Review Article

A New Kind of Shift Operators for Infinite Circular and Spherical Wells

Guo-Hua Sun,¹ K. D. Launey,² T. Dytrych,² Shi-Hai Dong,^{2,3} and J. P. Draayer²

¹ Centro Universitario Valle de Chalco, Universidad Autónoma del Estado de México,
 56615 Valle de Chalco Solidaridad, MEX, Mexico

² Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA

³ Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional,
 Edificio 9, UPALM, 07738 Mexico, DF, Mexico

Correspondence should be addressed to Shi-Hai Dong; dongsh2@yahoo.com

Received 2 April 2014; Accepted 4 May 2014; Published 22 May 2014

Academic Editor: Xiao-Yan Gu

Copyright © 2014 Guo-Hua Sun et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A new kind of shift operators for infinite circular and spherical wells is identified. These shift operators depend on all spatial variables of quantum systems and connect some eigenstates of confined systems of different radii R sharing energy levels with a common eigenvalue. In circular well, the momentum operators $P_{\pm} = P_x \pm iP_y$ play the role of shift operators. The P_x and P_y operators, the third projection of the orbital angular momentum operator L_z , and the Hamiltonian H form a complete set of commuting operators with the $SO(2)$ symmetry. In spherical well, the shift operators establish a novel relation between $\psi_{lm}(\mathbf{r})$ and $\psi_{(l\pm 1)(m\pm 1)}(\mathbf{r})$.

1. Introduction

It is well-known that algebraic methods have become the subject of interest in different fields. Systems with a dynamical symmetry can be treated algebraically [1–7]. Using the factorization method, we have established the ladder operators for some solvable potentials like the Morse potential, the Pöschl-Teller potential, the square-well potential, and others [8]. Contrary to traditional approaches [9], where an auxiliary variable was introduced, the ladder operators constructed in the framework of our approach are expressed only in terms of spatial coordinates. It should be pointed out that the ladder operators for those quantum systems depend on only one variable, that is, one-dimensional problem [8].

Motivated by the recently proposed factorization method [8], we generalize it to construct ladder operators that depend on all spatial variables. For this purpose, we derive shift operators for the circular [10–13] and spherical wells [14] directly from the normalized wave functions [15, 16]. The difference between the ladder and shift operators is that the former correspond to the same potential, but the

latter correspond to the same energy of different potential groups. Therefore, the latter method is named potential group approach. We start with a brief review, with the purpose of introducing the general formalism, and then we present the new shift operators that provide a novel insight into the problem. In particular, we include specific illustrations to show the generation of a series of potentials with different radii and the same energy.

This work is organized as follows. In Section 2 we review the exact solutions of the two quantum systems, namely, in the circular- and spherical-well cases. In Section 3 we present general formalism to establish the shift operators and illustrate how to generate a group of different potentials with same energy levels. Some concluding remarks are given in Section 4.

2. Exact Solutions

2.1. Circular Well. The circular and spherical wells are defined as $V(\mathbf{r}) = V(r) = 0$ for $r < R$ and $V(r) = \infty$ for $r \geq R$, where $\mathbf{r}$ denotes (r, ϕ) and (r, θ, ϕ) for circular and

TABLE 1: Zeros of the Bessel functions $J_m(x)$.

n th zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

spherical wells, respectively. The Schrödinger equation in relevant polar coordinates can be written as

$$-\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) \psi(r, \phi) = E \psi(r, \phi). \quad (1)$$

By taking $\psi(r, \phi) = R(r)e^{im\phi}/\sqrt{2\pi}$ ($m = 0, \pm 1, \pm 2, \dots$) and substituting it into (1), we have

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} - \frac{m^2}{r^2} R(r) = -k^2 R(r), \quad k = \sqrt{\frac{2\mu E}{\hbar^2}}. \quad (2)$$

By defining $z = kr$, it becomes the Bessel differential equation

$$\frac{d^2 R(z)}{dz^2} + \frac{1}{z} \frac{dR(z)}{dz} + \left(1 - \frac{m^2}{z^2} \right) R(z) = 0. \quad (3)$$

