Research Article
On the Deformation Retract of Kerr Spacetime and Its Folding

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The deformation retract of the Kerr spacetime is introduced using Lagrangian equations. The equatorial geodesics of the Kerr space have been discussed. The retraction of this space into itself and into geodesics has been presented. The deformation retract of this space into itself and after the isometric folding has been discussed. Theorems concerning these relations have been deduced.

1. Introduction

The real revolution in mathematical physics in the second half of twentieth century (and in pure mathematics itself) was algebraic topology and algebraic geometry [1]. In the nineteenth century, mathematical physics was essentially the classical theory of ordinary and partial differential equations. The variational calculus, as a basic tool for physicists in theoretical mechanics, was seen with great reservation by mathematicians until Hilbert set up its rigorous foundation by pushing forward functional analysis. This marked the transition into the first half of twentieth century, where, under the influence of quantum mechanics and relativity, mathematical physics turned mainly into functional analysis, complemented by the theory of Lie groups and by tensor analysis. All branches of theoretical physics still can expect the strongest impacts of use of the unprecedented wealth of results of algebraic topology and algebraic geometry of the second half of the twentieth century [1].

Today, the concepts and methods of topology and geometry have become an indispensable part of theoretical physics. They have led to a deeper understanding of many crucial aspects in condensed matter physics, cosmology, gravity, and particle physics. Moreover, several intriguing connections between only apparently disconnected phenomena have been revealed based on these mathematical tools [2, 3].

Topology enters general relativity through the fundamental assumption that spacetime exists and is organized as a manifold. This means that spacetime has a well-defined dimension, but it also carries with it the inherent possibility of modified patterns of global connectivity, such as distinguishing a sphere from a plane or a torus from a surface of higher genus. Such modifications can be present in the spatial topology without affecting the time direction, but they can also have a genuine spacetime character in which case the spatial topology changes with time [4]. The topology change in classical general relativity has been discussed in [5]. See [6] for some applications of differential topology in general relativity.

In general relativity, boundaries that are $S^1$ bundles over some compact manifolds arise in gravitational thermodynamics [7]. The trivial bundle $\Sigma = S^1 \times S^2$ is a classic example. Manifolds with complete Ricci-flat metrics admitting such boundaries are known; they are the Euclideanised Schwarzschild metric and the flat metric with periodic identification. York [8] shows that there are in general two or no Schwarzschild solutions depending on whether the squashing (the ratio of the radius of the $S^1$-fibre to that of the $S^2$-base) is below or above a critical value. York’s results in 4-dimension extend readily to higher dimensions.

The simplest example of nontrivial bundles arises in quantum cosmology in which the boundary is a compact $S^3$, that is, a nontrivial $S^1$ bundle over $S^2$. In the case of zero cosmological constant, regular 4 metrics admitting such an $S^3$ boundary are the Taub-Nut [9] and Taub-Bolt [10] metrics having zero and two-dimensional (regular) fixed point sets of the $U(1)$ action, respectively [7, II–13].
The Kerr metric describes the geometry of empty spacetime around a rotating uncharged axially symmetric black hole. The Kerr metric corresponds to the line element
\[
ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mr a \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} d\rho^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2}{\rho^2}\right) \sin^2 \theta d\phi^2,
\]
(1)
and the four-dimensional flat metric
\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.
\]
(7)

The Kerr metric is a vacuum solution of the Einstein equations, being valid in the absence of matter. If the black hole is not rotating \(a = 0\), the Kerr line element reduces to the Schwarzschild line element. The Kerr metric becomes asymptotically flat for \(r \gg M\), the Kerr line element reducesto the Schwarzschild line element.

The 5-dimensional case has been discussed in [27].

