Research Article

$(2 + 1)$ -Dimensional mKdV Hierarchy and Chirp Effect of Rossby Solitary Waves

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By constructing a kind of generalized Lie algebra, based on generalized Tu scheme, a new $(2 + 1)$ -dimensional mKdV hierarchy is derived which popularizes the results of $(1 + 1)$ -dimensional integrable system. Furthermore, the $(2 + 1)$ -dimensional mKdV equation can be applied to describe the propagation of the Rossby solitary waves in the plane of ocean and atmosphere, which is different from the $(1 + 1)$ -dimensional mKdV equation. By virtue of Riccati equation, some solutions of $(2 + 1)$ -dimensional mKdV equation are obtained. With the help of solitary wave solutions, similar to the fiber soliton communication, the chirp effect of Rossby solitary waves is discussed and some conclusions are given.

1. Introduction

In soliton theory, it is an important task to find new integrable hierarchies and their coupling systems. With the development of soliton theory, people began to concern about the $(2 + 1)$ -dimensional hierarchies. Seeking the Lax pair is a current way to get the $(2 + 1)$ -dimensional hierarchies. Different approaches to generate the integrable systems have been proposed [1–11]. Based on the generalized Tu scheme, many $(1 + 1)$ -dimensional and the corresponding Hamiltonian structures were obtained, such as the Dirac system [12, 13] and the NLS-mKdV system [14]. Tu Guizhang put forward a scheme for generating $(2 + 1)$ -dimensional hierarchies by using a residue operator with an associative algebra $A[\xi]$ which includes all pseudodifferential operators $\sum_{i=-\infty}^N a_i \xi^i$, where the operator ξ is defined by $\xi f = f\xi + (\partial_y f)$, $f \in A$. Recently, Zhang gave some $(2 + 1)$ -dimensional integrable hierarchies [15, 16], but it is difficult to solve these equations by using the usual ways which are mentioned in [15, 16]. Meanwhile, the applications of these equations were not mentioned. In this paper, we will use some tricks to deal with the coefficients of the spectral operator and

make some coefficients become unrelated to the variable y . Furthermore, we obtain some solutions of the $(2 + 1)$ -dimensional mKdV equation based on the classical Riccati equation. In particular, by employing these solutions, similar to fiber soliton communication, we will study the chirp effect of Rossby solitary waves.

Firstly, in order to get the $(2 + 1)$ -dimensional mKdV hierarchy, we recommend the following formulas:

$$\xi^n f = \sum_{i \geq 0} \begin{bmatrix} n \\ i \end{bmatrix} (\partial^i f) \xi^{n-i}, \quad (1)$$

$$f \xi^n = \sum_{i \geq 0} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} \xi^{n-i} (\partial^i f),$$

and introduce a residue operator

$$R : A[\xi] \longrightarrow A, \quad R\left(\sum a_i \xi^i\right) = a_{-1}. \quad (2)$$

Secondly, we introduce the Rossby solitary waves, which are crucial for the dynamics in ocean and atmosphere. Yang et al. [17] got the $(1 + 1)$ -dimensional mKdV equation

and used it to describe the Rossby solitary waves in the atmosphere and ocean. The results showed that the amplitude of Rossby solitary waves which propagate in a line satisfies the $(1 + 1)$ -dimensional mKdV equation. But actually as we all know that the Rossby solitary waves spread in a plane because of the terrain of the sea land in the ocean and the atmosphere, so we think the $(2 + 1)$ -dimensional mKdV equation is more suitable for describing the Rossby solitary waves.

Finally, we refer to the chirp effect which is caused by the excursion of the center wave and impacted by the dispersion and the nonlinear effect. As we all know, there exists frequency modulation effect in the light source, which is called chirp effect. Because the different parts of the pulse can produce different frequencies, the fiber soliton must be impacted by the chirp effect during the transmission. Similar to the fiber soliton communication, we think the chirp effect occurs also in the process of propagation of all kinds of waves which happen in the atmosphere and ocean. In [18], Song et al. used the nonlinear Schrödinger equation to discuss the chirp effect of internal solitary waves in ocean by means of the chirp effect in fiber solitary communication. Then in this paper, we use the same way to describe the chirp effect of the Rossby solitary waves based on the $(2 + 1)$ -dimensional mKdV equation.