It has two linearly independent solutions for each value of $|m|$. One is the regular $J_m(z)$ and the other is the singular $Y_m(z)$. Only the well-behaved $J_m(z)$ are physically acceptable, with the eigenvalues determined by $J_m(kR) = 0$. That is, kR is the n th root α_{mn} of the Bessel function $J_m(x)$ (see Table 1). In terms of this condition, the eigenvalues are then given by

$$E_{mn} = \frac{\hbar^2 k_{mn}^2}{2\mu} = \frac{\hbar^2}{2\mu R^2} \alpha_{mn}^2. \quad (4)$$

The corresponding eigenfunctions for the n th root are given by

$$\psi_m(r, \phi) = \mathcal{N}_m J_m(kr) e^{im\phi}, \quad \mathcal{N}_m = \frac{1}{R\pi^{1/2} |J_{m+1}(\alpha_{mn})|}. \quad (5)$$

Obviously, the complete spectrum for the full circular well corresponds to one set of $m = 0$ solutions but is doubly degenerate for each value of $|m| > 0$, since $\pm m$ give the same contributions. This corresponds physically to the equivalence of clockwise versus counterclockwise motions.

2.2. Infinite Spherical Well. When $r \leq R$, the Schrödinger equation in spherical coordinates is given by

$$-\frac{\hbar^2}{2\mu} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \psi(r, \theta, \phi) = E \psi(r, \theta, \phi). \quad (6)$$

For symmetrically central force fields, we can always express the wave functions as $\psi(r, \theta, \phi) = R(r)Y_{lm}(\theta, \phi) = e^{im\phi} R(r)\Theta(\theta)/\sqrt{2\pi}$ and substitute this into (6). The polar angle dependence θ leads to the associated Legendre equation

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta(\theta) = 0, \quad (7)$$

which is satisfied by the normalized associated Legendre functions $P_l^m(\cos \theta)$:

$$\Theta_{lm}(\theta) = \mathcal{P}_l^m(\cos \theta) = \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta), \quad (8)$$

$m \in [-l, l].$

Thus, we may define the spherical harmonics as

$$Y_l^m(\theta, \phi) \equiv (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}, \quad (9)$$

where the factor $(-1)^m$ is the so-called Condon-Shortley phase.

By separating the spherical harmonics, we obtain the radial equation as

$$-\frac{\hbar^2}{2\mu} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) R(r) = ER(r). \quad (10)$$

By defining $z = kr$, where k is given in (2), (10) is rearranged as spherical Bessel differential equation

$$\frac{d^2 R(z)}{dz^2} + \frac{2}{z} \frac{dR(z)}{dz} + \left(1 - \frac{l(l+1)}{z^2} \right) R(z) = 0 \quad (11)$$

with two linearly independent solutions $j_l(z)$ and $n_l(z)$. Only the well-behaved $j_l(z)$ are acceptable in physics. The eigenvalues are determined by the boundary condition $j_l(kR) = 0$ (see Table 2) as

$$E_{ln} = \frac{\hbar^2 k_{ln}^2}{2\mu} = \frac{\hbar^2}{2\mu R^2} \alpha_{ln}^2, \quad (12)$$

where kR is the n th root α_{ln} of the spherical Bessel function $j_l(x)$. The eigenfunctions of the quantum system for the n th root are given by

$$\psi_{lm}(r, \theta, \phi) = N_l j_l(kr) Y_{lm}(\theta, \phi), \quad N_l = \frac{\sqrt{2}}{R^{3/2} |j_{l+1}(\alpha_{ln})|}. \quad (13)$$

TABLE 2: Zeros of the spherical Bessel functions $j_l(x)$.

n th zero	$j_0(x)$	$j_1(x)$	$j_2(x)$	$j_3(x)$	$j_4(x)$	$j_5(x)$
1	3.1416	4.4934	5.7635	6.9879	8.1826	9.3558
2	6.2832	7.7253	9.0950	10.4171	11.7049	12.9665
3	9.4248	10.9041	12.3229	13.6980	15.0397	16.3547
4	12.5664	14.0662	15.5146	16.9236	18.3013	19.6532
5	15.7080	17.2208	18.6890	20.1218	21.5254	22.9046

3. Shift Operators

3.1. Circular Well. We now address the problem of finding the creation and annihilation operators for the wave functions (5) by the factorization method. As shown in [8], the shift operators can be constructed directly from the wave functions using only one variable, while in the present study they depend on all variables, r and φ .

