Research Article

Stabilization of the Wave Equation with Boundary Time-Varying Delay

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We study the stabilization of the wave equation with variable coefficients in a bounded domain and a time-varying delay term in the time-varying, weakly nonlinear boundary feedbacks. By the Riemannian geometry methods and a suitable assumption of nonlinearity, we obtain the uniform decay of the energy of the closed loop system.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ . It is assumed that Γ consists of two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$. Define

$$\mathcal{A}u = -\operatorname{div} A(x) \nabla u \quad \text{for } u \in H^1(\Omega), \quad (1)$$

where div is the divergence operator of the standard metric of $\mathbb{R}^n$. $A(x) = (a_{ij}(x))$ is symmetric, positively definite matrices for each $x \in \mathbb{R}^n$ and $a_{ij}(x)$ are smooth functions on $\mathbb{R}^n$.

We consider the stabilization of the wave equations with variable coefficients and time-varying delay in the dissipative boundary feedback as follows:

$$u_{tt} + \mathcal{A}u = 0 \quad (x, t) \in \Omega \times (0, +\infty),$$

$$u(x, t)|_{\Gamma_2} = 0 \quad t \in (0, +\infty),$$

$$\frac{\partial u(x, t)}{\partial \nu_{\mathcal{A}}} + \phi(t) (\mu g_1(u_t(x, t)) + \lambda g_2(u_t(x, t - \tau(t)))) = 0$$

$$(x, t) \in \Gamma_1 \times (0, +\infty),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega,$$

$$u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)) \quad (x, t) \in \Gamma_1 \times (0, \tau(0)), \quad (2)$$

where $\tau(t)$ satisfies

$$0 \leq \tau(t) \leq \tau_0, \quad d_0 \leq \tau'(t) \leq d < 1 \quad \forall t \geq 0, \quad (3)$$

where $\tau_0 > 0$ and d_0 and d are constants. $g_1 \in C(\mathbb{R})$ and there exist positive constants $c_1, p \geq 1$ such that

$$g_1(0) = 0, \quad sg_1(s) \geq |s|^2 \quad \text{for } s \in \mathbb{R}, \quad (4)$$

$$|g_1(s)| \leq c_1 |s| \quad \text{for } |s| > 1, \quad (5)$$

$$s^2 + (g_1(s))^2 \leq c_1 (sg_1(s))^{1/p} \quad \text{for } |s| \leq 1.$$

$g_2(s) \in C(\mathbb{R})$ satisfies

$$(g_2(s))^2 \leq sg_1(s) \quad \text{for } s \in \mathbb{R}, \quad (6)$$

and $\phi(t) \in C([-\tau(0), +\infty))$ satisfies

$$0 < \phi(t) \leq \phi_0 \quad \forall t \geq -\tau(0), \quad (7)$$

$$c_2 \phi(t) \leq \phi(t - \tau(t)) \leq c_3 \phi(t) \quad \forall t \geq 0, \quad (8)$$

$$\lim_{t \rightarrow \infty} \frac{F(t)}{t} = 0,$$

where ϕ_0, c_2 , and c_3 are positive constants and $F(t) = 1/\inf\{\phi(\rho) \mid 0 \leq \rho \leq t\}$.

$\partial u / \partial \nu_{\mathcal{A}}$ is the conormal derivative

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} = \langle A(x) u, \nu \rangle, \quad (9)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard metric of the Euclidean space $\mathbb{R}^n$ and $\nu(x)$ is the outside unit normal vector for each $x \in \Gamma$. Moreover, $\mu > 0$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and the initial data $(u_0, u_1, f_0, w_0, w_1, h_0)$ belongs to a suitable space.

There is a specific example for $\phi(t)$. Let $\phi_1 > 0$ be a constant. If $\phi(t)$ satisfies

$$\phi_1 \leq \phi(t) \leq \phi_0 \quad \forall t \geq 0, \quad (10)$$

then

$$\begin{aligned} \frac{\phi_1}{\phi_0} \phi(t) &\leq \phi(t - \tau(t)) \leq \frac{\phi_0}{\phi_1} \phi(t) \quad \forall t \geq 0, \\ \lim_{t \rightarrow \infty} \frac{F(t)}{t} &\leq \frac{\phi_0}{t} = 0. \end{aligned} \quad (11)$$

Conditions (7) and (8) hold.

In absence of delay ($\lambda = 0$), the problem (2) was studied by [1–8] and many others. The decay rate of the energy (when t goes to infinity) depends on the function ϕ and the growth of g_1 .