The aim of this paper is to get the $(2 + 1)$ -dimensional mKdV equation and describe the chirp effect of the Rossby solitary waves with the help of initial solitary wave solution of the $(2 + 1)$ -dimensional mKdV equation. The organization of the paper is as follows: in Section 2, we first formulate a special M to get a generalized Lie algebra which is different from the classical Lie algebra. The $(2 + 1)$ -dimensional mKdV hierarchy and the corresponding Hamiltonian structures [19–25] are presented according to the generalized Tu scheme and on the basis of the Riccati equation, the solitary wave solutions, periodical wave solutions, and Rational function wave solutions of the $(2 + 1)$ -dimensional mKdV equation are obtained and placed in Section 3. Section 4 is used to discuss the chirp effect of the Rossby solitary waves by choosing the initial solitary wave solution. Finally, some conclusions are placed in Section 5.

2. A Generalized Lie Algebra and $(2 + 1)$ -Dimensional mKdV Hierarchy

Consider the following set of Lie algebras:

$$\begin{aligned} e_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ e_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (3)$$

Let

$$M = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad (4)$$

and the commutator is defined as

$$[e_i, e_j] = e_i M e_j - e_j M e_i, \quad i, j = 1, 2, 3. \quad (5)$$

It is easy to compute that

$$\begin{aligned} [e_1, e_2] &= 2e_2, \\ [e_1, e_3] &= 2e_2 - 2e_3, \\ [e_2, e_3] &= -2e_1, \end{aligned} \quad (6)$$

which is different from the Lie algebra $\mathfrak{sl}(2)$. A loop algebra of the extended Lie algebra is presented as

$$\begin{aligned} e_i(n) &= e_i \lambda^n, \\ [e_{i(m)}, e_{j(n)}] &= [e_i, e_j] \lambda^{m+n}, \\ i, j &= 1, 2, 3. \end{aligned} \quad (7)$$

Consider an isospectral problem

$$\begin{aligned} \varphi_x &= U M \varphi, \\ \lambda_t &= 0, \quad U = (\lambda + u_1 + \xi) e_1 + u_2 e_2 + u_3 e_3, \\ \varphi_t &= M V \varphi, \\ V &= \sum_{m \geq 0} (A e_1(-n) + B e_2(-n) + C e_3(-n)). \end{aligned} \quad (8)$$

According to the generalized Tu scheme, we solve the stationary zero curvature equation

$$V_x = [U, V], \quad (9)$$

which gives rise to the fact that

$$\begin{aligned} A_x &= 2u_3 B - 2u_2 C, \\ C_x &= -2\lambda C - 2u_1 C - \xi C - C\xi + 2u_3 A, \\ B_x &= -2u_2 A + 2\lambda C + \xi C + C\xi + 2u_1 C - 2u_3 A + 2B\lambda \\ &\quad + 2u_1 B + B\xi + \xi B. \end{aligned} \quad (10)$$

Let $A = \sum A_m \lambda^{-m}$, $B = \sum B_m \lambda^{-m}$, and $C = \sum C_m \lambda^{-m}$; (10) is equal to the following formula:

$$\begin{aligned} A_{m,x} &= 2u_3 B_m - 2u_2 C_m, \\ A_{m,y} &= 0, \\ B_{m,x} &= -2u_2 A_m + 2C_{m+1} + C_{m,y} + 2\xi^* C_m + 2u_1 C_m \\ &\quad - 2u_3 A_m + 2B_{m+1} + 2u_1 B_m + B_{m,y} \\ &\quad + 2\xi^* B_m, \\ C_{m,x} &= -2C_{m+1} - 2u_1 C_m - C_{m,y} - 2\xi^* C_m + 2u_3 A_m, \end{aligned} \quad (11)$$