For this purpose, we start by acting with the following operator on the wave functions of (5):

$$\begin{aligned}
 & \left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y} \right) \psi_m(r, \varphi) \\
 &= \mathcal{N}_m e^{\pm i\varphi} \left[\frac{\partial J_m(kr)}{\partial r} \mp \frac{m}{r} J_m(kr) \right] e^{im\varphi} \\
 &= \mp \frac{\mathcal{N}_m}{\mathcal{N}_{m\pm 1}} \mathcal{N}_{m\pm 1} k J_{m\pm 1}(kr) e^{i(m\pm 1)\varphi} \\
 &= \mp \mathcal{E}^{\pm} k \psi_{m\pm 1}(r, \varphi), \quad \mathcal{E}^{\pm} = \frac{\mathcal{N}_m}{\mathcal{N}_{m\pm 1}},
 \end{aligned} \tag{14}$$

where we have used the recurrence relations among the Bessel functions:

$$\frac{dJ_m(z)}{dz} = \pm J_{m\mp 1}(z) \mp \frac{m}{z} J_m(z), \tag{15}$$

as well as the coordinate transformations $x = r \cos \varphi$ and $y = r \sin \varphi$.

Let us calculate matrix elements for some related functions. It follows from (14) that

$$\begin{aligned}
 \frac{\partial}{\partial x} \psi_m &= \frac{k}{2} (\mathcal{E}^- \psi_{m-1} - \mathcal{E}^+ \psi_{m+1}), \\
 2i \frac{\partial}{\partial y} \psi_m &= -k (\mathcal{E}^+ \psi_{m+1} + \mathcal{E}^- \psi_{m-1}),
 \end{aligned} \tag{16}$$

from which we have

$$\begin{aligned}
 2 \left\langle \psi_{n'} \left| \frac{\partial}{\partial x} \right| \psi_m \right\rangle &= k (\mathcal{E}^- \delta_{n', m-1} - \mathcal{E}^+ \delta_{n', m+1}), \\
 2i \left\langle \psi_{n'} \left| \frac{\partial}{\partial y} \right| \psi_m \right\rangle &= -k (\mathcal{E}^+ \delta_{n', m+1} + \mathcal{E}^- \delta_{n', m-1}).
 \end{aligned} \tag{17}$$

Let us study this problem from another point of view. From the definitions of the momentum operator $\mathbf{P} = -i\hbar \nabla$

and orbital angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{P}$, we can define the following operators:

$$\begin{aligned}
 P_{\pm} &= P_x \pm iP_y = -i\hbar e^{\pm i\varphi} \left[\frac{\partial}{\partial r} \pm \frac{(i/r) \partial}{\partial \varphi} \right], \\
 L_z &= xP_y - yP_x = -i\hbar \frac{\partial}{\partial \varphi},
 \end{aligned} \tag{18}$$

from which we have the following commutation relations:

$$\begin{aligned}
 [L_z, P_x] &= i\hbar P_y, & [L_z, P_y] &= -i\hbar P_x, \\
 [L_z, P_{\pm}] &= \pm \hbar P_{\pm}, & [P_+, P_-] &= 0, \\
 P_+ P_- &= P_x^2 + P_y^2 = P_- P_+.
 \end{aligned} \tag{19}$$

The Hamiltonian is given by

$$H = \frac{1}{2\mu} (P_x^2 + P_y^2) = \frac{1}{2\mu} P_+ P_- = \frac{1}{2\mu} P_- P_+, \tag{20}$$

from which we have

$$[L_z, H] = [P_x, H] = [P_y, H] = 0. \tag{21}$$

This means that the linear momentum operators $P_{x,y}$, L_z and the Hamiltonian H form a complete set of commuting operators with the SO(2) symmetry. The $P_{\pm}$ play the role of the shift operators as given by $P_{\pm} \psi_m(r, \varphi) = \mp i\hbar \mathcal{E}^{\pm} k \psi_{m\pm 1}(r, \varphi)$.

