2}{\alpha}.\end{aligned}\quad (19)$$

Consequently, returning back to the original variables, a solution of (3) is [23]

$$\begin{aligned}
 u(t, x, y, z) &= A_0 + A_1 a \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\} \\
 &+ A_2 a^2 \left\{ \frac{\cosh[a(\xi + C)] + \sinh[a(\xi + C)]}{1 - b \cosh[a(\xi + C)] - b \sinh[a(\xi + C)]} \right\}^2,
 \end{aligned} \quad (20)$$

where $\xi = \delta z + (1 - \alpha\delta)y - x + (\delta\beta - \delta\mu)t$ and C is an arbitrary constant of integration.

2.2.2. Solutions of (3) Using the Riccati Equation as the Simplest Equation. Here the balancing procedure gives $M = 2$ so the solutions of (14) are of the form

$$F(z) = A_0 + A_1 G + A_2 G^2. \quad (21)$$

Substituting (21) into (14) and using the Riccati equation [23], as before, we obtain the following algebraic system of equations in terms of A_0 , A_1 , and A_2 :

$$\begin{aligned}
 &-20\alpha A_2^2 b^2 - 120A_2 b^4 = 0, \\
 &-336A_2 ab^3 - 24A_1 b^4 - 24\alpha A_1 b^2 A_2 \\
 &-36\alpha A_2^2 ab = 0, \\
 &-42\alpha A_1 a A_2 b - 6\delta^2 A_2 b^2 - 12A_2 b^2 \\
 &-330A_2 a^2 b^2 - 6\alpha A_1^2 b^2 - 16\alpha A_2^2 a^2 \\
 &+ 12A_2 b^2 \delta\beta - 32\alpha A_2^2 db + 6A_2 b^2 \delta^2 \mu^2 \\
 &- 12A_2 b^2 \delta^2 \beta\mu - 240A_2 b^3 d - 60A_1 ab^3 \\
 &- 12\alpha A_0 A_2 b^2 = 0, \\
 &-2A_1 ad\delta^2 \beta\mu - 2\delta^2 A_2 d^2 + 2A_1 ad\delta\beta \\
 &- 4A_2 d^2 - A_1 a^3 d - 4\alpha A_0 A_2 d^2 - 2\alpha A_1^2 d^2 \\
 &- 2\alpha A_0 A_1 ad - \delta^2 A_1 ad - 16A_2 bd^3 \\
 &+ 2A_2 d^2 \delta^2 \mu^2 + 4A_2 d^2 \delta\beta - 8A_1 abd^2 \\
 &- 4A_2 d^2 \delta^2 \beta\mu - 2A_1 ad + A_1 ad\delta^2 \mu^2 \\
 &- 14A_2 a^2 d^2 = 0,
 \end{aligned}$$

$$\begin{aligned}
 &-20A_2 ab\delta^2 \beta\mu - 18\alpha A_1 a^2 A_2 - 36\alpha A_1 d A_2 b \\
 &- 10\delta^2 A_2 ab - 20A_2 ab - 10\alpha A_1^2 ab \\
 &- 2\delta^2 A_1 b^2 + 4A_1 b^2 \delta\beta - 4A_1 b^2 \delta^2 \beta\mu \\
 &- 4A_1 b^2 - 130A_2 a^3 b - 440A_1 adb^2 \\
 &+ 10A_2 ab\delta^2 \mu^2 - 28\alpha A_2^2 da - 4\alpha A_0 A_1 b^2 \\
 &- 50A_1 a^2 b^2 + 2A_1 b^2 \delta^2 \mu^2 - 20\alpha A_0 A_2 ab \\
 &+ 20A_2 ab\delta\beta - 40A_1 db^3 = 0,
 \end{aligned}$$