and here the symbol ξ^* means the conjugate of an element, where $\xi \in A[\xi]$. Set $A_0 = \alpha\xi^{-1}$, $B_0 = C_0 = 0$; we can get the other unknown numbers from (11):

$$\begin{aligned} A_1 &= 0, \\ B_1 &= \alpha u_2 \xi^{-1}, \\ C_1 &= \alpha u_3 \xi^{-1}, \\ B_2 &= \alpha \frac{(u_2 + u_3)_x}{2} \xi^{-1} - \alpha \frac{u_{2,y}}{2} \xi^{-1} - \alpha u_1 u_2 \xi^{-1} - \alpha u_2, \\ C_2 &= \alpha \frac{u_{3,y} - u_{3,x}}{2} \xi^{-1} - \alpha u_3 - \alpha u_1 u_3 \xi^{-1}, \\ A_2 &= \alpha \partial_x^- \left((u_3 u_2)_x + u_3 u_{2,y} - u_2 u_{3,y} + u_3 u_{3,x} \right) \xi^{-1}. \end{aligned} \quad (12)$$

For an arbitrary number n , indicating

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^n [A_n e_1(n-m) + B_n e_2(n-m) + C_n e_3(n-m)] \\ &= \lambda^n V - V_-^{(n)}, \end{aligned} \quad (13)$$

we can calculate that

$$-V_{+,x}^{(n)} + [U, V_+^{(n)}] = V_{-,x}^{(n)} - [U, V_-^{(n)}], \quad (14)$$

and a direct calculate can get the following formula:

$$\begin{aligned} -V_{+,x}^{(n)} + [U, V_+^{(n)}] &= -2(B_{n+1} + C_{n+1})e_2(0) \\ &\quad + 2C_{n+1}e_3(0). \end{aligned} \quad (15)$$

Taking $V^{(n)} = V_+^{(n)} - a_n e_1(0)$, then the zero curvature equation

$$U_t - V_x^{(n)} + [U, V_+^{(n)}] = 0 \quad (16)$$

admits that

$$\begin{aligned} u_t = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_t &= \begin{bmatrix} R(A_{n,x}) \\ R(B_{n,x} - 2u_1(B_n + C_n)) \\ R(C_{n,x} + 2u_1 C_n) \end{bmatrix} \\ &= \begin{bmatrix} -\partial & 0 & 0 \\ 0 & \partial & -\partial + 2u_1 \\ 0 & -\partial - 2u_1 & 0 \end{bmatrix} \begin{bmatrix} R(A_n) \\ -R(C_n) \\ -R(B_n + C_n) \end{bmatrix}, \end{aligned} \quad (17)$$

and then

$$\begin{bmatrix} R(A_n) \\ -R(C_n) \\ -R(B_n + C_n) \end{bmatrix} = L \begin{bmatrix} R(A_{n-1}) \\ -R(C_{n-1}) \\ -R(B_{n-1} + C_{n-1}) \end{bmatrix}, \quad (18)$$

where the recurrence operator L is given from (11):

$$L = \begin{bmatrix} 0 & Q & W \\ -u_3 & -u_1 - \frac{\partial_x}{2} - \frac{\partial_y}{2} - \xi^* & 0 \\ -(u_2 + u_3) & -\frac{\partial_x}{2} & -u_1 + \frac{\partial_x}{2} - \frac{\partial_y}{2} - \xi^* \end{bmatrix}, \quad (19)$$

where

$$\begin{aligned} Q &= -\partial^- \left[(u_3 + u_2) \partial_y + (2u_2 + 2u_3) \xi^* \right. \\ &\quad \left. + (2u_3 u_1 + 2u_2 u_1) + u_2 \partial_x \right], \\ W &= -\partial^- \left[u_3 (\partial_x - \partial_y - 2\xi^*) - 2u_1 u_3 \right]. \end{aligned} \quad (20)$$