The system (2) with constant coefficient (the case: $A(x)$ is a constant matrix on $\overline{\Omega}$) was studied by [9–11] and many other authors. For the system (2) with variable coefficients, the main tools to cope with the system (2) are the differential geometrical methods which were introduced by [12] and have been applied in many papers. See [13–22] and references cited therein. For a survey on the differential geometric methods, see [23, 24].

The authors in [11] considered the system (2) with constant coefficients operator and dissipative boundary conditions of time dependent delay and proved the exponential decay of the energy by combining the multiplier method with the use of suitable integral inequalities. Different from this paper, g_1 is assumed to be linearly bounded and ϕ is assumed to be a constant function in the paper [11].

Based on [11], the purpose of this paper is to solve the stability of the system (2) with variable coefficients and time-varying, weakly nonlinear terms. To obtain our stabilization result, we assume that

$$\left(\frac{1}{2c_2} + \frac{1}{2} \right) \frac{|\lambda|}{\sqrt{1-d}} < \mu, \quad (12)$$

where c_2 is defined in (8).

Define the energy of the system (2) by

$$\begin{aligned} E(t) &= \frac{1}{2} \int_{\Omega} \left(u_t^2 + \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} \right) dx \\ &\quad + \xi \int_{t-\tau(t)}^t \int_{\Gamma_1} \phi(\rho) u_{\rho}(x, \rho) g_1(u_{\rho}(x, \rho)) d\Gamma d\rho, \end{aligned} \quad (13)$$

where ξ is a positive constant satisfying

$$\frac{|\lambda|}{2c_2 \sqrt{1-d}} < \xi < \mu - \frac{|\lambda|}{2\sqrt{1-d}}. \quad (14)$$

We define

$$g = A^{-1}(x) \quad \text{for } x \in \mathbb{R}^n \quad (15)$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold with an inner product

$$\langle X, Y \rangle_g = \langle A^{-1}(x) X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g \quad X, Y \in \mathbb{R}_x^n. \quad (16)$$

Let D_g denote the Levi-Civita connection of the metric g . For the variable coefficients, the main assumptions are as follows.

Assumption A. There exists a vector field H on $\overline{\Omega}$ and a constant $\rho_0 > 0$ such that

$$D_g H(X, X) \geq \rho_0 |X|_g^2 \quad \text{for } X \in \mathbb{R}_x^n \quad x \in \overline{\Omega}. \quad (17)$$

Moreover, we assume that

$$\sup_{x \in \overline{\Omega}} \operatorname{div} H < \inf_{x \in \overline{\Omega}} \operatorname{div} H + 2\rho_0, \quad (18)$$

$$H \cdot \nu \leq 0 \quad x \in \Gamma_2, \quad H \cdot \nu \geq \delta \quad x \in \Gamma_1, \quad (19)$$

where δ is a positive constant.

Assumption (17) was introduced by [12] as a checkable assumption for the exact controllability of the wave equation with variable coefficients. Assumption A is also useful for the controllability and the stabilization of the quasilinear wave equation [15]. For the examples of the condition, see [12, 23].

Based on Assumption (17), Assumption A was given by [22] to study the stabilization of the wave equation with variable coefficients and boundary condition of memory type. The authors in [22] also constructed some examples of the condition based on the assumption that $A(x) = a(x)I$ or $A(x)$ is a perturbation of a symmetric positive definite matrix A .

Define

$$H_{\Gamma_2}^1(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Gamma_2} = 0\}. \quad (20)$$

To obtain the stabilization of the system (2), we assume that the system (2) is well posed such that

$$\begin{aligned} u &\in C^1([0, +\infty), L^2(\Omega)) \cap C([0, +\infty), H_{\Gamma_2}^1(\Omega)) \\ u_t &\in C([0, +\infty), L^2(\Gamma_1 \times (t, t - \tau(t)))) \end{aligned} \quad (21)$$

The main result of this paper is the following.

Theorem 1. Let Assumption A hold true. Then, there exists a constant C , such that

$$E(t) \leq C \left(\frac{E^{1/p}(0)}{(t + \tau_0)^{1/p}} + \frac{F(t + \tau_0)}{t + \tau_0} E(0) \right) \quad t > 0. \quad (22)$$

Remark 2. If $p = 1$ and $\phi(t)$ satisfies

$$\phi_1 \leq \phi(t) \leq \phi_0 \quad \forall t \geq 0, \quad (23)$$

where ϕ_0 and ϕ_1 are positive constants, then it follows from (13) that there exist constants $T_0 > 0$ and $0 < C_0 < 1$ such that

$$E(T_0) \leq C_0 E(0). \quad (24)$$

Then, the decay of the energy $E(t)$ is exponential. Methods in [21, 22] are useful for Theorem 1.