3. Equatorial Geodesics

We will be interested in the equatorial geodesics, that is, geodesics with \(\theta = \pi/2\). It is easy to show that such geodesics exist for the case of Kerr metric where \(\theta = \pi/2\) satisfies the \(\theta\)-component of the Euler Lagrange equations for the Lagrangian associated with the Kerr metric (1). Consider
\[
L = -\frac{1}{2} \left(1 - \frac{2Mr}{\rho^2}\right)t^2 - \frac{2Mra \sin^2 \theta}{\rho^2} \phi^2 + \frac{\rho^2}{2\Delta} r^2
\]
\[+ \frac{1}{2} \rho^2 \dot{\theta}^2 + \left(\rho^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2}\right) \sin^2 \theta \phi^2.
\]
(4)
The \(\theta\)-component of the Euler Lagrange equations gives
\[
\frac{d}{d\lambda} \left(\rho^2 \dot{\theta}\right) + \frac{2Mra \sin^2 \theta \cos \theta}{\rho^2} t^2 - \frac{a^2 \sin^2 \theta \cos \theta}{\Delta} r^2
\]
\[- \left(\sin \theta \cos \theta\right) \dot{\phi}^2 - \frac{4Mra}{\rho^2} \sin \theta \cos \left(\rho^2 + a^2 \sin^2 \theta\right) \phi
\]
\[+ \left(r^2 + a^2\right) \sin \theta \cos \theta \phi^2 + \frac{1}{\rho^2} \frac{4Mra^2}{\sin \theta \cos \theta}
\]
\[\times \left(a^2 \sin^4 \theta + 2 \left(r^2 + a^2 \cos \theta\right) \sin \theta\right) = 0.
\]
(5)

Comparing the Kerr line element,
\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 - \frac{4M a}{r} d\phi dt + \frac{r^2}{\Delta} dr^2 + \left(r^2 + a^2 + \frac{2M a^2}{r}\right) d\phi^2.
\]
(6)
And the four-dimensional flat metric
\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.
\]
(7)
The coordinates of the four-dimensional Kerr space (6) can be written as

\[
x_1 = \left( r^2 - 4Mr + (2Mr - 2M^2) \ln (r^2 - a^2 + 2Mr) \right)
\]
\[
+ \tan^{-1} \left( \frac{(M - r) \sqrt{a^2 - M^2}}{\sqrt{a^2 - M^2}} \right)
\times (8M^3 - 6a^2M + 2a^2r - 4M^2r) + (a^2 - 2M^2)
\times \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) + C_1 \right)^{1/2},
\]
\[
x_2 = \pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 a^2 + C_2},
\]
\[
x_3 = \pm \sqrt{\left( r^2 + a^2 \right)^2 + \frac{2Ma^2}{r}} \phi^2 + C_3,
\]
\[
x_4 = \pm \sqrt{\left( 1 - \frac{2M}{r} \right) t^2 + C_4}.
\]

In general relativity, the geodesic equation is equivalent to the Euler Lagrange equations

\[
\frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, 2, 3, 4
\]

associated to the Lagrangian

\[
L(x^M, \dot{x}^M) = \frac{1}{2} g_{MN} x^M \dot{x}^N.
\]

To find a geodesic which is a subset of the Kerr space, the Lagrangian could be written as

\[
L = -\frac{1}{2} \left( 1 - \frac{2M}{r} \right) \dot{t}^2 - \frac{2Ma}{r} \dot{\phi} + \frac{r^2}{2\Delta} \dot{r}^2
\]
\[
+ \frac{1}{2} \left( r^2 + a^2 + \frac{2Ma^2}{r} \right) \dot{\phi}^2.
\]

There is no explicit dependence on \( t \) or \( \phi \); thus \( \partial L / \partial \dot{t} = 0 \) and \( \partial L / \partial \dot{\phi} = 0 \) are constants of motion; that is,

\[
- \left( 1 - \frac{2M}{r} \right) \dot{t} - \frac{2Ma}{r} \dot{\phi} = K_1,
\]
\[
- \frac{2Ma}{r} \dot{t} + \left( r^2 + a^2 + \frac{2Ma^2}{r} \right) \dot{\phi} = K_2.
\]

