Research Article

A Modified Three-Level Average Linear-Implicit Finite Difference Method for the Rosenau-Burgers Equation

Jiraporn Janwised, Ben Wongsaijai, Thanasak Mouktonglang, and Kanyuta Poochinapan

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to Kanyuta Poochinapan; kanyuta@hotmail.com

Received 23 January 2014; Accepted 16 March 2014; Published 17 April 2014

Academic Editor: Pavel Kurasov

Copyright © 2014 Jiraporn Janwised et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We introduce a new technique, a three-level average linear-implicit finite difference method, for solving the Rosenau-Burgers equation. A second-order accuracy on both space and time numerical solution of the Rosenau-Burgers equation is obtained using a five-point stencil. We prove the existence and uniqueness of the numerical solution. Moreover, the convergence and stability of the numerical solution are also shown. The numerical results show that our method improves the accuracy of the solution significantly.

1. Introduction

A nonlinear wave phenomenon is the important area of scientific research. There are mathematical models which describe the dynamic of wave behaviors such as the KdV equation, the Rosenau equation, and many others. The KdV equation cannot explain the wave-wave and wave-wall interactions for the model of the dynamics of compact discrete systems. Therefore, Rosenau [1, 2] presented the novel model, which is more suitable than the KdV equation, as follows:

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \quad (1)$$

The existence and uniqueness of the solution for this equation were proved by Park [3]. Many methods have been applied to find a numerical solution of the Rosenau equation such as a discontinuous Galerkin method [4], a finite element Galerkin method [5], and a finite difference method [6–8]. Numerical solutions and error estimates in $\|\cdot\|$ and $\|\cdot\|_\infty$ norms were obtained for the Rosenau equation in one space variable [9].

As for Burgers' equation,

$$u_t - u_{xx} + uu_x = 0, \quad (2)$$

this equation has been studied in the evolution equation describing a wave propagation. Moreover, the simulation for Burgers' equation was the very first step of conceptual understanding of the method for the computations of complex flow.

The existence and uniqueness of the generalized Burgers' equation have been shown with certain conditions.

In this paper, we consider the following initial-boundary value problem of the generalized Rosenau-Burgers equation:

$$u_t + u_{xxxxt} - \alpha u_{xx} + \beta u_x + (u^p)_x = 0, \quad (3)$$

$$0 \leq x \leq 1, \quad 0 \leq t \leq T,$$

with an initial condition

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \quad (4)$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad (5)$$

$$0 \leq t \leq T,$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, and $p \geq 2$ is an integer.

This equation was proposed in 1989 to describe the wave in shallow water. It differs from Burgers' equation by an additional strongly dissipative term u_{xxxxt} . The behavior of the solution to the Rosenau-Burgers equation with the Cauchy problem has been well studied for the past years [10–13]. Several second-order accuracy finite difference methods in space were used for finding numerical solutions on both linear and nonlinear terms [14–20].

Hu et al. [14] have proposed the Crank-Nicolson difference scheme, nonlinear scheme, for the Rosenau-Burgers equation. Hu et al. [18] have proposed a three-level average implicit finite difference scheme for the Rosenau-Burgers equation. The schemes are obviously implicit and require a heavy calculation for each iteration. Pan and Zhang [20] have proposed a three-level linear-implicit difference scheme. The schemes, we have mentioned above, are second-order accuracy on both time and space.

In this paper, we propose a modified three-level average linear-implicit finite difference method for the Rosenau-Burgers equation. By comparing with the existence second-order accuracy finite difference scheme on a test problem, our new technique gives a better maximal error of the numerical solutions. A second-order accuracy on both space and time numerical solution of the equation is obtained using a five-point stencil.

This paper is organized into 7 sections. In Section 2, we describe our modified finite different scheme. In Section 3, we discuss the solvability of our scheme. The existence and uniqueness are also proven in this section. In Section 4, we give complete proofs on the convergence and stability of the finite difference scheme which is second-order accuracy on both space and time. The numerical results are given in Section 5 to confirm and illustrate our theoretical analysis. Then we finish our paper by concluding remarks.

2. Modified Finite Difference Scheme

In this section, we give a complete description of our modified finite difference scheme and an algorithm for the formulation of the problem (3)–(5). We first describe our solution domain and its grid. We define the solution domain to be $Q = \{(x, t) \mid 0 \leq x \leq 1, 0 \leq t \leq T\}$, which is covered by a uniform grid $Q_h = \{(x_i, t_n) \mid x_i = ih, t_n = n\tau, i = 0, \dots, J, n = 0, \dots, N\}$, with spacings $h = 1/J$ and $\tau = T/N$. Denote $u_i^n \approx u(ih, n\tau)$ and $Z_h^0 = \{u = (u_i) \mid u_0 = u_J = 0, i = -1, 0, \dots, J, J+1\}$. Throughout this paper, we will denote C as a generic constant independent of step sizes τ and h . For nonnegative integer k , let $H^k(\Omega)$ denote the usual Sobolev space of real-valued functions defined on Ω . We define the following Sobolev space:

$$H_0^k(\Omega) = \left\{ u \in H^k(\Omega) : \begin{aligned} &\frac{\partial^i u}{\partial u^i} = 0 \text{ on } \partial\Omega, \quad i = 0, 1, \dots, k-1 \end{aligned} \right\}. \quad (6)$$

