The Nondifferentiable Solution for Local Fractional Tricomi Equation Arising in Fractal Transonic Flow by Local Fractional Variational Iteration Method

Ai-Min Yang,1,2 Yu-Zhu Zhang,2,3 and Xiao-Long Zhang1

1 College of Science, Hebei United University, Tangshan 063009, China
2 College of Mechanical Engineering, Yanshan University, Qinhuangdao 066004, China
3 College of Metallurgy and Energy, Hebei United University, Tangshan 063009, China

Correspondence should be addressed to Yu-Zhu Zhang; zyz@heuu.edu.cn

Received 8 May 2014; Accepted 29 May 2014; Published 19 June 2014

Academic Editor: Xiao-Jun Yang

Copyright © 2014 Ai-Min Yang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We present the nondifferentiable approximate solution for local fractional Tricomi equation arising in fractal transonic flow by local fractional variational iteration method. Some illustrative examples are shown and graphs are also given.

1. Introduction

In this paper, we study the local fractional Tricomi equation given as follows:

\[ y^\alpha \frac{\partial^{2\alpha} u(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x, y)}{\partial y^{2\alpha}} = 0, \]  

(1)

where the quantity \( u(x, y) \) is the nondifferentiable function and the operator is local fractional operator suggested as follows [1–3]:

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \Delta^\alpha \left( u(x,t) - u(x,t_0) \right) \left( t - t_0 \right)^\alpha, \]  

(2)

where

\[ \Delta^\alpha \left( u(x,t) - u(x,t_0) \right) \equiv \Gamma (1 + \alpha) \left[ u(x,t) - u(x,t_0) \right]. \]  

(3)

Local fractional derivative was applied to deal with nondifferentiable phenomena arising in mathematical physics [4–9]. When the fractal dimension \( \alpha \) is equal to 1, we obtain the following differential equation:

\[ y^\alpha \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0, \]  

(4)

which is structured by Tricomi [10]. The Tricomi equation was used to describe the transonic flow [10–22].

2. Local Fractional Calculus Theory

In this section, we present the local fractional calculus theory, which is used in the present paper.

Definition 1 (see [1, 2]). One has the function \( f(x) \in C_\alpha (a, b) \), if

\[ |f(x) - f(x_0)| < e^\delta, \quad 0 < \alpha \leq 1, \]  

(5)

is valid, where \( |x - x_0| < \delta \), for \( \varepsilon, \delta > 0 \) and \( \varepsilon \in R \).
Definition 2 (see [1, 4–9]). Let \( f(x) \) satisfy condition (5). The local fractional integral of \( f(x) \) of order \( \alpha \) in the interval \([a, b]\) is defined through

\[
a^\alpha I_b^a f(x) = \frac{1}{\Gamma (1 + \alpha)} \int_a^b f(t) (dt)^\alpha
\]

\[
= \frac{1}{\Gamma (1 + \alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j) (\Delta t_j)^\alpha,
\]

where the partitions of the interval \([a, b]\) are \((t_j, t_{j+1})\), \( j = 0, \ldots, N - 1 \), \( t_0 = a \), and \( t_N = b \) with \( \Delta t_j = t_{j+1} - t_j \) and \( \Delta t = \max(\Delta t_0, \Delta t_1, \Delta t_2, \ldots) \).

Definition 3 (see [1, 4–9]). Let \( f(x) \) satisfy condition (5). The inverse formula of (6) is given as follows:

\[
\int_0^x \left( \frac{\lambda}{\Gamma (1 + \alpha)} \right) \left( \frac{\Gamma (1 + \alpha) \Gamma (1 + (n+2)\alpha)}{\Gamma (1 + (n+1)\alpha)} \right) (ds)^\alpha = u(x).
\]

The formulas of local fractional derivative and integral used in the paper are presented as follows [1, 6, 7]:

\[
\frac{d^\alpha x}{dx^\alpha} = x^n (1 - \alpha)(x - x_0)^n (1 - \alpha) = 0,
\]

\[
D_x^\alpha g(x) = \frac{\partial f(x)}{\partial x^\alpha} = \frac{\partial f(x)}{\partial x}.
\]