So we have the following set of equations:

\[
\frac{d}{d\lambda} \left( \frac{r^2 \dot{\phi}}{r^2 + a^2 - 2Mr} \right) - \left( \frac{M}{r^2} \right) \dot{r} \dot{t} + \frac{r^2}{r^2 + a^2 - 2Mr} \dot{r}^2 + \left( r - \frac{Ma^2}{r^2} \right) \dot{\phi}^2 = 0,
\]
\[
\frac{d}{d\lambda} \left[ - \left( 1 - \frac{2M}{r} \right) \dot{t} - 2 \frac{Ma}{r} \dot{\phi} \right] = 0,
\]
\[
\frac{d}{d\lambda} \left[ - \frac{2Ma}{r} \dot{t} + \left( r^2 + a^2 + \frac{2Ma^2}{r} \right) \dot{\phi} \right] = 0.
\]

If the constant \( a \) is zero, we have

\[
x_1 = \left( r^2 - 4Mr + (2Mr - 2M^2) \right)
\times \ln \left( - \frac{r^2 + 2Mr}{r^2 + a^2} \right) + \frac{\tan^{-1} \left( \frac{(M - r) \sqrt{a^2 - M^2}}{\sqrt{a^2 - M^2}} \right)}{M}
\times (8M^3 - 6a^2M + 2a^2r - 4M^2r) + (a^2 - 2M^2)
\times \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) + C_1 \right)^{1/2},
\]
\[
x_2 = \pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 a^2 + C_2},
\]
\[
x_3 = \pm \sqrt{\left( r^2 + a^2 \right)^2 + \frac{2Ma^2}{r}} \phi^2 + C_3,
\]
\[
x_4 = \pm \sqrt{\left( 1 - \frac{2M}{r} \right) t^2 + C_4}.
\]

Since \( x_1^2 + x_2^2 + x_3^2 - x_4^2 > 0 \) which is the great circle \( S_1 \) in the Kerr space \( K \), this geodesic is a retraction in Kerr space; \( ds^2 > 0 \). This is a retraction.

For \( a = 0 \), (12) becomes

\[
- \left( 1 - \frac{2M}{r} \right) \dot{t} = K_1,
\]
\[
\dot{r}^2 + \dot{\phi}^2 = K_2.
\]

If \( K_1 = 0 \), then \( t = \text{const} \). If the constant is zero, then

\[
x_1 = \frac{1}{2} r^2 + M^2 - Mr + M \ln (r - M) - M^2 \ln (r - M),
\]
\[
x_2 = \pm \sqrt{\frac{\pi^2}{4} r^2 + C_2},
\]
\[
x_3 = \pm \sqrt{r^2 \dot{\phi}^2 + C_3},
\]
\[
x_4 = \pm \sqrt{C_4}.
\]

Since \( x_1^2 + x_2^2 + x_3^2 - x_4^2 > 0 \) which is the great circle \( S_1 \) in the Kerr space \( K \), this geodesic is a retraction in Kerr space; \( ds^2 > 0 \).
If $K_2 = 0$, then $\phi = \text{const.}$ If the constant is zero, then

\[
\begin{align*}
&x_1 = \frac{1}{2}r^2 + M^2 - Mr + M \ln (r - M) r - M^2 \ln (r - M), \\
x_2 = \pm \sqrt{\frac{\pi^2}{4} r^2 + C_2}, \\
x_3 = \pm \sqrt{C_3}, \\
x_4 = \pm \sqrt{\left(1 - \frac{2M}{r}\right) t^2 + C_4}.
\end{align*}
\]

(17)

Since $x_1^2 + x_2^2 + x_3^2 - x_4^2 > 0$ which is the great circle $S_3$ in the Kerr space $K$, this geodesic is a retraction in Kerr space; $ds^2 > 0$.

From the above discussion, the following theorem has been proved.