We use the following notations for the simplicity:

$$\begin{aligned} \bar{u}_i^n &= \frac{u_i^{n+1} + u_i^{n-1}}{2}, & (u_i^n)_{\hat{t}} &= \frac{u_i^{n+1} - u_i^{n-1}}{2\tau}, \\ (u_i^n)_x &= \frac{u_{i+1}^n - u_i^n}{h}, & (u_i^n)_{\bar{x}} &= \frac{u_i^n - u_{i-1}^n}{h}, \\ (u_i^n)_{\bar{x}} &= \frac{u_{i+1}^n - u_{i-1}^n}{2h}, & (u_i^n)_{x\bar{x}} &= \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \end{aligned}$$

$$\begin{aligned} (u^n, v^n) &= h \sum_{i=1}^{J-1} u_i^n v_i^n, & \|u^n\|^2 &= (u^n, u^n), \\ \|u^n\|_{\infty} &= \max_{1 \leq i \leq J-1} |u_i^n|. \end{aligned} \quad (7)$$

Since $(u^p)_x = p/(p+1)[u^{p-1}u_x + (u^p)_x]$, the following finite difference scheme solves the problem (3)–(5):

$$\begin{aligned} (u_i^n)_{\hat{t}} + \frac{h^2}{6} (u_i^n)_{x\bar{x}\hat{t}} - \frac{\alpha h^2}{12} (\bar{u}_i^n)_{xx\bar{x}\bar{x}} + (u_i^n)_{xx\bar{x}\bar{x}\hat{t}} - \alpha (\bar{u}_i^n)_{x\bar{x}} \\ + \beta (\bar{u}_i^n)_{\bar{x}} + \psi(u_i^n, \bar{u}_i^n) = 0, \quad 1 \leq i \leq J-1, \end{aligned} \quad (8)$$

$$u_0^n = u_J^n = 0, \quad (u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0, \quad 1 \leq n \leq N, \quad (9)$$

$$u_i^0 = u_0(x_i), \quad (10)$$

where

$$\psi(u_i^n, \bar{u}_i^n) = \frac{p}{p+1} \left[(u_i^n)^{p-1} (\bar{u}_i^n)_{\bar{x}} + ((u_i^n)^{p-1} \bar{u}_i^n)_{\bar{x}} \right]. \quad (11)$$

The following lemmas are some properties of the above finite difference scheme which can be obtained directly from the definition. They are essential for existence, uniqueness, convergence, and stability of our numerical solution.

Lemma 1 (Hu et al. [14]). *For any two mesh functions $u, v \in Z_h^0$, we have*

$$\begin{aligned} (u_x, v) &= -(u, v_{\bar{x}}), & (v, u_{x\bar{x}}) &= -(v_x, u_x), \\ (u, u_{x\bar{x}}) &= -(u_x, u_x) = -\|u_x\|^2. \end{aligned} \quad (12)$$

Furthermore, if $(u_0)_{x\bar{x}} = (u_J)_{x\bar{x}} = 0$, then

$$(u, u_{xx\bar{x}\bar{x}}) = \|u_{x\bar{x}}\|^2. \quad (13)$$

Lemma 2 (discrete Sobolev's inequality [9]). *There exist constants C_1 and C_2 such that*

$$\|u^n\|_{\infty} \leq C_1 \|u^n\| + C_2 \|u_x^n\|. \quad (14)$$

The following theorem guarantees that the numerical solution obtained from scheme (8)–(9) is bounded.

Theorem 3. *Suppose $u_0 \in H_0^2[0, L]$. Then there is an estimation for the solution u^n of the scheme (8)–(9) that satisfies*

$$\|u^n\| \leq C, \quad \|u_{x\bar{x}}^n\| \leq C \quad (15)$$

which imply $\|u^n\|_{\infty} \leq C$ for some $C \in \mathbb{R}$.

Proof. Consider the inner product between (8) and $2\bar{u}^n \equiv (u^{n+1} + u^{n-1})$. According to Lemma 1, we have

$$\begin{aligned} \frac{1}{2\tau} \left(\|u^{n+1}\|^2 - \|u^{n-1}\|^2 \right) - \frac{h^2}{12\tau} \left(\|u_x^{n+1}\|^2 - \|u_x^{n-1}\|^2 \right) \\ - \frac{\alpha h^2}{6} \|\bar{u}_{x\bar{x}}^n\|^2 + \frac{1}{2\tau} \left(\|u_{x\bar{x}}^{n+1}\|^2 - \|u_{x\bar{x}}^{n-1}\|^2 \right) \\ + 2\alpha \|u_x^n\|^2 + \beta (\bar{u}_x^n, 2\bar{u}^n) + (\psi(u^n, \bar{u}^n), 2\bar{u}^n) = 0. \end{aligned} \quad (16)$$

By a direct calculation and the boundary condition (9), we have

$$\begin{aligned} (\bar{u}_x^n, 2\bar{u}^n) &= 2h \sum_{i=1}^{J-1} \left[\frac{1}{2h} (\bar{u}_{i+1}^n - \bar{u}_{i-1}^n) \bar{u}_i^n \right] \\ &= \sum_{i=1}^{J-1} \bar{u}_i^n \bar{u}_{i+1}^n - \sum_{i=1}^{J-1} \bar{u}_{i-1}^n \bar{u}_i^n = 0. \end{aligned} \quad (17)$$