$$\begin{aligned}
 &-12A_1 da\delta^2 \beta\mu - 4A_1 bd\delta^2 \beta\mu + 2A_1 bd\delta^2 \mu^2 \\
 &+ 6A_2 da\delta^2 \mu^2 + 4A_1 bd\delta\beta + 12A_2 da\delta\beta \\
 &- 4\alpha A_0 A_1 bd - 12\alpha A_0 A_2 da - 2A_1 a^2 \delta^2 \beta\mu \\
 &+ A_1 a^2 \delta^2 \mu^2 + 2A_1 a^2 \delta\beta - 2\delta^2 A_1 bd - 6\delta^2 A_1 da \\
 &- 6\alpha A_1^2 da - 12\alpha A_1 d^2 A_2 - 2\alpha A_0 A_1 a^2 \\
 &- 120A_2 ad^2 b - 22A_1 a^2 bd - A_1 a^4 \\
 &- 2A_1 a^2 - 4A_1 bd - \delta^2 A_1 a^2 - 12A_2 da \\
 &- 16A_1 b^2 d^2 - 30A_2 a^3 d = 0, \\
 &-16A_2 db\delta^2 \beta\mu - 6A_1 ab\delta^2 \beta\mu + 8A_2 db\delta^2 \mu^2 \\
 &- 8A_2 a^2 \delta^2 \beta\mu + 6A_1 ab\delta\beta + 16A_2 db\delta\beta \\
 &- 30\alpha A_1 d A_2 a - 6\alpha A_0 A_1 ab - 16\alpha A_0 A_2 db \\
 &+ 3A_1 ab\delta^2 \mu^2 + 4A_2 a^2 \delta^2 \mu^2 + 8A_2 a^2 \delta\beta \\
 &- 3\delta^2 A_1 ab - 8\delta^2 A_2 db - 8\alpha A_1^2 db - 8\alpha A_0 A_2 a^2 \\
 &- 60A_1 adb^2 - 232A_2 a^2 db - 8A_2 a^2 \\
 &- 16A_2 a^4 - 4\alpha A_2^2 a^2 - 15A_1 a^3 b - 16A_2 db \\
 &- 136A_2 b^2 d^2 - 12\alpha A_2^2 d^2 - 4\delta^2 A_2 a^2 \\
 &- 6A_2 ab = 0.
 \end{aligned} \quad (22)$$

Solving the above equations yields

$$\begin{aligned}
 A_0 &= \frac{-a^2 - 8bd - \delta^2 - 2 + 2\delta\beta + \delta^2 \mu^2 - 2\delta^2 \beta\mu}{2\alpha}, \\
 A_1 &= \frac{-6ab}{\alpha}, \\
 A_2 &= \frac{-6b^2}{\alpha},
 \end{aligned} \quad (23)$$

and, consequently, the solutions of (3) are

$$\begin{aligned}
 u(t, x, y, z) &= A_0 + A_1 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left[\frac{1}{2} \theta (\xi + C) \right] \right\} \\
 &\quad + A_2 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left[\frac{1}{2} \theta (\xi + C) \right] \right\}^2, \\
 u(t, x, y, z) &= A_0 \\
 &\quad + A_1 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left(\frac{1}{2} \theta \xi \right) \right. \\
 &\quad \left. + \frac{\operatorname{sech}(\theta \xi / 2)}{C \cosh(\theta \xi / 2) - (2b/\theta) \sinh(\theta \xi / 2)} \right\} \\
 &\quad + A_2 \left\{ -\frac{a}{2b} - \frac{\theta}{2b} \tanh \left(\frac{1}{2} \theta \xi \right) \right. \\
 &\quad \left. + \frac{\operatorname{sech}(\theta \xi / 2)}{C \cosh(\theta \xi / 2) - (2b/\theta) \sinh(\theta \xi / 2)} \right\}^2,
 \end{aligned} \tag{24}$$

where $\xi = \delta z + (1 - \alpha \delta) y - x + (\delta \beta - \delta \mu) t$ and C is an arbitrary constant of integration.