2. Basic Inequality of the System

In this section, we work on Ω with two metrics at the same time: the standard dot metric $\langle \cdot, \cdot \rangle$ and the Riemannian metric $g = \langle \cdot, \cdot \rangle_g$ given by (15).

If $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g , via the Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X \rangle_g, \quad (25)$$

where X is any vector field on $(\mathbb{R}^n, g)$. The following lemma provides further relations between the two metrics; see [12], Lemma 3.

Lemma 3. Let $x = (x_1, \dots, x_n)$ be the natural coordinate system in $\mathbb{R}^n$. Let f, h be functions and let $\mathcal{H}, X$ be vector fields. Then,

(a)

$$\langle H(x), A(x) X(x) \rangle_g = \langle (x), X(x) \rangle, \quad x \in \mathbb{R}^n; \quad (26)$$

(b)

$$\nabla_g f = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}(x) f_{x_j} \right) \frac{\partial}{\partial x_i} = A(x) \nabla f, \quad x \in \mathbb{R}^n, \quad (27)$$

where ∇f is the gradient of f in the standard metric;

(c)

$$\nabla_g f(h) = \langle \nabla_g f, \nabla_g h \rangle_g = \langle \nabla f, A(x) \nabla h \rangle, \quad x \in \mathbb{R}^n, \quad (28)$$

where the matrix $A(x)$ is given in formula (1).

To prove Theorem 1, we still further need several lemmas. Define

$$E_0(t) = \frac{1}{2} \int_{\Omega} \left(u_t^2 + |\nabla_g u|_g^2 \right) dx. \quad (29)$$

Then, we have

$$E(t) = E_0(t) + \xi \int_{t-\tau(t)}^t \int_{\Gamma_1} u_t^2(x, \rho) d\Gamma d\rho. \quad (30)$$

Lemma 4. Suppose that condition (14) holds true. Let (u, w) be the solution of system (2). Then, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} E(0) - E(T) \\ \geq C_1 \int_0^T \int_{\Gamma_1} \phi(t) \left[u_t(x, t) g_1(u_t(x, t)) + u_t(x, t - \tau(t)) \right. \\ \left. \times g_1(u_t(x, t - \tau(t))) \right] d\Gamma dt, \end{aligned} \quad (31)$$

where $T \geq 0$. Assertion (31) implies that $E(t)$ is decreasing.

Proof. Differentiating (13), we obtain

$$\begin{aligned} E'(t) &= \int_{\Omega} (u_t u_{tt} + \nabla_g u \cdot \nabla u_t) dx \\ &+ \int_{\Gamma_1} \xi \phi(t) u_t(x, t) g_1(u_t(x, t)) d\Gamma \\ &- \int_{\Gamma_1} \xi \phi(t - \tau(t)) (1 - \tau'(t)) u_t \\ &\times (x, t - \tau(t)) g_1(u_t(x, t - \tau(t))) d\Gamma. \end{aligned} \quad (32)$$

Applying Green's formula and by integrating by parts with (3) and (8), we arrive at

$$\begin{aligned} E'(t) &= \int_{\Gamma_1} \phi(t) [-\mu u_t g_1(u_t) - \lambda u_t g_2(u_t(x, t - \tau(t)))] d\Gamma \\ &+ \int_{\Gamma_1} \xi \phi(t) u_t(x, t) g_1(u_t(x, t)) d\Gamma \\ &- \int_{\Gamma_1} \xi \phi(t - \tau(t)) (1 - \tau'(t)) u_t(x, t - \tau(t)) g_1 \\ &\times (u_t(x, t - \tau(t))) d\Gamma \\ &\leq \int_{\Gamma_1} \phi(t) \left(-\mu u_t g_1(u_t) + \frac{|\lambda|}{2\sqrt{1-d}} u_t^2 + \xi u_t g_1(u_t) \right) d\Gamma \\ &+ \int_{\Gamma_1} \phi(t) \left[-c_2 (1-d) \xi u_t(x, t - \tau(t)) g_1 \right. \\ &\times (u_t(x, t - \tau(t))) \\ &\left. + \frac{\sqrt{1-d} |\lambda|}{2} g_2^2(u_t(x, t - \tau(t))) \right] d\Gamma. \end{aligned} \quad (33)$$