Furthermore, by using the definition of the inner product

$$\begin{aligned} &(\psi(u^n, \bar{u}^n), 2\bar{u}^n) \\ &= \frac{2p}{p+1} h \sum_{i=1}^{J-1} \left[(u_i^n)^{p-1} (\bar{u}_i^n)_{\hat{x}} + ((u_i^n)^{p-1} \bar{u}_i^n)_{\hat{x}} \right] \bar{u}_i^n \\ &= \frac{p}{p+1} \sum_{i=1}^{J-1} \left[(u_i^n)^{p-1} (\bar{u}_{i+1}^n - \bar{u}_{i-1}^n) + (u_{i+1}^n)^{p-1} \bar{u}_{i+1}^n \right. \\ &\quad \left. - (u_{i-1}^n)^{p-1} \bar{u}_{i-1}^n \right] \bar{u}_i^n \\ &= \frac{p}{p+1} \sum_{i=1}^{J-1} \left[(u_i^n)^{p-1} \bar{u}_{i+1}^n \bar{u}_i^n - (u_{i-1}^n)^{p-1} \bar{u}_i^n \bar{u}_{i-1}^n \right] \\ &\quad - \frac{p}{p+1} \sum_{i=1}^{J-1} \left[(u_i^n)^{p-1} \bar{u}_i^n \bar{u}_{i-1}^n - (u_{i+1}^n)^{p-1} \bar{u}_{i+1}^n \bar{u}_i^n \right] = 0. \end{aligned} \quad (18)$$

From (17) and (18), (16) can be rewritten as

$$\begin{aligned} &\frac{1}{2\tau} (\|u^{n+1}\|^2 - \|u^{n-1}\|^2) - \frac{h^2}{12\tau} (\|u_x^{n+1}\|^2 - \|u_x^{n-1}\|^2) \\ &\quad - \frac{\alpha h^2}{6} \|\bar{u}_{x\bar{x}}^n\|^2 + \frac{1}{2\tau} (\|u_{x\bar{x}}^{n+1}\|^2 - \|u_{x\bar{x}}^{n-1}\|^2) \\ &\quad + 2\alpha \|\bar{u}_x^n\|^2 = 0. \end{aligned} \quad (19)$$

Therefore,

$$\begin{aligned} &(\|u^{n+1}\|^2 - \|u^{n-1}\|^2) - \frac{h^2}{6} (\|u_x^{n+1}\|^2 - \|u_x^{n-1}\|^2) \\ &\quad + (\|u_{x\bar{x}}^{n+1}\|^2 - \|u_{x\bar{x}}^{n-1}\|^2) \leq \frac{\alpha \tau h^2}{6} (\|u_{x\bar{x}}^{n+1}\|^2 + \|u_{x\bar{x}}^{n-1}\|^2). \end{aligned} \quad (20)$$

We now define

$$\begin{aligned} E^n &\equiv (\|u^n\|^2 + \|u^{n-1}\|^2) - \frac{h^2}{6} (\|u_x^n\|^2 + \|u_x^{n-1}\|^2) \\ &\quad + (\|u_{x\bar{x}}^n\|^2 + \|u_{x\bar{x}}^{n-1}\|^2). \end{aligned} \quad (21)$$

Then inequality (20) can be rewritten as follows:

$$E^{n+1} - E^n \leq \tau C (E^{n+1} + E^n). \quad (22)$$

If τ is sufficiently small which satisfies $\tau \leq (k-2)/kC$ and $k > 2$, then

$$\begin{aligned} E^{n+1} &\leq \frac{(1+\tau C)}{(1-\tau C)} E^n \leq (1+\tau kC) E^n \leq (1+\tau kC)^n E^1 \\ &\leq \exp(kCT) E^1. \end{aligned} \quad (23)$$

Hence,

$$\begin{aligned} &(\|u^{n+1}\|^2 + \|u^n\|^2) - \frac{h^2}{6} (\|u_x^{n+1}\|^2 + \|u_x^n\|^2) \\ &\quad + (\|u_{x\bar{x}}^{n+1}\|^2 + \|u_{x\bar{x}}^n\|^2) \leq C. \end{aligned} \quad (24)$$

By using Lemma 1 and the Cauchy-Schwarz inequality, we arrive at

$$\|u_x^n\|^2 \leq \frac{1}{2} (\|u^n\|^2 + \|u_{x\bar{x}}^n\|^2). \quad (25)$$

Then, we get

$$\left(1 - \frac{h^2}{12}\right) [(\|u^{n+1}\|^2 + \|u^n\|^2) + (\|u_{x\bar{x}}^{n+1}\|^2 + \|u_{x\bar{x}}^n\|^2)] \leq C. \quad (26)$$

If h is sufficiently small in which $(1 - h^2/12) > 0$, we arrive at that

$$\|u^{n+1}\| \leq C, \quad \|u_{x\bar{x}}^{n+1}\| \leq C. \quad (27)$$

From (25), it follows that $\|u_x^{n+1}\| \leq C$. By Lemma 2, it is obvious that $\|u^{n+1}\|_{\infty} \leq C$ and that completes the proof. $\square$

3. Solvability

In this section, we prove the solvability of a solution for scheme (8). This guarantees the existence and uniqueness of our numerical solution.