If we define $\varepsilon_{mn} = 2\mu E_{mn}/\hbar^2 = (\alpha_{mn}/R_{mn})^2$, then for fixed energy and an initial radius R_{m1} (chosen without loss of generality) we have

$$\begin{aligned}
 R_{m2} &= \frac{\alpha_{m2}}{\alpha_{m1}} R_{m1}, & R_{m3} &= \frac{\alpha_{m3}}{\alpha_{m1}} R_{m1}, \\
 R_{m4} &= \frac{\alpha_{m4}}{\alpha_{m1}} R_{m1}, & R_{m5} &= \frac{\alpha_{m5}}{\alpha_{m1}} R_{m1}, \dots
 \end{aligned} \tag{22}$$

The different radii $R_{m1, m2, m3, \dots}$ correspond to different circular wells for the same energy level. This group of circular disks with different radii R_{mm} is thus constructed for a given m . For example, when $m = 0$, we have $R_{02} = 2.29545 R_{01}$, $R_{03} = 3.59851 R_{01}$, $R_{04} = 4.90331 R_{01}$, $R_{05} = 6.20879 R_{01}$, and so forth. This has been illustrated in Figure 1. For different m , we can construct many different groups of circular rings.

3.2. Spherical Well. Let us discuss the infinite spherical well case. Likewise, we construct the shift operators by considering all variables, r , θ , and ϕ , in $\psi_{lm}(r, \theta, \phi)$. In order to

FIGURE 1: Different-radius circular wells constructed by shift operators for $m = 0$ case.

construct the shift operators, we proceed in two steps. First, we obtain the shift operators for the radial part. Second, we find the shift operators for the spherical harmonics $Y_{lm}(\theta, \phi)$. The combination of the two results provides the final solution.

Let us perform the first step. Based on the recurrence relations among the spherical Bessel functions [17]

$$\frac{dj_l(z)}{dz} = \begin{cases} j_{l-1}(z) - \frac{l+1}{z} j_l(z), \\ -j_{l+1}(z) + \frac{l}{z} j_l(z), \end{cases} \quad (23)$$

we have

$$A^+ = -\left(\frac{d}{dr} - \frac{l}{r}\right), \quad A^- = \frac{d}{dr} + \frac{l+1}{r}, \quad (24)$$

with the properties

$$\begin{aligned} A^+ \psi_{lm}(r, \theta, \phi) &= N_l k j_{l+1}(kr) Y_{lm}(\theta, \phi), \\ A^- \psi_{lm}(r, \theta, \phi) &= N_l k j_{l-1}(kr) Y_{lm}(\theta, \phi). \end{aligned} \quad (25)$$

Now, let us derive the shift operators for the spherical harmonics $Y_{lm}(\theta, \phi)$. For fixed orbital quantum number l , the ladder operators for m are well-known [14]:

$$L_{\pm} = L_x \pm iL_y = \pm e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \phi} \right). \quad (26)$$

Acting with them on $Y_{lm}(\theta, \phi)$ leads to

$$\begin{aligned} Y_{l(m\pm 1)}(\theta, \phi) &= \frac{e^{\mp i\phi}}{\sqrt{(l\pm m)(l\mp m+1)}} \\ &\times \left(\mp \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right) Y_{lm}(\theta, \phi). \end{aligned} \quad (27)$$

FIGURE 2: Connecting paths between the states (l, m) and $(l \pm 1, m \pm 1)$.