3. Conservation Laws for (3)

We utilize the new conservation theorem due to Ibragimov [24] to obtain conservation laws for the (3 + 1)-dimensional Boussinesq equation (3) written as

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2\alpha u_x^2 - 2\alpha u u_{xx} - u_{xxxx} = 0. \tag{25}$$

For details of notations, definitions, and theorems the reader is referred to [24].

In Section 2.1 we derived the following eight Lie point symmetries of equation (25):

$$\begin{aligned}
 X_1 &= \frac{\partial}{\partial x}, \\
 X_2 &= \frac{\partial}{\partial t}, \\
 X_3 &= \frac{\partial}{\partial y}, \\
 X_4 &= \frac{\partial}{\partial z}, \\
 X_5 &= y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y},
 \end{aligned}$$

$$\begin{aligned}
 X_6 &= z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}, \\
 X_7 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \\
 X_8 &= -2\alpha t \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} \\
 &\quad + (1 + 2\alpha u) \frac{\partial}{\partial u}.
 \end{aligned} \tag{26}$$

Corresponding to each of these eight Lie point symmetries we shall construct eight conserved vectors. By definition [24] the adjoint equation of (25) is given by

$$\begin{aligned}
 E^*(t, x, u, v, \dots, u_{xxxx}, v_{xxxx}) \\
 = \frac{\delta}{\delta u} \left[v \left(u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2\alpha u_x^2 \right. \right. \\
 \left. \left. - 2\alpha u u_{xx} - u_{xxxx} \right) \right] = 0,
 \end{aligned} \tag{27}$$

which gives

$$v_{tt} - v_{xx} - v_{yy} - v_{zz} - 2\alpha u v_{xx} - v_{xxxx} = 0. \tag{28}$$

Here $v = v(t, x, y, z)$ is a new dependent variable. Clearly, (25) is not self-adjoint. The Lagrangian for the system of (25) and (28) is given by

$$\begin{aligned}
 \mathcal{L} &= v \left(u_{tt} - u_{xx} - u_{yy} - u_{zz} - 2\alpha u_x^2 \right. \\
 &\quad \left. - 2\alpha u u_{xx} - u_{xxxx} \right).
 \end{aligned} \tag{29}$$

(i) Consider first the translation symmetry $X_1 = \partial/\partial x$. In this case the operator Y_1 [24] is the same as X_1 and the Lie characteristic function $W = -u_x$. Thus the components [24] T^i , $i = 1, 2, 3, 4$, of the conserved vector $T = (T^1, T^2, T^3, T^4)$ are given by

$$\begin{aligned}
 T^1 &= u_x v_t - v u_{tx}, \\
 T^2 &= v u_{tt} - v u_{yy} - v u_{zz} - u_x v_x \\
 &\quad - 2\alpha u u_x v_x - u_x v_{xxx} + v_{xx} u_{xx} - v_x u_{xxx}, \\
 T^3 &= -u_x v_y + v u_{xy}, \\
 T^4 &= -u_x v_z + v u_{xz}.
 \end{aligned} \tag{30}$$

(ii) The second translation symmetry $X_2 = \partial/\partial t$ gives $W = -u_t$. Hence the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$\begin{aligned}
 T^1 &= -v u_{xx} - v u_{yy} - v u_{zz} - 2\alpha v u_x^2 - 2\alpha v u_{xx} \\
 &\quad - v u_{xxxx} + u_t v_t, \\
 T^2 &= 2\alpha v u_t u_x - u_t v_x - 2\alpha u u_t v_x - u_t v_{xxx} \\
 &\quad + v u_{tx} + 2\alpha v u_{tx} + v_{xx} u_{tx} - v_x u_{txx} + v u_{txxx}, \\
 T^3 &= -v_y u_t + v u_{ty}, \\
 T^4 &= -v_z u_t + v u_{tz}.
 \end{aligned} \tag{31}$$