Theorem 4. *The finite difference scheme (8)–(10) is uniquely solvable.*

Proof. To prove the theorem, we proceed by the mathematical induction. We assume that $u^0, u^1, \dots, u^n$ satisfy the difference scheme (8). Indeed, u^1 can be computed by an available second-order accuracy method. Next we prove that there exists u^{n+1} which satisfied (8). Consider

$$\begin{aligned} &\frac{1}{2\tau} u_i^{n+1} - \left(\frac{\alpha}{2} - \frac{h^2}{12\tau}\right) (u_i^{n+1})_{x\bar{x}} + \left(\frac{1}{2\tau} - \frac{\alpha h^2}{24}\right) (u_i^{n+1})_{xx\bar{x}\bar{x}} \\ &\quad + \frac{\beta}{2} (u_i^{n+1})_{\hat{x}} + \frac{1}{2} \psi(u_i^n, u_i^{n+1}) = 0, \quad 1 \leq i \leq J-1, \end{aligned} \quad (28)$$

where

$$\psi(u_i^n, u_i^{n+1}) = \frac{p}{p+1} [(u_i^n)^{p-1} (u_i^{n+1})_{\hat{x}} + ((u_i^n)^{p-1} u_i^{n+1})_{\hat{x}}]. \quad (29)$$

By taking the inner product of (28) with u^{n+1} and using Lemma 1, we obtain

$$\begin{aligned} &\frac{1}{2\tau} \|u^{n+1}\|^2 + \left(\frac{\alpha}{2} - \frac{h^2}{12\tau}\right) \|u_x^{n+1}\|^2 + \left(\frac{1}{2\tau} - \frac{\alpha h^2}{24}\right) \|u_{x\bar{x}}^{n+1}\|^2 \\ &\quad + \frac{1}{2} (\psi(u^n, u^{n+1}), u^{n+1}) = 0. \end{aligned} \quad (30)$$

Notice that

$$\begin{aligned}
& (\psi(u^n, u^{n+1}), u^{n+1}) \\
&= \frac{hp}{p+1} \sum_{i=1}^{J-1} [(u_i^n)^{p-1} (u_i^{n+1})_{\hat{x}} + ((u_i^n)^{p-1} u_i^{n+1})_{\hat{x}}] u_i^{n+1} \\
&= \frac{p}{2(p+1)} \sum_{i=1}^{J-1} [(u_i^n)^{p-1} (u_{i+1}^{n+1} - u_{i-1}^{n+1}) \\
&\quad + (u_{i+1}^n)^{p-1} u_{i+1}^{n+1} - (u_{i-1}^n)^{p-1} u_{i-1}^{n+1}] u_i^{n+1} \\
&= \frac{p}{2(p+1)} \sum_{i=1}^{J-1} [(u_i^n)^{p-1} u_{i+1}^{n+1} u_i^{n+1} - (u_{i-1}^n)^{p-1} u_{i-1}^{n+1} u_i^{n+1}] \\
&\quad - \frac{p}{2(p+1)} \sum_{i=1}^{J-1} [(u_i^n)^{p-1} u_{i-1}^{n+1} u_i^{n+1} - (u_{i+1}^n)^{p-1} u_{i+1}^{n+1} u_i^{n+1}] = 0.
\end{aligned} \tag{31}$$

Hence,

$$\|u^{n+1}\|^2 + \left(\alpha\tau - \frac{h^2}{6}\right) \|u_{xx}^{n+1}\|^2 + \left(1 - \frac{\alpha\tau h^2}{12}\right) \|u_{xx}^{n+1}\|^2 = 0. \tag{32}$$

Similar to the proof of inequality (25), (32) can be rewritten as

$$\left(1 - \frac{h^2}{12}\right) (\|u^{n+1}\|^2 + \|u_{xx}^{n+1}\|^2) \leq \frac{\alpha\tau h^2}{12} \|u_{xx}^{n+1}\|^2. \tag{33}$$

For h and τ are sufficiently small which satisfies $1 - (h^2/12)(1 - \alpha\tau) > 0$, we obtain

$$\|u^{n+1}\|^2 + \|u_{xx}^{n+1}\|^2 \leq 0. \tag{34}$$

It follows that

$$\|u^{n+1}\|^2 = \|u_{xx}^{n+1}\|^2 = 0. \tag{35}$$

This implies that there uniquely exists a trivial solution satisfying (8)–(10). Hence, u^{n+1} is uniquely solvable. This completes the proof. $\square$

4. Convergence and Stability

In this section, the second-order rate of convergence and stability of scheme (8)–(9) are guaranteed and explicitly proved. Let $e_i^n = v_i^n - u_i^n$, where v_i^n and u_i^n are the solutions of the problem (3)–(5) and the problem (8)–(9), respectively. We arrive at the following error equation:

$$\begin{aligned}
r_i^n &= (e_i^n)_{\hat{t}} + \frac{h^2}{6} (e_i^n)_{xx\hat{t}} - \frac{\alpha h^2}{12} (\bar{e}_i^n)_{xx\bar{x}\bar{x}} + (e_i^n)_{xx\bar{x}\bar{x}\hat{t}} - \alpha (\bar{e}_i^n)_{xx} \\
&\quad + \beta (\bar{e}_i^n)_{\hat{x}} + \frac{p}{p+1} [(v_i^n)^{p-1} (\bar{v}_i^n)_{\hat{x}} + ((v_i^n)^{p-1} \bar{v}_i^n)_{\hat{x}}] \\
&\quad - \frac{p}{p+1} [(u_i^n)^{p-1} (\bar{u}_i^n)_{\hat{x}} + ((u_i^n)^{p-1} \bar{u}_i^n)_{\hat{x}}],
\end{aligned} \tag{36}$$

where r_i^n denotes the truncation error. By the Taylor expansion, we easily obtain that $r_i^n = O(\tau^2 + h^2)$ holds as $\tau, h \rightarrow 0$. The following lemmas are well known and useful for the proofs of the convergence and stability.