It follows from (3), (4), (12), and (14) that

$$\begin{aligned} E'(t) &\leq -C \phi(t) \int_{\Gamma_1} [u_t(x, t) g_1(u_t(x, t)) \\ &+ u_t(x, t - \tau(t)) \\ &\times g_1(u_t(x, t - \tau(t)))] d\Gamma, \end{aligned} \quad (34)$$

where $C > 0$ satisfies

$$C = \min \left\{ \mu - \frac{|\lambda|}{2\sqrt{1-d}} - \xi, c_2 (1-d) \xi - \frac{\sqrt{1-d} |\lambda|}{2} \right\}. \quad (35)$$

Then, inequality (31) follows directly from (34) integrating from 0 to T . $\square$

3. Proofs of Theorem 1

From Proposition 2.1 in [12], we have the following identities.

Lemma 5. Suppose that $u(x, t)$ solves equation $u_{tt} + \mathcal{A}u = 0$, $(x, t) \in \Omega \times (0, +\infty)$ and that $\mathcal{H}$ is a vector field defined on $\bar{\Omega}$. Then, for $T \geq 0$,

$$\begin{aligned} & \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} \mathcal{H}(u) d\Gamma dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left(u_t^2 - |\nabla_g u|_g^2 \right) \mathcal{H} \cdot \nu d\Gamma dt \\ &= (u_t, \mathcal{H}(u))_0^T + \int_0^T \int_{\Omega} D_g \mathcal{H}(\nabla_g u, \nabla_g u) dx dt \\ &+ \frac{1}{2} \int_0^T \int_{\Omega} \left(u_t^2 - |\nabla_g u|_g^2 \right) \operatorname{div} \mathcal{H} dx dt. \end{aligned} \quad (36)$$

Moreover, assume that $P \in C^1(\bar{\Omega})$. Then,

$$\begin{aligned} & \int_0^T \int_{\Omega} \left(u_t^2 - |\nabla_g u|_g^2 \right) P dx dt \\ &= (u_t, uP)_0^T + \frac{1}{2} \int_0^T \int_{\Omega} \nabla_g P (u^2) dx dt \\ &- \int_0^T \int_{\Gamma} Pu \frac{\partial u}{\partial \nu_{\mathcal{A}}} d\Gamma dt. \end{aligned} \quad (37)$$

Lemma 6. Suppose that all assumptions in Theorem 1 hold true. Let u be the solution of the system (2). Then, there exists a positive constant C for which

$$\begin{aligned} & \int_0^T E(t) dt \leq CE_0(T + \tau_0) \\ &+ C \int_0^{T+\tau_0} \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \right) d\Gamma dt \\ &+ C \int_0^{T+\tau_0} \int_{\Gamma} \phi(t) u_t(x, t - \tau(t)) g_1 \\ &\times (u_t(x, t - \tau(t))) d\Gamma dt, \end{aligned} \quad (38)$$

where $T \geq 0$.

Proof. Let θ be a positive constant satisfying

$$\frac{1}{2} \sup_{x \in \bar{\Omega}} \operatorname{div} H < \theta < \frac{1}{2} \inf_{x \in \bar{\Omega}} \operatorname{div} H + \rho_0. \quad (39)$$

Set

$$\mathcal{H} = H, \quad P = \theta - \rho_0. \quad (40)$$

Substituting identity (37) into identity (36), we have

$$\begin{aligned} \Pi_{\Gamma} &= (u_t, H(u) + Pu)_0^T \\ &+ \int_0^T \int_{\Omega} \left(D_g H(\nabla_g u, \nabla_g u) - \rho_0 |\nabla_g u|_g^2 \right) dx dt \\ &+ \int_0^T \int_{\Omega} \left(\left(\frac{1}{2} \operatorname{div} H + \rho_0 - \theta \right) u_t^2 \right. \\ &\quad \left. + \left(\theta - \frac{1}{2} \operatorname{div} H \right) |\nabla_g u|_g^2 \right) dx dt, \end{aligned} \quad (41)$$

where

$$\begin{aligned} \Pi_{\Gamma} &= \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu_{\mathcal{A}}} (H(u) + uP) d\Gamma dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma} \left(u_t^2 - |\nabla_g u|_g^2 \right) H \cdot \nu d\Gamma dt. \end{aligned} \quad (42)$$