(iii) For the third symmetry $X_3 = \partial/\partial y$, we have $W = -u_y$ and the corresponding components of the conserved vector are

$$\begin{aligned} T^1 &= v_t u_y - v u_{ty}, \\ T^2 &= -u_y v_x + 2\alpha v u_y u_x - 2\alpha u u_y v_x - u_y v_{xxx} \\ &\quad + v u_{xy} + 2\alpha v u_{xy} + v_{xx} u_{xy} - v_x u_{xxy} + v u_{xxy}, \\ T^3 &= v u_{tt} - v u_{xx} - v u_{zz} - 2\alpha v u_x^2 \\ &\quad - 2\alpha v u_{xx} - v u_{xxx} - u_y v_y, \\ T^4 &= -v_z u_y + v u_{yz}. \end{aligned} \quad (32)$$

(iv) The fourth symmetry $X_4 = \partial/\partial z$ gives $W = -u_z$ and the corresponding components of the conserved vector are

$$\begin{aligned} T^1 &= v_t u_z - v u_{tz}, \\ T^2 &= -u_z v_x + 2\alpha v u_z u_x - 2\alpha u u_z v_x - u_z v_{xxx} \\ &\quad + v u_{xz} + 2\alpha v u_{xz} + v_{xx} u_{xz} - v_x u_{xxz} + v u_{xxz}, \\ T^3 &= -v_y u_z + v u_{yz}, \\ T^4 &= v u_{tt} - v u_{xx} - v u_{yy} - 2\alpha v u_x^2 - 2\alpha v u_{xx} \\ &\quad - v u_{xxx} - u_z v_z. \end{aligned} \quad (33)$$

(v) For the symmetry $X_5 = y\partial/\partial z - z\partial/\partial y$, we have $W = -y u_z + z u_y$ and the corresponding components of the conserved vector, as before, are given by

$$\begin{aligned} T^1 &= y u_z v_t - z u_y v_t - y v u_{tz} + z v u_{ty}, \\ T^2 &= 2\alpha y v u_x u_z - y u_z v_x - 2\alpha y u u_z v_x - y u_z v_{xxx} \\ &\quad - 4\alpha z v u_x u_y + z u_y v_x + 2\alpha z v u_y u_x \\ &\quad + 2\alpha z u u_y v_x + z u_y v_{xxx} + y v u_{xz} + 2\alpha y v u_{xz} \\ &\quad + y u_{xz} v_{xx} - z v u_{xy} - 2\alpha z v u_{xy} \\ &\quad - z u_{xy} v_{xx} - y v_x u_{xxz} - z v_x u_{xxy} - y v u_{xxxz} - z v u_{xxxy}, \\ T^3 &= -z v u_{tt} + z v u_{xx} + z v u_{zz} + 2\alpha z v u_x^2 \\ &\quad + 2\alpha z v u_{xx} + z v u_{xxx} - y v_y u_z + z v_y u_y \\ &\quad + v u_t + y v u_{yz} + v u_z, \\ T^4 &= y v u_{tt} - z v u_{xx} - z v u_{yy} - 2\alpha y v u_x^2 \\ &\quad - 2\alpha y v u_{xx} - y v u_{xxx} - y v_z u_z + z v_z u_y \\ &\quad - v u_y - z v u_{yz}. \end{aligned} \quad (34)$$

(vi) Likewise, the symmetry $X_6 = z\partial/\partial t + t\partial/\partial z$ gives $W = -z u_t - t u_z$ and the corresponding components of the conserved vector are given by