Lemma 5 (Zheng and Hu [16]). Suppose that $u \in H_0^2[0, L]$. Then the solution of the initial-boundary value problem (3)–(5) satisfies

$$\|u\|_{L_2} \leq C, \quad \|u_{xx}\|_{L_2} \leq C, \quad \|u\|_{L_\infty} \leq C, \tag{37}$$

for a constant C .

Lemma 6 (discrete Gronwall inequality [9]). Suppose $w(k)$, $\rho(k)$ are nonnegative mesh functions and $\rho(k)$ is a nondecreasing function. If $C > 0$ and

$$w(k) \leq \rho(k) + C\tau \sum_{l=0}^{k-1} w(l), \quad \forall k, \tag{38}$$

then

$$w(k) \leq \rho(k) e^{C\tau k}, \quad \forall k. \tag{39}$$

The following theorem guarantees the convergence of our scheme with the convergence rate of $O(\tau^2 + h^2)$.

Theorem 7. Suppose $u_0 \in H_0^2[0, L]$. Then the solution u^n of scheme (8)–(9) converges to the solution of the problem (3)–(5) in the sense of $\|\cdot\|_\infty$ and the rate of convergence is $O(\tau^2 + h^2)$.

Proof. By taking the inner product of (8) and $2\bar{e}^n \equiv e^{n+1} + e^{n-1}$ and using the fact that $(e_{\hat{x}}^n, 2\bar{e}^n) = 0$, we get

$$\begin{aligned}
& \frac{1}{2\tau} (\|e^{n+1}\|^2 - \|e^{n-1}\|^2) - \frac{h^2}{12\tau} (\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2) \\
& \quad - \frac{\alpha h^2}{6} \|\bar{e}_{xx}^n\|^2 + \frac{1}{2\tau} (\|e_{xx}^{n+1}\|^2 - \|e_{xx}^{n-1}\|^2) + 2\alpha \|\bar{e}_x^n\|^2 \\
& \quad = (r^n, 2\bar{e}^n - (M_1 + M_2, 2\bar{e}^n)),
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
M_1 &= \frac{p}{p+1} [(v^n)^{p-1} (\bar{v}^n)_{\hat{x}} - (u^n)^{p-1} (\bar{u}^n)_{\hat{x}}], \\
M_2 &= \frac{p}{p+1} [((v^n)^{p-1} \bar{v}^n)_{\hat{x}} - ((u^n)^{p-1} \bar{u}^n)_{\hat{x}}].
\end{aligned} \tag{41}$$

According to Lemma 5, Theorem 3, and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& (M_1, 2\bar{e}^n) \\
&= \frac{2p}{p+1} h \sum_{i=1}^{J-1} [(v_i^n)^{p-1} (\bar{v}_i^n)_{\hat{x}} - (u_i^n)^{p-1} (\bar{u}_i^n)_{\hat{x}}] \bar{e}_i^n \\
&= \frac{2p}{p+1} h \sum_{i=1}^{J-1} [(v_i^n)^{p-1} (\bar{e}_i^n)_{\hat{x}} \bar{e}_i^n] \\
&\quad + \frac{2p}{p+1} h \sum_{i=1}^{J-1} [(v_i^n)^{p-1} - (u_i^n)^{p-1}] (\bar{u}_i^n)_{\hat{x}} \bar{e}_i^n
\end{aligned}$$

$$\begin{aligned}
&= \frac{2p}{p+1} h \sum_{i=1}^{J-1} \left[(\nu_i^n)^{p-1} (\bar{e}_i^n)_{\bar{x}} \bar{e}_i^n \right] \\
&\quad + \frac{2p}{p+1} h \sum_{i=1}^{J-1} \left[e_i^n \sum_{k=0}^{p-2} (\nu_i^n)^{p-2-k} (u_i^n)^k \right] (\bar{u}_i^n)_{\bar{x}} \bar{e}_i^n \\
&\leq C \left(\|\bar{e}_{\bar{x}}^n\|^2 + \|\bar{e}^n\|^2 + \|e^n\|^2 \right) \\
&\leq C \left(\|e_{\bar{x}}^{n+1}\|^2 + \|e_{\bar{x}}^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \right). \tag{42}
\end{aligned}$$

Similar to the proof of (42), we have also

$$\begin{aligned}
(M_2, 2\bar{e}^n) &\leq C \left(\|\bar{e}_{\bar{x}}^n\|^2 + \|\bar{e}^n\|^2 + \|e^n\|^2 \right) \\
&\leq C \left(\|e_{\bar{x}}^{n+1}\|^2 + \|e_{\bar{x}}^{n-1}\|^2 \right. \\
&\quad \left. + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \right). \tag{43}
\end{aligned}$$

Furthermore,

$$\|e_{\bar{x}}^n\|^2 \leq \|e_x^n\|^2 \leq \frac{1}{2} \left(\|e^n\|^2 + \|e_{x\bar{x}}^n\|^2 \right), \tag{44}$$

$$(r^n, 2\bar{e}^n) \leq \|r^n\|^2 + \frac{1}{2} \left(\|e^{n+1}\|^2 + \|e^{n-1}\|^2 \right). \tag{45}$$