When $n = 2$, a simple reduction of (17) gets the following generalized $(2+1)$ -dimensional equation:

$$\begin{aligned} u_{1,t} &= \alpha \left((u_3 u_2)_x + u_3 u_{2,y} - u_2 u_{3,y} + u_3 u_{3,x} \right), \\ u_{2,t} &= \alpha \left(\frac{(u_2 + u_3)_{xx}}{2} - \frac{u_{2,xy}}{2} - (u_1 u_2)_x - u_1 u_{2,x} \right. \\ &\quad \left. + u_1 u_{2,y} - u_1 u_{3,y} + 2u_1^2 u_2 + 2u_1^2 u_3 \right), \\ u_{3,t} &= \alpha \left(\frac{u_{3,xy} - u_{3,xx}}{2} - (u_1 u_3)_x + u_1 (u_{3,y} - u_{3,x}) \right. \\ &\quad \left. - 2u_1^2 u_3 \right). \end{aligned} \quad (21)$$

When taking $n = 3$, $u_2 = u_3$, and $u_1 = 0$, we obtain the $(2+1)$ -dimensional mKdV hierarchy:

$$u_{2,t} = \alpha u_{2,xxx} + 6u_2^2 u_{2,x} - \alpha u_{2,xyy} - \alpha u_{2,xyy}, \quad (22)$$

and if $\alpha = 1$, then (22) can be translated into the classical $(2+1)$ -dimensional mKdV equation:

$$u_{2,t} = u_{2,xxx} + 6u_2^2 u_{2,x} - u_{2,xyy} - u_{2,xyy}. \quad (23)$$

In what follows, we will study the Hamiltonian structures of the hierarchy (17). In order to seek the Hamiltonian structure, we introduce the linear function by

$$\langle a, b \rangle = 2(a_1 b_1 - a_3 b_2 - a_2 b_3 - a_3 b_3), \quad (24)$$

for all $a = a_1 e_1(0) + a_2 e_2(0) + a_3 e_3(0)$, $b = b_1 e_1(0) + b_2 e_2(0) + b_3 e_3(0)$. The isospectral operator can be written as

$$\begin{aligned} U &= \begin{bmatrix} \lambda + u_1 + \xi + u_3 & 2u_3 \\ u_2 & \lambda + u_1 + \xi - u_3 \end{bmatrix}, \\ V &= \begin{bmatrix} A + C & 2C \\ B & A - C \end{bmatrix}. \end{aligned} \quad (25)$$

Therefore, we can get

$$\begin{aligned}\frac{\partial U}{\partial u_1} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e_1(0), \\ \frac{\partial U}{\partial u_2} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e_2(0), \\ \frac{\partial U}{\partial u_3} &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = e_3(0), \\ \frac{\partial U}{\partial \lambda} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e_1(0).\end{aligned}\quad (26)$$

It is easy to verify that (26) satisfies the variational identity

$$\frac{\delta}{\delta u} R \langle V, U_\lambda \rangle = \lambda^{-r} \frac{\partial}{\partial \lambda} \lambda^r R \langle V, U_u \rangle, \quad (27)$$

where

$$\langle V, U_\lambda \rangle = \text{tr}(V, U_\lambda). \quad (28)$$

Then the (2 + 1)-dimensional hierarchy (17) can be written as the following Hamiltonian form:

$$u_{tn} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_{tn} = J \begin{bmatrix} R(A_n) \\ R(-C_n) \\ R(-B_n - C_n) \end{bmatrix} = J \frac{\delta H_n}{\delta u}. \quad (29)$$

3. The Solutions of (2+1)-Dimensional mKdV Equation

In order to get the solutions of (23), we introduce the classical Riccati equation [26–29]

$$P'(\xi) = a + cP^2(\xi), \quad (30)$$

which has the following special solutions:

- (1) when $a = 1, c = -1, P(\xi) = \tanh \xi, \coth \xi$;
- (2) when $a = 1, c = 1, P(\xi) = \tan \xi$;
- (3) when $a = -1, c = -1, P(\xi) = \cot \xi$;
- (4) when $a = 1, c = -4, P(\xi) = \tanh(\xi)/(1 + \tanh^2(\xi))$;
- (5) when $a = 1, c = 4, P(\xi) = \tan \xi/(1 - \tan^2 \xi)$;
- (6) when $a = -1, c = -4, P(\xi) = \cot \xi/(1 - \cot^2 \xi)$;
- (7) when $a = 1/2, c = 1/2, P(\xi) = \tan \xi \pm \sec \xi, \csc \xi - \cot \xi$;
- (8) when $a = -1/2, c = -1/2, P(\xi) = \sec \xi - \tan \xi, \cot \xi \pm \csc \xi$;
- (9) when $a = 1/2, c = -1/2, P(\xi) = \tanh(\xi)/(1 + \text{sech}(\xi)), \tanh(\xi) \pm i \text{sech}(\xi)$;
- (10) when $a = 0, c \neq 0, P(\xi) = -1/(c\xi + c_0)$.

As to (23), firstly, let

$$u = \gamma(x, y, t) P(\xi(x, y, t)), \quad (31)$$

where $P(\xi)$ satisfies Riccati equation (30). Substitute (31) into (23); we can get the following formula:

$$\begin{aligned}\sum_{i=1}^6 R_i F_i(P, P', P'', \dots) \\ = R_1 P'''' + R_2 P' P^2 + R_3 P^3 + R_4 P + R_5 P'' + R_6 P',\end{aligned}\quad (32)$$

where

$$\begin{aligned}R_1 &= \gamma_x^3 - \gamma_x^2 \xi_y - \gamma_x \xi_y^2, \\ R_2 &= 6\gamma^3 \xi_x, \\ R_3 &= 6\gamma^2 \gamma_x, \\ R_4 &= \gamma_{xxx} - \gamma_{xyy} - \gamma_{xxy} - \gamma_t, \\ R_5 &= 3\gamma_x \xi_x^2 + 3\gamma_x \xi_x \xi_{xx} - 2\gamma_x \xi_x \xi_y - \gamma_{xx} \xi_y - \gamma_y \xi_x^2 \\ &\quad - 2\gamma_x \xi_x \xi_{xy} - 2\gamma_y \xi_x \xi_y - \gamma_{yy} \xi_x - \gamma_x \xi_y^2 \\ &\quad - 2\gamma \xi_{xy} \xi_{xy}, \\ R_6 &= 3\gamma_{xx} \xi_x + 3\gamma_x \xi_{xx} + \gamma \xi_{xxx} - \gamma_{xx} \xi_y - 2\gamma_{xy} \xi_x \\ &\quad - 2\gamma_x \xi_{xy} - \gamma_y \xi_{xx} - \gamma \xi_{xxy} - \gamma_{yy} \xi_x - 2\gamma_{xy} \xi_y \\ &\quad - 2\gamma_y \xi_{xy} - \gamma_x \xi_{yy} - \gamma \xi_{xxy} - \gamma \xi_t.\end{aligned}\quad (33)$$

Secondly, we try to get a Riccati equation; let

$$\begin{aligned}\sum_{i=1}^6 R_i F_i(P, P', P'', \dots) &= R_1 P'''' + R_2 P' P^2 + R_3 P^3 \\ &\quad + R_4 P'' + R_5 P' + R_6 P = R_1 \partial_{\xi\xi} (P' - a - cP^2) \\ &\quad + 2cPR_1 \partial_{\xi} (P' - a - cP^2) \\ &\quad + 2cP'R_1 (P' - a - cP^2) + R_4 \partial_{\xi} (P' - a - cP^2) \\ &\quad + 2cR_4 P (P' - a - cP^2) \\ &= (R_1 \partial_{\xi\xi} + 2cPR_1 \partial_{\xi} + 2cP'R_1 + R_4 \partial_{\xi} + 2cR_4 P) \\ &\quad \cdot (P' - a - cP^2).\end{aligned}\quad (34)$$

Thirdly, compare the coefficient of (32) and (34); we can get the following formula:

$$\begin{aligned}\xi_x &= \xi_y, \\ 6\gamma c^2 \xi_x^3 &= 6\gamma^3 \xi_x, \\ 2ac\gamma \xi_x^3 &= -3\gamma_{xx} \xi_x - 3\gamma_x \xi_{xx} - \gamma \xi_{xxx} - \gamma \xi_t, \\ 2ac(3\gamma_x \xi_x^2 + 3\gamma \xi_x \xi_{xx}) &= -\gamma_{xxx} - \gamma_t, \\ 2c^2(3\gamma_x \xi_x^2 + 3\gamma \xi_x \xi_{xx}) &= 6\gamma^2 \gamma_x.\end{aligned}\quad (35)$$

At last, we can get a special solution $\xi = b(x + y) - 2acb^3t$ of (35), where b is a real number.