$$\begin{aligned} T^1 &= -z v u_{xx} - z v u_{yy} - z v u_{zz} - 2\alpha z v u_x^2 \\ &\quad - 2\alpha z v u_{xx} - z v u_{xxx} + z v_t u_t + t v_t u_z \\ &\quad - v u_z - t v u_{tz}, \\ T^2 &= 2\alpha z v u_x u_t - z u_t v_x - 2\alpha z u u_t v_x - z u_t v_{xxx} \\ &\quad + 2\alpha t v u_x u_z - t u_z v_x - 2\alpha t u u_z v_x \\ &\quad - t u_z v_{xxx} + 2v u_{tx} + 2\alpha z v u_{tx} + z u_{tx} v_{xx} \\ &\quad + t v u_{xz} + 2\alpha t v u_{xz} \\ &\quad + t u_{xz} v_{xx} - z v_x u_{txx} - t v_x u_{xxz} + z v u_{txxx} \\ &\quad + t v u_{xxxz}, \\ T^3 &= -z u_t v_y - t u_z v_y + z v u_{ty} + t v u_{yz}, \\ T^4 &= t v u_{tt} - t v u_{xx} - t v u_{yy} - 2\alpha t v u_x^2 \\ &\quad - 2\alpha t v u_{xx} - t v u_{xxx} - z v_t u_z - t v_z u_z \\ &\quad + v u_t + z v u_{tz}. \end{aligned} \quad (35)$$

(vii) As before, the symmetry $X_7 = y\partial/\partial t + t\partial/\partial y$ yields $W = -y u_t - t u_y$ and the corresponding components of the conserved vector are given by

$$\begin{aligned} T^1 &= -y v u_{xx} - y v u_{yy} - y v u_{zz} - 2\alpha y v u_x^2 \\ &\quad - 2\alpha y v u_{xx} - y v u_{xxx} + y v_t u_t + t v_t u_y \\ &\quad - v u_y - t v u_{ty}, \\ T^2 &= 2\alpha y v u_x u_t - y u_t v_x - 2\alpha y u u_t v_x - y u_t v_{xxx} \\ &\quad + 4\alpha t v u_x u_y - t u_y v_x - 2\alpha t u u_y v_x \\ &\quad - 2\alpha t v u_x u_y - t u_y v_{xxx} + y v u_{tx} + 2\alpha y v u_{tx} \\ &\quad + y u_{tx} v_{xx} + t v u_{xy} + 2\alpha t v u_{xy} \\ &\quad + t u_{xy} v_{xx} - y v_x u_{txx} - t v_x u_{xxy} + y v u_{txxx} \\ &\quad + t v u_{xxxy}, \\ T^3 &= t v u_{tt} - t v u_{xx} - t v u_{yy} - 2\alpha t v u_x^2 \\ &\quad - 2\alpha t v u_{xx} - t v u_{xxx} - y v_y u_t - t v_y u_y \\ &\quad + v u_t + y v u_{ty}, \\ T^4 &= -y u_t v_z - t u_y v_z + y v u_{tz} + t v u_{yz}. \end{aligned} \quad (36)$$

(viii) Finally, for the symmetry

$$X_8 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} + (1 + 2\alpha u) \frac{\partial}{\partial u}, \quad (37)$$

the value of Y_8 is not the same as X_8 and in fact is given by

$$Y_8 = -2\alpha t \frac{\partial}{\partial t} - x\alpha \frac{\partial}{\partial x} - 2\alpha y \frac{\partial}{\partial y} - 2\alpha z \frac{\partial}{\partial z} + (1 + 2\alpha u) \frac{\partial}{\partial u} + \alpha v \frac{\partial}{\partial v}. \quad (38)$$

The Lie characteristic function $W = 1 + 2\alpha u + 2\alpha t u_t + x\alpha u_x + 2\alpha y u_y + 2\alpha z u_z$ and, consequently, the conserved vector T has components given by