By substituting (42)–(45) into (40), we obtain

$$\begin{aligned}
&\frac{1}{2\tau} \left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2 \right) \\
&\quad - \frac{h^2}{12\tau} \left(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2 \right) - \frac{\alpha h^2}{6} \|\bar{e}_{x\bar{x}}^n\|^2 \\
&\quad + \frac{1}{2\tau} \left(\|e_{x\bar{x}}^{n+1}\|^2 - \|e_{x\bar{x}}^{n-1}\|^2 \right) + 2\alpha \|\bar{e}_x^n\|^2 \\
&\leq \|r^n\|^2 + C \left(\|e_{x\bar{x}}^{n+1}\|^2 + \|e_{x\bar{x}}^n\|^2 + \|e_{x\bar{x}}^{n-1}\|^2 \right. \\
&\quad \left. + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \right). \tag{46}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(\|e^{n+1}\|^2 - \|e^{n-1}\|^2 \right) - \frac{h^2}{6} \left(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2 \right) \\
&\quad + \left(\|e_{x\bar{x}}^{n+1}\|^2 - \|e_{x\bar{x}}^{n-1}\|^2 \right) \leq 2\tau \|r^n\|^2 \\
&\quad + C\tau \left(\|e_{x\bar{x}}^{n+1}\|^2 + \|e_{x\bar{x}}^n\|^2 + \|e_{x\bar{x}}^{n-1}\|^2 \right. \\
&\quad \left. + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2 \right). \tag{47}
\end{aligned}$$

Let

$$\begin{aligned}
E^n &= \left(\|e^n\|^2 + \|e^{n-1}\|^2 \right) - \frac{h^2}{6} \left(\|e_x^n\|^2 + \|e_x^{n-1}\|^2 \right) \\
&\quad + \left(\|e_{x\bar{x}}^n\|^2 + \|e_{x\bar{x}}^{n-1}\|^2 \right), \tag{48} \\
B^n &= \left(\|e^n\|^2 + \|e^{n-1}\|^2 \right) + \left(\|e_{x\bar{x}}^n\|^2 + \|e_{x\bar{x}}^{n-1}\|^2 \right).
\end{aligned}$$

From (44), then (47) can be rewritten as

$$E^{n+1} - E^n \leq 2\tau \|r^n\|^2 + \tau C (E^{n+1} + E^n). \tag{49}$$

That is,

$$(1 - C\tau) (E^{n+1} - E^n) \leq 2\tau \|r^n\|^2 + 2C\tau E^n. \tag{50}$$

If τ is sufficiently small which satisfies $1 - C\tau > 0$, then

$$E^{n+1} - E^n \leq C\tau \|r^n\|^2 + C\tau E^n. \tag{51}$$

Summing up from 1 to n , we have

$$E^{n+1} - E^1 \leq C\tau \sum_{k=1}^n \|r^k\|^2 + C\tau \sum_{k=1}^n E^k. \tag{52}$$

Then

$$E^{n+1} \leq E^1 + C\tau \sum_{k=1}^n \|r^k\|^2 + C\tau \sum_{k=1}^n E^k. \tag{53}$$

Using (44), we obtain that

$$\frac{1}{2} B^n \leq E^n \leq B^n. \tag{54}$$

Equations (53) and (54) yield

$$\frac{1}{2} B^{n+1} \leq B^1 + C\tau \sum_{k=1}^n \|r^k\|^2 + C\tau \sum_{k=1}^n B^k, \tag{55}$$

which is equivalent to

$$B^{n+1} \leq 2B^1 + C\tau \sum_{k=1}^n \|r^k\|^2 + C\tau \sum_{k=1}^n B^k. \tag{56}$$

Notice that

$$\tau \sum_{k=1}^n \|r^k\|^2 \leq n\tau \max_{1 \leq k \leq n} \|r^k\|^2 \leq T \cdot O(\tau^2 + h^2)^2. \tag{57}$$

Since we can approximate u^1 using any available second-order accuracy method, we have $B^1 = O(\tau^2 + h^2)^2$. Hence

$$B^{n+1} \leq O(\tau^2 + h^2)^2 + C\tau \sum_{k=1}^n B^k. \tag{58}$$

According to Lemma 6, $B^{n+1} \leq O(\tau^2 + h^2)^2$ implies

$$\|e^{n+1}\| \leq O(\tau^2 + h^2), \quad \|e_{x\bar{x}}^{n+1}\| \leq O(\tau^2 + h^2). \tag{59}$$

It follows from (44) that

$$\|e_x^{n+1}\| \leq O(\tau^2 + h^2). \tag{60}$$

By using Lemma 2, we have

$$\|e^{n+1}\|_{\infty} \leq O(\tau^2 + h^2). \tag{61}$$

This completes the proof of Theorem 7. $\square$

FIGURE 1: Numerical solutions at $t = 10$, $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 4$.

FIGURE 2: Numerical solutions at $t = 10$, $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 8$.

Theorem 8. Under the conditions of Theorem 7, the solution of scheme (8)-(9) is stable with respect to $\|\cdot\|_{\infty}$.

5. Numerical Experiments

In this section, we present numerical experiments on a test problem

$$u_t + u_{xxxxt} - \alpha u_{xx} + \beta u_x + (u^p)_x = 0, \quad (62)$$

$$0 \leq x \leq 1, \quad 0 \leq t \leq 10,$$

with an initial condition

$$u(x, 0) = u_0(x) = x^4(1-x)^4, \quad 0 \leq x \leq 1, \quad (63)$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad (64)$$

$$0 \leq t \leq 10,$$

to confirm and illustrate the accuracy of our method. Since the exact solution is not known, the finest grid ($h = 1/256$) is used as a reference solution (pseudanalytical solution). We make comparisons between schemes proposed in [18, 20] with our scheme (8), which is also second-order in space and time. The errors from the three schemes are presented in Tables 1, 2, and 3. For $p = 2, 4$, and 8 , it is clear that our scheme gives better approximation than both schemes proposed in [18, 20].