**Theorem 1.** The retraction of the Kerr space is a geodesic in the Kerr space.

### 4. Deformation Retract of Kerr Space

The deformation retract of the Kerr space $K$ is defined as

\[
\phi : K \times I \longrightarrow K,
\]

(18)

where $I$ is the closed interval $[0, 1]$. The retraction of the Kerr space $K$ is defined as

\[
R : K \longrightarrow S_1, S_2, S_3.
\]

(19)

Then, the deformation retract of the Kerr space $K$ into a geodesic $S_1 \subset K$ is defined by

\[
\begin{align*}
\phi(M, c) &= \cos \frac{\pi c}{2} \left\{ \left( r^2 - 4Mr + (2Mr - 2M^2) \right) \right. \\
&\quad \times \ln \left( -r^2 - a^2 + 2Mr \right) \\
&\quad + \tan^{-1} \left( \frac{(M - r) / \sqrt{a^2 - M^2}}{\sqrt{a^2 - M^2}} \right) \\
&\quad \times \left( 8M^3 - 6a^2 M + 2a^2 r - 4M^2 r \right) \\
&\quad + \left( a^2 - 2M^2 \right) \\
&\quad \left. \times \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) \right\}^{1/2} \\
&\quad \pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 8a^2 + C_2}, \\
&\quad \pm \sqrt{(r^2 + a^2) + \frac{2Ma^2}{r}} \phi^2 + C_3, \\
&\quad \pm \sqrt{\left(1 - \frac{2M}{r}\right) t^2 + C_4} \right\},
\end{align*}
\]

(20)

where

\[
\phi(M, 0) = \left\{ \left( r^2 - 4Mr + (2Mr - 2M^2) \right) \right.
\]

\[
\times \ln \left( -r^2 - a^2 + 2Mr \right) \\
\times \tan^{-1} \left( \frac{8M^3 - 6a^2 M + 2a^2 r - 4M^2 r}{(M - a)(M + a)} \right) \\
\times \left( a^2 - 2M^2 \right) \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) \\
\left. \times \left(1 - \frac{2M}{r}\right) t^2 + C_4 \right\}^{1/2},
\]

(21)
The deformation retract of the Kerr space into a geodesic \( S_2 \subset K \) is defined by

\[
\phi(M, c) = (1 - c) \frac{\pi c}{2} \left\{ \left( r^2 - 4Mr + (2Mr - 2M^2) \right) \times \ln \left( -r^2 - a^2 + 2Mr \right) + \frac{\tan^{-1} \left( (M - r) \sqrt{a^2 - M^2} \right)}{\sqrt{a^2 - M^2}} \times \left( 8M^3 - 6a^2 M + 2a^2 r - 4M^2 r \right) + \left( a^2 - 2M^2 \right) \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) \right. \\
+ \left. C_1 \right\}^{1/2}, \pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 a^2 + C_2},
\]

\[
\pm \sqrt{\left( r^2 + a^2 \right) + \frac{2Ma^2}{r} \phi^2 + C_3},
\]

\[
\pm \sqrt{\left( 1 - \frac{2M}{r} \right) t^2 + C_4} \right\}.
\]

(22)

Now we are going to discuss the folding of the Kerr space \( K \):

\[
f : K \rightarrow K,
\]

(24)

where

\[
f \left( x_1, x_2, x_3, x_4 \right) = \left( |x_1|, x_2, x_3, x_4 \right).
\]

(25)

An isometric folding of the Kerr space into itself may be defined by

\[
\phi(M, c) \left\{ \left( r^2 - 4Mr + (2Mr - 2M^2) \ln \left( -r^2 - a^2 + 2Mr \right) + \frac{\tan^{-1} \left( (M - r) \sqrt{a^2 - M^2} \right)}{\sqrt{a^2 - M^2}} \times \left( 8M^3 - 6a^2 M + 2a^2 r - 4M^2 r \right) + \left( a^2 - 2M^2 \right) \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) + C_1 \right\}^{1/2},
\]

\[
\pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 a^2 + C_2},
\]

\[
\pm \sqrt{\left( r^2 + a^2 \right) + \frac{2Ma^2}{r} \phi^2 + C_3},
\]

\[
\pm \sqrt{\left( 1 - \frac{2M}{r} \right) t^2 + C_4} \right\}.
\]