Now, our key issue is to find the shift operators for l when m is fixed. According to (7), the θ equation can be modified as

$$\begin{aligned} -\sin^2 \theta \frac{d^2 \Theta(\theta)}{d\theta^2} - \sin \theta \cos \theta \frac{d\Theta(\theta)}{d\theta} - l(l+1) \sin^2 \theta \Theta(\theta) \\ = -m^2 \Theta(\theta). \end{aligned} \quad (28)$$

Referring to the technique in [4], we obtain the following ladder operators for fixed m :

$$\begin{aligned} \Theta_{(l+1)m}(\theta) &= -\frac{c_{(l+1)m}}{c_{lm}} \frac{1}{\sqrt{(l+1+m)(l+1-m)}} \\ &\times \left[-\sin \theta \frac{d}{d\theta} - (l+1) \cos \theta \right] \Theta_{lm}(\theta), \\ \Theta_{(l-1)m}(\theta) &= -\frac{c_{(l-1)m}}{c_{lm}} \frac{1}{\sqrt{(l+m)(l-m)}} \\ &\times \left[\sin \theta \frac{d}{d\theta} - l \cos \theta \right] \Theta_{lm}(\theta), \end{aligned} \quad (29)$$

where c_{lm} is the ratio coefficients for $\Theta_{lm}(\theta)$. The introduction of “-” agrees with the standard phase conventions for spherical harmonics.

To calculate c_{lm} , we can calculate $\Theta_{(l+1)(m+1)}(\theta)$ by stepping from (l, m) to $(l+1, m+1)$ along two different paths, 1 and 2 (see Figure 2), in the (l, m) parameter space. Path 1 steps first from (l, m) to $(l, m+1)$ via the m step-up operator (27)

and then steps from $(l, m+1)$ to $(l+1, m+1)$ with the l step-up operator (29). After some algebraic manipulations, one has

$$\begin{aligned} \Theta_{(l+1)(m+1)}(\theta) &= \frac{c_{(l+1)(m+1)}}{c_{l(m+1)}} \frac{1}{\sqrt{(l+m+1)(l+m+2)}} \\ &\times \left[\cos \theta \frac{d}{d\theta} - \frac{m}{\sin \theta} - (l+1) \sin \theta \right] \Theta_{lm}(\theta). \end{aligned} \quad (30)$$

In the calculation, we used the following identity together with the θ equation (28):

$$\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} = \frac{m^2}{\sin^2 \theta} - l(l+1). \quad (31)$$

In a similar way, following path 2, that is, we step first from (l, m) to $(l+1, m)$ via the l step-up operator (29) and then step from $(l+1, m)$ to $(l+1, m+1)$ with the m step-up operator (27). As a result, one has

$$\begin{aligned} \Theta_{(l+1)(m+1)}(\theta) &= \frac{c_{(l+1)m}}{c_{lm}} \frac{1}{\sqrt{(l+m+1)(l+m+2)}} \\ &\times \left[\cos \theta \frac{d}{d\theta} - \frac{m}{\sin \theta} - (l+1) \sin \theta \right] \Theta_{lm}(\theta). \end{aligned} \quad (32)$$

Comparing (30) and (32), we have $c_{(l+1)m}/c_{lm} = c_{(l+1)(m+1)}/c_{l(m+1)}$, which implies that the ratio is independent of the quantum number m . This is because m is fixed but l changes in (28). Consequently, we can calculate it by setting $m = l$. Substituting this into (30) or (32), we have

$$\begin{aligned} \Theta_{(l+1)(l+1)}(\theta) &= \mathcal{N}_{l+1} \sin^{(l+1)} \theta \\ &= \frac{c_{(l+1)l}}{c_{ll}} \frac{1}{\sqrt{(2l+1)(2l+2)}} \\ &\times \left[\cos \theta \frac{d}{d\theta} - \frac{l}{\sin \theta} - (l+1) \sin \theta \right] \mathcal{N}_l \sin^l \theta \\ &= -\frac{c_{(l+1)l}}{c_{ll}} \sqrt{\frac{2l+1}{2l+2}} \mathcal{N}_l \sin^{(l+1)} \theta. \end{aligned} \quad (33)$$