FIGURE 3: Numerical solutions at $t = 10$, $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 4$, and $p = 4$.

The corresponding errors with respect to $\|\cdot\|$ and $\|\cdot\|_{\infty}$ -norm are listed in Tables 4, 5, and 6 for $p = 2, 4$, and 8 . The rate of convergence is computed using two grids according to the formula

$$\text{rate} = \log_2 \frac{\|e_h\|}{\|e_{h/2}\|}. \quad (65)$$

Since we have five grids, we can get four different estimations of the convergent rates. As shown in Tables 4, 5, and 6 for one particular choice of the parameters, the three numerically estimated rates are presented and they are close to the

TABLE 1: The maximal errors of numerical solutions at $t = 10$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 2$.

Methods	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
[18]	$5.319866e - 4$	$1.276895e - 4$	$3.018276e - 5$	$6.026481e - 6$
[20]	$4.483472e - 5$	$1.110691e - 5$	$2.644709e - 6$	$5.290243e - 7$
Scheme (8)	$3.831638e - 5$	$9.264042e - 6$	$2.207048e - 6$	$4.408479e - 7$

TABLE 2: The maximal errors of numerical solutions at $t = 10$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 4$.

Methods	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
[18]	$5.319788e - 4$	$1.276876e - 4$	$3.018218e - 5$	$6.026111e - 6$
[20]	$4.482287e - 5$	$1.110366e - 5$	$2.644021e - 6$	$5.288865e - 7$
Scheme (8)	$3.830394e - 5$	$9.261242e - 6$	$2.206324e - 6$	$4.407013e - 7$

FIGURE 4: Numerical solutions at $t = 10$, $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 4$, and $p = 8$. FIGURE 6: Numerical solutions at different time with $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 8$. FIGURE 5: Numerical solutions at different time with $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 4$. FIGURE 7: Numerical solutions at different time with $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 4$, and $p = 4$.

TABLE 3: The maximal errors of numerical solutions at $t = 10$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 8$.

Methods	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
[18]	$5.319785e - 4$	$1.276873e - 4$	$3.018188e - 5$	$6.025843e - 6$
[20]	$4.482286e - 5$	$1.110366e - 5$	$2.644016e - 6$	$5.288811e - 7$
Scheme (8)	$3.830394e - 5$	$9.261242e - 6$	$2.206324e - 6$	$4.407013e - 7$

TABLE 4: The errors of numerical solutions at $t = 10$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 2$.

h	$\ e\ _{\infty}$	Rate	$\ e\ $	Rate
1/8	$1.674577e - 4$	—	$9.800635e - 5$	—
1/16	$3.831638e - 5$	2.127764	$2.291144e - 5$	2.096807
1/32	$9.264042e - 6$	2.048248	$5.554049e - 6$	2.044456
1/64	$2.207048e - 6$	2.069524	$1.315895e - 6$	2.077495
1/128	$4.408479e - 7$	2.323765	$2.628915e - 7$	2.323505

TABLE 5: The errors of numerical solutions at $t = 10$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 4$.

h	$\ e\ _{\infty}$	Rate	$\ e\ $	Rate
1/8	$1.673863e - 4$	—	$9.800669e - 5$	—
1/16	$3.830394e - 5$	2.127617	$2.291161e - 5$	2.096801
1/32	$9.261242e - 6$	2.048215	$5.554093e - 6$	2.044456
1/64	$2.206324e - 6$	2.069561	$1.315904e - 6$	2.077497
1/128	$4.407013e - 7$	2.323772	$2.628917e - 7$	2.323514

TABLE 6: The errors of numerical solutions at $t = 10$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, and $p = 8$.

h	$\ e\ _{\infty}$	Rate	$\ e\ $	Rate
1/8	$1.673863e - 4$	—	$9.800669e - 5$	—
1/16	$3.830394e - 5$	2.127617	$2.291161e - 5$	2.096801
1/32	$9.261242e - 6$	2.048215	$5.554093e - 6$	2.044456
1/64	$2.206324e - 6$	2.069561	$1.315905e - 6$	2.077496
1/128	$4.407013e - 7$	2.323772	$2.628922e - 7$	2.323512

theoretically predicted ones which are second-order rates of convergence.

Figures 1, 2, 3, and 4 show the numerical solutions at $t = 10$ with $p = 4$ and 8. The graphs of the numerical solutions of Hu et al. [18], Pan and Zhang [20] schemes, and the proposed scheme are presented. It is to confirm that the approximated solutions are coinciding.

Figures 5, 6, 7, and 8 present the numerical solutions computed by the finite difference scheme (8) with $\tau = 0.1$, $h = 1/32$ at $t = 0, 2, 4, 6$, and 8 when $\alpha = 2$ and 4, respectively. It is clear that the amplitude of the numerical solution decreases over time. In Figures 9, 10, 11, and 12, numerical solutions are presented for a fixed set of h , τ , and t with different values of β and α , respectively. The graph shows that β does not contribute to the height of the amplitude of the numerical solution. On the other hand, the larger the value of α , the smaller the amplitude of the numerical solution.