(23)
The deformation retract of the folded Kerr space $K$ into the folded $S_1$ is

$$\phi f (M, c) = \cos \frac{\pi c}{2} \left\{ \left| \begin{array}{c}
\left( r^2 - 4Mr + (2Mr - 2M^2) \\
\times \ln \left( -r^2 - a^2 + 2Mr \right) \\
+ \frac{\tan^{-1} \left( (M - r) / \sqrt{a^2 - M^2} \right)}{\sqrt{a^2 - M^2}} \\
\times (8M^3 - 6a^2M + 2a^2r - 4M^2r) \\
+ (a^2 - 2M^2) \\
\times \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) + C_1 \right)^{1/2} \\
\pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 8a^2 + C_2}, \\
\pm \sqrt{\left( r^2 + a^2 \right) + \frac{2Ma^2}{r}} \phi^2 + C_3, \\
\pm \sqrt{\left( 1 - \frac{2M}{r} \right) t^2 + C_4} \times I
\end{array} \right| \right\}$$

with

$$\phi f (M, c) = \cos \frac{\pi c}{2} \left\{ \left| \begin{array}{c}
\left( r^2 - 4Mr + (2Mr - 2M^2) \\
\times \ln \left( -r^2 - a^2 + 2Mr \right) \\
+ \frac{\tan^{-1} \left( (M - r) / \sqrt{a^2 - M^2} \right)}{\sqrt{a^2 - M^2}} \\
\times (8M^3 - 6a^2M + 2a^2r - 4M^2r) \\
+ (a^2 - 2M^2) \\
\times \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) + C_1 \right)^{1/2} \\
\pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 8a^2 + C_2}, \\
\pm \sqrt{\left( r^2 + a^2 \right) + \frac{2Ma^2}{r}} \phi^2 + C_3, \\
\pm \sqrt{\left( 1 - \frac{2M}{r} \right) t^2 + C_4} \times I
\end{array} \right| \right\}$$

\[ (26) \]
\[-M^2 \ln (r - M), \pm \sqrt{\frac{\pi^2}{4} r^2 + C_2}, \]
\[\pm r^2 \phi^2 + C_3, \pm \left(1 - \frac{2M}{r}\right) t^2 + C_4\].

(28)

The deformation retract of the folded Kerr space $K$ into the folded $S_2$ is

$$\phi (M, c) = (1 - c) \left\{ \left( r^2 - 4Mr + (2Mr - 2M^2) \right) \times \ln \left( -r^2 - a^2 + 2Mr \right) \right. $$
$$+ \frac{\tan^{-1} \left( (M - r) / \sqrt{a^2 - M^2} \right)}{\sqrt{a^2 - M^2}} \times \left( 8M^3 - 6a^2 M + 2a^2 r - 4M^2 r \right) $$
$$\left. + (a^2 - 2M^2) \right\} \ln \left( \frac{2Mr - r^2 - a^2}{(M - a)(M + a)} \right) + C_1 \right\}^{1/2},$$
$$\pm \sqrt{\frac{\pi^2}{4} r^2 + \pi^2 a^2 + C_2},$$
$$\pm \sqrt{\left( r^2 + a^2 \right) + \frac{2Ma^2}{r} \phi^2 + C_3},$$
$$\pm \sqrt{\left( 1 - \frac{2M}{r}\right) t^2 + C_4} \right\}.$$

(29)

Therefore, the following theorem has been proved.

**Theorem 2.** The deformation retract of the isometric folding of Kerr space and any folding homeomorphic to this type of folding is different from the deformation retract of Kerr space.

5. Conclusion

The deformation retract of the Kerr space has been investigated by making use of Lagrangian equations. The equatorial geodesics of the Kerr space have been discussed. The retraction of this space into itself and into geodesics has been presented. The deformation retraction of the Eguchi-Hanson space is a geodesic which is found to be a great circle. The deformation retract of the isometric folding of Kerr space and any folding homeomorphic to this type of folding is found to be different from the deformation retract of Kerr space.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.
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