Then we obtain the solutions of (23) according to different conditions.

Condition 1. Solitary wave solutions are as follows:

(1) when $a = 1, c = -1$:

$$\begin{aligned} u_1 &= -b \tanh(b(x + y) - 2b^3t), \\ u_2 &= -b \coth(b(x + y) - 2b^3t), \\ u_3 &= -b \operatorname{sech}(b(x + y) + b^3t); \end{aligned} \quad (36)$$

(2) when $a = 1/2, c = -1/2$:

$$\begin{aligned} u_4 &= -\frac{1}{2}b \tanh\left(b(x + y) - \frac{1}{2}b^3t\right) \\ &\quad + i \operatorname{sech}\left(b(x + y) - \frac{1}{2}b^3t\right), \\ u_5 &= -\frac{1}{2}b \frac{\tanh(b(x + y) - (1/2)b^3t)}{1 + \operatorname{sech}(b(x + y) - (1/2)b^3t)}; \end{aligned} \quad (37)$$

(3) when $a = 1, c = -4$:

$$u_6 = -4b \frac{\tanh(b(x + y) - 2b^3t)}{1 + \tanh(b(x + y) - 2b^3t)}. \quad (38)$$

Condition 2. Periodical wave solutions are as follows:

(4) when $a = 1, c = 1$:

$$u_7 = b \tan(b(x + y) - 2b^3t); \quad (39)$$

(5) when $a = -1, c = -1$:

$$u_8 = -b \cot(b(x + y) + 2b^3t); \quad (40)$$

(6) when $a = 1/2, c = 1/2$:

$$\begin{aligned} u_9 &= \frac{1}{2}b \left[\tan\left(b(x + y) - \frac{1}{2}b^3t\right) \right. \\ &\quad \left. \pm \sec\left(b(x + y) - \frac{1}{2}b^3t\right) \right], \\ u_{10} &= \frac{1}{2}b \left[\csc\left(b(x + y) - \frac{1}{2}b^3t\right) \right. \\ &\quad \left. - \cot\left(b(x + y) - \frac{1}{2}b^3t\right) \right]; \end{aligned} \quad (41)$$

(7) when $a = -1/2, c = -1/2$:

$$\begin{aligned} u_{11} &= -\frac{1}{2}b \left[\sec\left(b(x + y) + \frac{1}{2}b^3t\right) \right. \\ &\quad \left. - \tan\left(b(x + y) + \frac{1}{2}b^3t\right) \right], \\ u_{12} &= -\frac{1}{2}b \left[\cot\left(b(x + y) + \frac{1}{2}b^3t\right) \right. \\ &\quad \left. - \csc\left(b(x + y) + \frac{1}{2}b^3t\right) \right]; \end{aligned} \quad (42)$$

(8) when $a = 1, c = 4$:

$$u_{13} = 4b \frac{\tan(b(x + y) - 2b^3t)}{1 - \tan(b(x + y) - 2b^3t)}. \quad (43)$$

Condition 3. Rational function wave solution is as follows:

(9) $a = 0, c \neq 0$:

$$u_{14} = -cb \frac{1}{cb(x + y) + t_0}. \quad (44)$$

4. The Chirp Effect of Rossby Solitary Waves

In this part, we begin to discuss the application of the $(2 + 1)$ -dimensional mKdV equation. As to the Rossby solitary waves in the atmosphere, Yang et al. got the $(1 + 1)$ -dimensional mKdV equation to describe the propagation of Rossby solitary waves in a line in the atmosphere and ocean, but in fact, as we all know, the propagation of Rossby solitary waves in a plane is more suitable for reflecting the real condition of atmosphere and ocean. So in this paper, we use the $(2 + 1)$ -dimensional mKdV equation for analysis of the chirp effect of the Rossby solitary waves with the help of chirp effect in fiber soliton communication.