FIGURE 8: Numerical solutions at different time with $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 4$, and $p = 8$. FIGURE 9: Numerical solutions at different α with $h = 1/32$, $\tau = 0.1$, $\beta = 1$, and $p = 4$.

FIGURE 10: Numerical solutions at different α with $h = 1/32$, $\tau = 0.1$, $\beta = 1$, and $p = 8$.

FIGURE 12: Numerical solutions at different β with $h = 1/32$, $\tau = 0.1$, $\alpha = 1$, and $p = 8$.

FIGURE 11: Numerical solutions at different β with $h = 1/32$, $\tau = 0.1$, $\alpha = 1$, and $p = 4$.

FIGURE 13: Absolute error distribution at $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, $p = 4$, and $t = 10$.

Absolute error distributions for the three methods are plotted at $t = 10$ in Figures 13, 14, 15, and 16. Clearly, our proposed method gives a better approximate solution than the schemes proposed in [18, 20], especially at the peak of the solution. As the results in some of the applications where the characterization of the solution at the peak needs to be precise our proposed method is highly recommended.

6. Concluding Remarks

We introduce a three-level average linear-implicit finite difference method for solving the Rosenau-Burgers equation. We prove the existence and uniqueness of the numerical

FIGURE 14: Absolute error distribution at $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, $p = 8$, and $t = 10$.

FIGURE 16: Absolute error distribution at $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, $p = 8$, and $t = 10$.

FIGURE 15: Absolute error distribution at $h = 1/32$, $\tau = 0.1$, $\beta = 1$, $\alpha = 2$, $p = 4$, and $t = 10$.

solution. The convergence and stability of the numerical solution are also shown. The quantitative comparison of the numerical results from previously known methods shows that our method improves the accuracy of the solution significantly. In addition, our results provide the most precise peak amplitude.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research was supported by Chiang Mai University.

References

- [1] P. Rosenau, "A quasi continuous description of a nonlinear transmission line," *Physica Scripta*, vol. 34, pp. 827–829, 1986.
- [2] P. Rosenau, "Dynamics of dense discrete systems," *Progress of Theoretical Physics*, vol. 79, pp. 1028–1042, 1988.
- [3] M. A. Park, "On the Rosenau equation," *Applied Mathematics and Computation*, vol. 9, pp. 145–152, 1990.
- [4] S. M. Choo, S. K. Chung, and K. I. Kim, "A discontinuous Galerkin method for the Rosenau equation," *Applied Numerical Mathematics*, vol. 58, no. 6, pp. 783–799, 2008.
- [5] H. Y. Lee, M. R. Ohm, and J. Y. Shin, "Finite element Galerkin solutions for the Rosenau equation," *Applicable Analysis*, vol. 2, pp. 153–160, 1997.
- [6] K. Omrani, F. Abidi, T. Achouri, and N. Khiari, "A new conservative finite difference scheme for the Rosenau equation," *Applied Mathematics and Computation*, vol. 201, no. 1-2, pp. 35–43, 2008.
- [7] K. Zheng and J. Hu, "Two conservative difference schemes for the generalized rosenau equation," *Boundary Value Problems*, vol. 2010, Article ID 543503, 2010.
- [8] M. Wang, D. Li, and P. Cui, "A conservative finite difference scheme for the generalized rosenau equation," *International Journal of Pure and Applied Mathematics*, vol. 71, no. 4, pp. 539–549, 2011.
- [9] Y. Zhou, *Application of Discrete Functional Analysis to the Finite Difference Method*, International Academic Publishers, Beijing, China, 1990.
- [10] M. Mei, "Long-time behavior of solution for Rosenau-Burgers equation (I)," *Applicable Analysis*, vol. 63, pp. 315–330, 1996.
- [11] M. Mei, "Long-time behavior of solution for Rosenau-Burgers equation (II)," *Applicable Analysis*, vol. 68, pp. 333–356, 1998.

- [12] L. Liu and M. Mei, "A better asymptotic profile of Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 131, no. 1, pp. 147–170, 2002.
- [13] L. Liu, M. Mei, and Y. S. Wong, "Asymptotic behavior of solutions to the Rosenau-Burgers equation with a periodic initial boundary," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 8, pp. 2527–2539, 2007.
- [14] B. Hu, Y. Xu, and J. Hu, "Crank-Nicolson finite difference scheme for the Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 311–316, 2008.
- [15] D. Li, Z. Wang, Y. Wu, and Y. Lu, "A finite difference simulation for Rosenau-Burgers equation," in *Proceedings of the International Conference on Information Engineering and Computer Science (ICIECS '09)*, pp. 1–4, Wuhan, China.
- [16] K. Zheng and J. Hu, "Crank-Nicolson difference scheme for the generalized Rosenau-Burgers equation," *International Journal of Mathematics and Computer Science*, vol. 5, no. 4, 2009.
- [17] W. Ma, A. Yang, and Y. Wang, "A second-order accurate linearized difference scheme for the Rosenau-Burgers equation," *Journal of Information and Computational Science*, vol. 7, no. 8, pp. 1793–1800, 2010.
- [18] J. Hu, B. Hu, and Y. Xu, "Average implicit linear difference scheme for generalized Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 217, no. 19, pp. 7557–7563, 2011.
- [19] M. O. Korpusov, "On the blow-up of solutions of the Benjamin-Bona-Mahony-Burgers and Rosenau-Burgers equations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 75, no. 4, pp. 1737–1743, 2012.
- [20] X. Pan and L. Zhang, "A new finite difference scheme for the Rosenau-Burgers equation," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8917–8924, 2012.