From (13), we obtain the stationary condition as follows:

\[
\left. 1 - \left( \frac{\lambda}{\Gamma (1 + \alpha)} \right)^\alpha \right|_{s=x} = 0, \quad \left. \frac{\lambda}{\Gamma (1 + \alpha)} \right|_{s=x} = 0.
\]

Then, the fractal Lagrange multiplier is

\[
\frac{\lambda}{\Gamma (1 + \alpha)} = \frac{(s - x)^\alpha}{\Gamma (1 + \alpha)}.
\]

Making use of (12) and (15), we have the local fractional interaction formula as follows:

\[
\left. u_{n+1} (x) = u_n (x) + \frac{1}{\Gamma (1 + \alpha)} \right|_{s=x} \left. \left( \frac{\lambda}{\Gamma (1 + \alpha)} \right)^\alpha \right|_{s=x} = 0.
\]

Therefore, from (16), we get the solution given by

\[
u (x) = \lim_{n \to \infty} u_n (x).
\]
### 4. The Initial-Boundary Value Problems for Local Fractional Tricomi Equation

In this section, we discuss the initial-boundary value problems for local fractional Tricomi equation.

**Example 1.** Let us consider the initial-boundary value conditions for the local fractional Tricomi equation as follows:

\[
\begin{align*}
    u(0, y) &= 0, \quad (18) \\
    u(l, y) &= 0, \quad (19) \\
    u(x, 0) &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \quad (20) \\
    \partial_\alpha u(x, 0) &= 0. \quad (21)
\end{align*}
\]

From (18), (20), and (21), we have

\[
u_{n+1}(x, y) = u_n(x, y) + I_y(\alpha) \left\{ \frac{(s-y)^\alpha}{\Gamma(1+\alpha)} \times \left( \frac{s^\alpha}{\Gamma(1+\alpha)} \frac{\partial^{2\alpha} u_n(x, s)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u_n(x, s)}{\partial y^{2\alpha}} \right) \right\}, \quad (22)
\]

where the initial value is given by

\[
u_0(x, y) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (23)
\]

From (22), we present the first approximate formula as follows:

\[
u_1(x, y) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{y^{3\alpha}}{\Gamma(1+3\alpha)} \quad (24)
\]

and its graph is shown in Figure 1.

The second approximate term is

\[
u_2(x, y) = \nu_1(x, y) + I_y(\alpha) \left\{ \frac{(s-y)^\alpha}{\Gamma(1+\alpha)} \times \left( \frac{s^\alpha}{\Gamma(1+\alpha)} \frac{\partial^{2\alpha} u_1(x, s)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u_1(x, s)}{\partial y^{2\alpha}} \right) \right\} \quad (25)
\]

\[
u_2(x, y) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{3y^{3\alpha}}{\Gamma(1+3\alpha)}
\]

and its graph is given in Figure 2.
The third approximation is presented as follows:

\[ u_3(x, y) = u_2(x, y) + 0 I_y^{(\alpha)} \]  
\[ \times \left\{ \frac{(s-y)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{s^\alpha}{\Gamma(1+\alpha)} \frac{\partial^2 u_2(x,s)}{\partial x^{2\alpha}} + \frac{\partial^2 u_2(x,s)}{\partial y^{2\alpha}} \right\} + \frac{3s^\alpha}{\Gamma(1+\alpha)} \right\} \]  
\[ = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{3y^{3\alpha}}{\Gamma(1+3\alpha)} \]  
and its graph is illustrated in Figure 3.

The fourth approximation reads as follows:

\[ u_4(x, y) = u_3(x, y) + 0 I_y^{(\alpha)} \]  
\[ \times \left\{ \frac{(s-y)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{s^\alpha}{\Gamma(1+\alpha)} \frac{\partial^2 u_3(x,s)}{\partial x^{2\alpha}} + \frac{\partial^2 u_3(x,s)}{\partial y^{2\alpha}} \right\} + \frac{7s^\alpha}{\Gamma(1+\alpha)} \right\} \]  
\[ = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{7y^{3\alpha}}{\Gamma(1+3\alpha)} \]  
(27)

and its graph is presented in Figure 4.