According to the normalization factor $\mathcal{N}_l$ given in [14]

$$\mathcal{N}_l = (-1)^l \frac{\sqrt{(2l)!}}{2^l l!} \sqrt{\frac{2l+1}{4\pi}}, \quad (34)$$

we have $\mathcal{N}_{l+1}/\mathcal{N}_l = -\sqrt{(2l+3)/(2l+2)}$. Therefore, in terms of (33) we obtain $c_{(l+1)l}/c_{ll} = \sqrt{(2l+3)/(2l+1)} = c_{(l+1)m}/c_{lm}$.

The corresponding relations (29) become

$$\begin{aligned} \Theta_{(l+1)m}(\theta) &= \sqrt{\frac{(2l+3)/(2l+1)}{(l+1+m)(l+1-m)}} \\ &\times \left[\sin \theta \frac{d}{d\theta} + (l+1) \cos \theta \right] \Theta_{lm}(\theta), \\ \Theta_{(l-1)m}(\theta) &= \sqrt{\frac{(2l-1)/(2l+1)}{(l+m)(l-m)}} \\ &\times \left[-\sin \theta \frac{d}{d\theta} + l \cos \theta \right] \Theta_{lm}(\theta). \end{aligned} \quad (35)$$

Thus, relation (32) can be written as

$$\Theta_{(l+1)(m+1)}(\theta) = B^+ \Theta_{lm}(\theta), \quad (36)$$

where

$$\begin{aligned} B^+ &= \sqrt{\frac{(2l+3)/(2l+1)}{(l+1+m)(l+2+m)}} \\ &\times \left[\cos \theta \frac{d}{d\theta} - \frac{m}{\sin \theta} - (l+1) \sin \theta \right]. \end{aligned} \quad (37)$$

Similarly, let us look for the relation between $\Theta_{(l-1)(m-1)}(\theta)$ and $\Theta_{lm}(\theta)$. To this end, we can calculate $\Theta_{(l-1)(m-1)}(\theta)$ by stepping from (l, m) to $(l-1, m-1)$ along two different paths, 3 and 4, in the (l, m) parameter space (see Figure 1).

Following path 3, we step first from (l, m) to $(l-1, m)$ via the l step-down operator (29) and then step from $(l-1, m)$ to $(l-1, m-1)$ with the m step-down operator (27). After some algebraic manipulations, one has

$$\begin{aligned} \Theta_{(l-1)(m-1)}(\theta) &= \frac{c_{(l-1)m}}{c_{lm}} \frac{1}{(l-m) \sqrt{(l+m)(l+m-1)}} \\ &\times \left[\sin \theta \left(\frac{d^2}{d\theta^2} + \cot \theta \frac{d}{d\theta} \right) - (l-m) \cos \theta \frac{d}{d\theta} \right. \\ &\quad \left. - ml \frac{\cos^2 \theta}{\sin \theta} + l \sin \theta \right] \\ &= -\frac{c_{(l-1)m}}{c_{lm}} \frac{1}{\sqrt{(l+m)(l+m-1)}} \\ &\times \left[\cos \theta \frac{d}{d\theta} + \frac{m}{\sin \theta} + l \sin \theta \right] \Theta_{lm}(\theta). \end{aligned} \quad (38)$$