We decompose Π_{Γ} as

$$\Pi_{\Gamma} = \Pi_{\Gamma_1} + \Pi_{\Gamma_2}. \quad (43)$$

Since $u|_{\Gamma_2} = 0$, we obtain $\nabla_{\Gamma} u|_{\Gamma_2} = 0$; that is,

$$\nabla_g u = \frac{\partial u}{\partial \nu_{\mathcal{A}}} \frac{\nu_{\mathcal{A}}}{|\nu_{\mathcal{A}}|_g^2} \quad \text{for } x \in \Gamma_2. \quad (44)$$

Similarly, we have

$$H(u) = \left\langle H, \nabla_g u \right\rangle_g = \frac{\partial u}{\partial \nu_{\mathcal{A}}} \frac{H \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} \quad \text{for } x \in \Gamma_2. \quad (45)$$

Using formulas (44) and (45) in formula (42) on the portion Γ_2 , with (19), we obtain

$$\Pi_{\Gamma_2} = \frac{1}{2} \int_0^T \int_{\Gamma_2} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \frac{H \cdot \nu}{|\nu_{\mathcal{A}}|_g^2} d\Gamma dt \leq 0. \quad (46)$$

From (19), we have

$$\begin{aligned} \Pi_{\Gamma_1} &= \int_0^T \int_{\Gamma_1} \frac{\partial u}{\partial \nu_{\mathcal{A}}} (H(u) + uP) d\Gamma dt \\ &+ \frac{1}{2} \int_0^T \int_{\Gamma_1} \left(u_t^2 - |\nabla_g u|_g^2 \right) H \cdot \nu d\Gamma dt \\ &\leq C_{\varepsilon} \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 d\Gamma dt + \varepsilon \int_0^T \int_{\Gamma_1} \left(u^2 + |\nabla_g u|_g^2 \right) d\Gamma dt \\ &+ \int_0^T \int_{\Gamma_1} \left(Cu_t^2 - \delta |\nabla_g u|_g^2 \right) d\Gamma dt \\ &\leq C \int_0^T \int_{\Gamma_1} \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 d\Gamma dt + \varepsilon E_0(t) + C \int_0^T \int_{\Gamma_1} u_t^2 d\Gamma dt. \end{aligned} \quad (47)$$

Substituting formulas (46) and (47) into formula (41), with (39), we obtain

$$\begin{aligned} \int_0^T E_0(t) dt &\leq C(E_0(0) + E_0(T)) \\ &+ C \int_0^T \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \right) d\Gamma dt. \end{aligned} \quad (48)$$

Let $\varrho = \rho - \tau(\rho)$, and from (3), (7), (8), and (30), we have

$$\begin{aligned}
& \int_0^{T+\tau_0} E_0(t) dt \\
& + \int_0^{T+\tau_0} \int_{\Gamma} \phi(t) u_t(x, t - \tau(t)) g_1 \times (u_t(x, t - \tau(t))) d\Gamma dt \\
& \geq \int_0^{T+\tau_0} E_0(t) dt + \frac{1}{\tau_0} \int_0^T \int_t^{t+\tau_0} \int_{\Gamma_1} \phi(\rho) u_{\rho} \\
& \quad \times (x, \rho - \tau(\rho)) g_1(u_t(x, \rho - \tau(\rho))) d\Gamma d\rho dt \\
& = \int_0^{T+\tau_0} E_0(t) dt + \frac{1}{\tau_0} \int_0^T \int_{t-\tau(t)}^{t+T_1-\tau(t+\tau_0)} \int_{\Gamma_1} \phi(\rho) u_{\rho} h_0(x, \varrho) g_1 \\
& \quad \times (u_t(x, \varrho)) d\Gamma \frac{d\varrho}{1 - \tau'(\rho)} dt \\
& \geq \int_0^{T+\tau_0} E_0(t) dt + \frac{c_2}{(1 - d_0)\tau_0} \int_0^T \int_{t-\tau(t)}^t \int_{\Gamma_1} \phi(\varrho) u_{\varrho}(x, \varrho) g_1 \\
& \quad \times (u_t(x, \varrho)) d\Gamma d\varrho dt \\
& \geq \min \left\{ 1, \frac{c_2}{(1 - d_0)\tau_0} \right\} \int_0^T E(t) dt.
\end{aligned} \tag{49}$$