$$\begin{aligned} T^1 &= 2\alpha t v u_{xx} + 2\alpha t v u_{yy} + 2\alpha t v u_{zz} + 4\alpha^2 t v u_x^2 \\ &\quad + 4\alpha^2 t v u_{xx} + 2\alpha t v u_{xxxx} - v_t \\ &\quad - 2\alpha u v_t - 2\alpha t u_t v_t - \alpha x u_x v_t - 2\alpha y u_y v_t \\ &\quad - 2\alpha z u_z v_t + 4\alpha v u_t + \alpha x v u_{tx} \\ &\quad + 2\alpha y v u_{ty} + 2\alpha z v u_{tz}, \\ T^2 &= -\alpha x v u_{tt} + \alpha x v u_{yy} + \alpha x v u_{zz} - 3\alpha v u_x \\ &\quad + v_x + 4\alpha u v_x + v_{xxx} - 8\alpha^2 u v u_x \\ &\quad + 4\alpha^2 u^2 v_x + 2\alpha u v_{xxx} + 2\alpha t u_t v_x - 4\alpha^2 t v u_t u_x \\ &\quad + 4\alpha^2 t u u_t v_x + 2\alpha t u_t v_{xxx} \\ &\quad + \alpha x u_x v_x + 2\alpha^2 x u u_x v_x + \alpha x u_x v_{xxx} \\ &\quad + 2\alpha y u_y v_x - 4\alpha^2 y v u_x u_y + 4\alpha^2 y u u_y v_x \\ &\quad + 2\alpha y u_y v_{xxx} + 2\alpha z u_z v_x - 4\alpha^2 z v u_x u_z \\ &\quad + 4\alpha^2 z u u_z v_x + 2\alpha z u_z v_{xxx} - 2\alpha t v u_{tx} \\ &\quad - 4\alpha^2 t v u_{tx} - 2\alpha t u_{tx} v_{xx} - 2\alpha^2 u v u_x \\ &\quad - 3\alpha u_x v_{xx} - \alpha x u_{xx} v_{xx} - 2\alpha y v u_{xy} \\ &\quad - 4\alpha^2 y v u_{xy} - 2\alpha y u_{xy} v_{xx} - 2\alpha z v u_{xz} \\ &\quad - 4\alpha^2 z v u_{xz} - 2\alpha z u_{xz} v_{xx} + 4\alpha v_x u_{xx} \\ &\quad + 2\alpha t v_x u_{txx} + \alpha x v_x u_{xxx} + 2\alpha y v_x u_{xxy} \\ &\quad + 2\alpha z v_x u_{xxz} - 2\alpha t v u_{txxx} - 5\alpha v u_{xxx} \\ &\quad - 2\alpha y v u_{xxxy} - 2\alpha z v u_{xxxz}, \end{aligned}$$

$$\begin{aligned} T^3 &= -2\alpha y v u_{tt} + 2\alpha y v u_{xx} + 2\alpha y v u_{zz} \\ &\quad + 4\alpha^2 y v u_x^2 + 4\alpha^2 y v u_{xx} + 2\alpha y v u_{xxxx} + v_y \\ &\quad + 2\alpha u v_y + 2\alpha t u_t v_y + \alpha x u_x v_y + 2\alpha y u_y v_y \\ &\quad + 2\alpha z u_z v_y - 4\alpha v u_y - 2\alpha t v u_{ty} \\ &\quad - \alpha x v u_{xy} - 2\alpha z v u_z, \\ T^4 &= -2\alpha z v u_{tt} + 2\alpha z v u_{xx} + 2\alpha z v u_{yy} \\ &\quad + 4\alpha^2 z v u_x^2 + 4\alpha^2 z v u_{xx} + 2\alpha z v u_{xxxx} + v_z \\ &\quad + 2\alpha u v_z + 2\alpha t u_t v_z + \alpha x u_x v_z + 2\alpha y u_y v_z \\ &\quad + 2\alpha z u_z v_z - 4\alpha v u_z - 2\alpha t v u_{tz} \\ &\quad - \alpha x v u_{xz} - 2\alpha y v u_{yz}. \end{aligned} \quad (39)$$

Remark. Each conserved vector T obtained above contains the arbitrary solution v of the adjoint equation (28) and hence gives an infinite number of conservation laws.

4. Conclusions

Exact solutions of the (3 + 1)-dimensional Boussinesq equation (3) were obtained with the aid of Lie point symmetries of (3) as well as the simplest equation method. The solutions obtained were solitary waves and nontopological soliton. Furthermore, the conservation laws for the (3 + 1)-dimensional Boussinesq equation were also constructed by utilizing the new conservation theorem due to Ibragimov [24].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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