Following path 4, we step first from (l, m) to $(l, m-1)$ via the m step-down operator (27) and then step from $(l, m-1)$

to $(l-1, m-1)$ with the l step-down operator (29). Similarly, one has

$$\begin{aligned}
 \Theta_{(l-1)(m-1)}(\theta) &= \frac{c_{(l-1)(m-1)}}{c_{l(m-1)}} \frac{1}{(l-m+1) \sqrt{(l+m)(l+m-1)}} \\
 &\times \left[\sin \theta \frac{d^2}{d\theta^2} - \frac{m}{\sin \theta} + m \cos \theta \frac{d}{d\theta} \right. \\
 &\quad \left. - l \cos \theta \frac{d}{d\theta} - ml \frac{\cos^2 \theta}{\sin \theta} \right] \\
 &= -\frac{c_{(l-1)(m-1)}}{c_{l(m-1)}} \frac{1}{\sqrt{(l+m)(l+m-1)}} \\
 &\times \left[\cos \theta \frac{d}{d\theta} + \frac{m}{\sin \theta} + l \sin \theta \right] \Theta_{lm}(\theta). \quad (39)
 \end{aligned}$$

Comparing (38) and (39), we have $c_{(l-1)(m-1)}/c_{l(m-1)} = c_{(l-1)m}/c_{lm}$. Likewise, setting $m = l$ and substituting it into (38) yield

$$\begin{aligned}
 \Theta_{(l-1)(l-1)}(\theta) &= \mathcal{N}_{l-1} \sin^{(l-1)} \theta \\
 &= -\frac{c_{(l-1)l}}{c_{ll}} \frac{1}{\sqrt{(2l)(2l-1)}} \\
 &\times \left[\cos \theta \frac{d}{d\theta} + \frac{l}{\sin \theta} + l \sin \theta \right] \mathcal{N}_l \sin^l \theta \quad (40) \\
 &= -\frac{c_{(l-1)l}}{c_{ll}} \sqrt{\frac{2l}{2l-1}} \mathcal{N}_l \sin^{(l-1)} \theta.
 \end{aligned}$$

The relation $\mathcal{N}_{l-1}/\mathcal{N}_l = -\sqrt{2l/(2l+1)}$ yields $c_{ll}/c_{(l-1)l} = \sqrt{(2l+1)/(2l-1)} = c_{lm}/c_{(l-1)m}$. Thus, based on this result we have

$$\Theta_{(l-1)(m-1)}(\theta) = B^- \Theta_{lm}(\theta), \quad (41)$$

with

$$B^- = \sqrt{\frac{(2l-1)/(2l+1)}{(l+m)(l+m-1)}} \left[-\cos \theta \frac{d}{d\theta} - \frac{m}{\sin \theta} - l \sin \theta \right]. \quad (42)$$

Using the operators $B^\pm$ and $A^\pm$, we can express the shift operators as follows:

$$B^\pm A^\pm \psi_{lm}(r, \theta, \phi) = \mathcal{E}_\pm k \psi_{(l\pm 1)(m\pm 1)}(r, \theta, \phi), \quad (43)$$

with $\mathcal{E}_\pm = N_l/N_{l\pm 1}$. Consequently, the relation between $\psi_{lm}(r, \theta, \phi)$ and $\psi_{(l\pm 1)(m\pm 1)}(r, \theta, \phi)$ is established for the same k but different-radius spherical wells.

As treated in the circular well, define $\varepsilon_{ln} = 2\mu E_{ln}/\hbar^2 = (\alpha_{ln}/R_{ln})^2$. For fixed energy and an initial radius R_{l1} , we have

$$\begin{aligned}
 R_{l2} &= \frac{\alpha_{l2}}{\alpha_{l1}} R_{l1}, & R_{l3} &= \frac{\alpha_{l3}}{\alpha_{l1}} R_{l1}, \\
 R_{l4} &= \frac{\alpha_{l4}}{\alpha_{l1}} R_{l1}, & R_{l5} &= \frac{\alpha_{l5}}{\alpha_{l1}} R_{l1}, \dots \quad (44)
 \end{aligned}$$

FIGURE 3: Different-radius spherical wells constructed by shift operators for $l = 0$ case illustrated in cross section.

The different radii $R_{l1, l2, l3, \dots}$ correspond to different-radius spherical wells with the same energy levels. This group of spherical wells is thus constructed via the potential group approach for fixed l . For example, when $l = 0$, we have $R_{02} = 2R_{01}$, $R_{03} = 3R_{01}$, $R_{04} = 4R_{01}$, $R_{05} = 5R_{01}$, and so forth. This is a special case. The reason is that for $l = 0$, the zeros of the spherical Bessel function $j_{0n}(x)$ are equal to $n\pi$ as illustrated in Figure 3. For different l , we can construct many different groups of spherical wells.