Since

$$\begin{aligned}
E_0(0) &= E_0(T + \tau_0) - \int_0^{T+\tau_0} \int_{\Gamma_1} u_t \frac{\partial u}{\partial \nu_{\mathcal{A}}} d\Gamma dt \\
&\leq E_0(T + \tau_0) + \frac{1}{2} \int_0^{T+\tau_0} \int_{\Gamma_1} \left(u_t^2 + \frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 d\Gamma dt,
\end{aligned} \tag{50}$$

substituting formula (49) into formula (48), we obtain

$$\begin{aligned}
& \int_0^T E(t) dt \\
& \leq CE_0(T + \tau_0) + C \int_0^{T+\tau_0} \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \right) d\Gamma dt \\
& \quad + \int_0^{T+\tau_0} \int_{\Gamma} \phi(t) u_t(x, t - \tau(t)) g_1 \\
& \quad \times (u_t(x, t - \tau(t))) d\Gamma dt.
\end{aligned} \tag{51}$$

Inequality (38) holds. $\square$

Proof of Theorem 1. Since $E(t)$ is decreasing, from (38), for sufficiently large T , we have

$$\begin{aligned}
& TE(T) \\
& \leq C \int_0^{T+\tau_0} \int_{\Gamma_1} \left(u_t^2 + \left(\frac{\partial u}{\partial \nu_{\mathcal{A}}} \right)^2 \right) d\Gamma dt \\
& \quad + C \int_0^{T+\tau_0} \int_{\Gamma} \phi(t) u_t(x, t - \tau(t)) g_1(u_t(x, t - \tau(t))) d\Gamma dt \\
& \leq C \int_0^{T+\tau_0} \int_{\Gamma_1} (\phi(t))^2 [(g_1(u_t))^2 + (g_2(u_t))^2] d\Gamma dt \\
& \quad + C \int_0^{T+\tau_0} \int_{\Gamma_1} u_t^2(x, t) d\Gamma dt \\
& \quad + C \int_0^{T+\tau_0} \int_{\Gamma} \phi(t) u_t(x, t - \tau(t)) g_1(u_t(x, t - \tau(t))) d\Gamma dt
\end{aligned}$$

$$\begin{aligned}
& \leq C \int_0^{T+\tau_0} \int_{\Gamma_1} (\phi(t))^2 [(g_1(u_t))^2 + (g_2(u_t))^2] d\Gamma dt \\
& \quad + CF(T + \tau_0) \int_0^{T+\tau_0} \int_{\Gamma_1} \phi(t) u_t^2(x, t) d\Gamma dt \\
& \quad + C \int_0^{T+\tau_0} \int_{\Gamma} \phi(t) u_t(x, t - \tau(t)) g_1(u_t(x, t - \tau(t))) d\Gamma dt,
\end{aligned} \tag{52}$$

where $F(t)$ is defined in (8). With (4)–(8) and (31), we deduce that

$$\begin{aligned}
& TE(T) \\
& \leq C \int_0^{T+\tau_0} \int_{\Gamma_1} \phi(t) (s^2 + (g_1(s))^2) d\Gamma dt \\
& \quad + CF(T + \tau_0) (E(0) - E(T + \tau_0)) \\
& \leq C \int_0^{T+\tau_0} \int_{\{x \in \Gamma_1, |u_t(x, t)| \leq 1\}} \phi(t) (u_t g_1(u_t))^{1/p} d\Gamma dt \\
& \quad + CF(T + \tau_0) (E(0) - E(T + \tau_0)) \\
& \leq C \int_0^{T+\tau_0} \int_{\Gamma_1} \phi(t) (u_t g_1(u_t))^{1/p} d\Gamma dt \\
& \quad + CF(T + \tau_0) (E(0) - E(T + \tau_0)) \\
& \leq C \text{meas}(\Gamma_1) \int_0^{T+\tau_0} \phi(t) dt \\
& \quad \cdot \left(\frac{1}{\text{meas}(\Gamma_1) \int_0^{T+\tau_0} \phi(t) dt} \right. \\
& \quad \times \left. \int_0^{T+\tau_0} \int_{\Gamma_1} \phi(t) u_t g(u_t) d\Gamma dt \right)^{1/p} \\
& \quad + CF(T + \tau_0) (E(0) - E(T + \tau_0)) \\
& \leq C(T + \tau_0)^{1-1/p} \cdot (E(0) - E(T + \tau_0))^{1/p} \\
& \quad + CF(T + \tau_0) (E(0) - E(T + \tau_0)).
\end{aligned} \tag{53}$$

Therefore,

$$E(t) \leq C \left(\frac{E^{1/p}(0)}{(T + \tau_0)^{1/p}} + \frac{F(T + \tau_0)}{T + \tau_0} E(0) \right). \tag{54}$$

Note that $E(t)$ is decreasing; estimate (22) holds. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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