The fifth approximation is as follows:

\[ u_5(x, y) = u_4(x, y) + 0 I_y^{(\alpha)} \]  
\[ \times \left\{ \frac{(s-y)^\alpha}{\Gamma(1+\alpha)} \left\{ \frac{s^\alpha}{\Gamma(1+\alpha)} \frac{\partial^2 u_4(x,s)}{\partial x^{2\alpha}} + \frac{\partial^2 u_4(x,s)}{\partial y^{2\alpha}} \right\} + \frac{15s^\alpha}{\Gamma(1+\alpha)} \right\} \]  
\[ = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{15y^{3\alpha}}{\Gamma(1+3\alpha)} \]  
(28)

and its graph is shown in Figure 5.

After successive iterative processes, we obtain the nondifferentiable series solution as follows:

\[ u(x, y) = \lim_{i \to \infty} u_i(x, y) \]  
\[ = \lim_{n \to \infty} \left\{ \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2^n - 1}{\Gamma(1+3\alpha)} \right\}, \]  
(29)
which is the local fractional divergent series. Therefore, we can obtain the approximate solution.

Example 2. The initial-boundary value conditions for the local fractional Tricomi equation are presented as follows:

\[
\begin{align*}
  u(0, y) &= 0, \\
  u(l, y) &= 0, \\
  u(x, 0) &= x^{\alpha} \frac{\Gamma(1 + \alpha)}{\Gamma(1 + \alpha)}, \\
  \frac{\partial x^{\alpha} u(x, 0)}{\partial x^{\alpha}} &= \frac{x^{\alpha}}{\Gamma(1 + \alpha)}. 
\end{align*}
\]

In view of (16), (32), and (33), we obtain the local fractional iterative formula as follows:

\[
\begin{align*}
  u_{n+1}(x, y) &= u_n(x, y) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \\
  &= u_0(x, y) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \\
  &= u_0(x, y) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \left( \frac{(s - y)^{\alpha}}{\Gamma(1 + \alpha)} \right) \\
  &= \left( \frac{s^{\alpha}}{\Gamma(1 + \alpha)} \frac{\partial^{2\alpha} u_0(x, s)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u_0(x, s)}{\partial y^{2\alpha}} \right) \\
  &= \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)}.
\end{align*}
\]

From (34) and (35), we give the first approximation as follows:

\[
\begin{align*}
  u_1(x, y) &= u_0(x, y) + \frac{\partial^{\alpha}}{\partial y^{\alpha}} \\
  &= \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)} \\
  &= \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)}.
\end{align*}
\]

Hence, from (36), we arrive at the following results:

\[
\begin{align*}
  u_0(x, y) &= u_1(x, y) = u_2(x, y) = \cdots = u_n(x, y). 
\end{align*}
\]

Therefore, we get the exact solution with nondifferential term as follows:

\[
\begin{align*}
  u(x, y) &= \lim_{n \to \infty} u_n(x, y) \\
  &= \lim_{n \to \infty} \left\{ \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)} \right\} \\
  &= \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)} \\
  &= \frac{x^{\alpha}}{\Gamma(1 + \alpha)} + \frac{x^{\alpha}}{\Gamma(1 + \alpha)} \frac{y^{\alpha}}{\Gamma(1 + \alpha)}.
\end{align*}
\]

and its graph is shown in Figure 6.

5. Conclusions

The initial-boundary value problems for local fractional Tricomi equation arising in fractal transonic flow based upon the local fractional derivatives are discussed. The solutions with nondifferentiable terms are obtained by using the local fractional variational iteration method and their graphs are also given to show the implement of the present method.
Conflict of Interests

The authors declare that they have no competing interests in this paper.

Acknowledgments

This work was supported by National Scientific and Technological Support Projects (no. 2012BAE09B00), the National Natural Science Foundation of China (no. 51274270), and the National Natural Science Foundation of Hebei Province (no. E2013209215).

References

Submit your manuscripts at http://www.hindawi.com