4. Conclusions

We have provided a brief review and a new insight on constructing the shift operators for infinite circular and spherical wells using the potential group approach. The name of this method relates to the fact that the shift operators connect those quantum systems with different potentials but with the same energy spectrum. In particular, we have constructed shift operators that depend on all spatial variables. By considering two special cases $m = 0$ (for circular wells) and $l = 0$ (for spherical wells), we have illustrated the construction of the different-radius wells for the same energy level. For different m and l , we have obtained a series of shift operators with a given quantum number m or l . Before ending this work, we give a useful remark on (22) and (44). That is, these two equations define the corresponding eigenstates and energy eigenvalue for the potentials with the successive radii but do not hold for the entire spectrum.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the referees for positive and invaluable suggestions which have improved the paper greatly. This work is partially supported by SIP-20140772-IPN, Mexico, as well as in part by US National Science Foundation (Grant no. OCI-0904874), and US Department of Energy (Grant no. DE-SC0005248). This work is dedicated to the 75th anniversary of Professor Eugenio Ley-koo.

References

- [1] A. Perelomov, *Generalized Coherent States and their Applications*, Springer, New York, NY, USA, 1986.
- [2] R. G. Wybourne, *Classical Groups for Physicists*, Wiley, New York, NY, USA, 1974.
- [3] E. Schrödinger, "A method of determining quantum mechanical eigenvalues and eigenfunctions," *Proceedings of the Royal Irish Academy*, vol. 46, no. 9, p. 183, 1940.
- [4] L. Infeld and T. E. Hull, "The factorization method," *Reviews of Modern Physics*, vol. 23, pp. 21–68, 1951.
- [5] W. Miller, *Lie Theory and Special Functions*, Academic Press, New York, NY, USA, 1968.
- [6] M. Humi, "Factorisation of separable partial differential equations," *Journal of Physics A: Mathematical and General*, vol. 20, no. 14, pp. 4577–4585, 1987.
- [7] M. Humi, "Dilatations and factorizable equations," *Journal of Physics A: Mathematical and General*, vol. 38, no. 28, article 6351, 2005.
- [8] S. H. Dong, *Factorization Method in Quantum Mechanics*, vol. 150, Springer, Amsterdam, The Netherlands, 2007.
- [9] M. Berrondo and A. Palma, "The algebraic approach to the Morse oscillator," *Journal of Physics A: Mathematical and General*, vol. 13, no. 3, pp. 773–780, 1980.
- [10] R. W. Robinett, "Quantum mechanics of the two-dimensional circular billiard plus baffle system and half-integral angular momentum," *European Journal of Physics*, vol. 24, no. 3, article 231, 2003.
- [11] R. W. Robinett, "Quantum wave packet revivals," *Physics Reports*, vol. 392, no. 1-2, pp. 1–119, 2004.
- [12] K. Volke-Sepulveda and E. Ley-Koo, "General construction and connections of vector propagation invariant optical fields: TE and TM modes and polarization states," *Journal of Optics A: Pure and Applied Optics*, vol. 8, no. 10, article 867, 2006.
- [13] R. L. Liboff, *Introductory Quantum Mechanics*, Addison-Wesley, Reading, Mass, USA, 1980.
- [14] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, Academic Press, New York, NY, USA, 5th edition, 2001.
- [15] G. H. Sun and S. H. Dong, "New type operators for circular well potential in two dimensions," *Physics Letters A*, vol. 374, no. 40, pp. 4112–4114, 2010.
- [16] G. H. Sun and S. H. Dong, "New type shift operators for three-dimensional infinite well potential," *Modern Physics Letters A*, vol. 26, no. 5, pp. 351–358, 2011.
- [17] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, NY, USA, 5th edition, 1994.