As to the $(2 + 1)$ -dimensional mKdV equation (22)

$$u_{2,t} = \alpha u_{2,xxx} + 6u_2^2 u_{2,x} - \alpha u_{2,xyy} - \alpha u_{2,xyy}, \quad (45)$$

where α is the coefficient of dispersion part. On the basis of initial wave form

$$u = u_0 \tanh\left(\frac{x + y}{l}\right). \quad (46)$$

In the transmission of Rossby wave, the nonlinear part and the dispersive part act as an important function. In fiber soliton communication, chirp effect is caused by the excursion of the center wave; it is impacted by the dispersion and the nonlinear effect. Then we discuss the chirp effect of the Rossby solitary waves.

Firstly, we alone consider the dispersion function; then (22) can be changed into

$$u_t = \alpha u_{xxx} - \alpha u_{xyy} - \alpha u_{xyy}, \quad (47)$$

and we only observe the time t from 0 to Δt , where Δt is an infinitesimal variable; then we can get the approximate solution of (47):

$$\begin{aligned} u(\Delta t, x, y) &= u_0 \tanh\left(\frac{x + y}{l}\right) \\ &\quad \cdot \exp\left(\sqrt{2}\left(-\frac{\alpha \Delta t}{l^2} + 2\frac{\alpha \Delta t}{l^2} \tanh^{5/2}\left(\frac{x + y}{l}\right)\right)\right), \end{aligned} \quad (48)$$

and then the phase of the wave is

$$\varphi_D = \sqrt{2}\left(-\frac{\alpha \Delta t}{l^2} + 2\frac{\alpha \Delta t}{l^2} \tanh^{5/2}\left(\frac{x + y}{l}\right)\right), \quad (49)$$

from (49), we can get the chirp caused by the dispersion:

$$\begin{aligned}\Delta v_D &= -\nabla \varphi_D \\ &= -10\sqrt{2} \left(\frac{\alpha \Delta t}{l^2} \tanh^{3/2} \left(\frac{x+y}{l} \right) \operatorname{sech}^2 \left(\frac{x+y}{l} \right) \right).\end{aligned}\quad (50)$$

Secondly, we consider the function of the nonlinear effect; then (22) can be changed into the following formula:

$$u_t = 6u^2 u_x, \quad (51)$$

likewise, the time is still from 0 to Δt , where Δt is an infinitesimal variable; substituting (46) into (51), we can get the approximate solution of (51):

$$\begin{aligned}u(\Delta t, x, y) &= u_0 \tanh \left(\frac{x+y}{l} \right) \\ &\cdot \exp \left(6\sqrt{2}u_0^{5/2} \tanh^{5/2} \left(\frac{x+y}{l} \right) \Delta t \right),\end{aligned}\quad (52)$$

accordingly, the phase of the wave is

$$\varphi_N = 6\sqrt{2}u_0^{5/2} \tanh^{5/2} \left(\frac{x+y}{l} \right) \Delta t, \quad (53)$$

then the chirp caused by nonlinear effect is

$$\begin{aligned}\Delta v_N &= -\nabla \varphi_N \\ &= -30\sqrt{2}u_0^{5/2} \tanh^{3/2} \left(\frac{x+y}{l} \right) \operatorname{sech}^2 \left(\frac{x+y}{l} \right) \Delta t.\end{aligned}\quad (54)$$

According to (54) and (50), we can get the whole chirp

$$\begin{aligned}\Delta v_S &= \Delta v_N + \Delta v_D = \left(-10\sqrt{2} \frac{\alpha}{l^2} - 30\sqrt{2}u_0^{5/2} \right) \\ &\cdot \tanh^{3/2} \left(\frac{x+y}{l} \right) \operatorname{sech}^2 \left(\frac{x+y}{l} \right) \Delta t,\end{aligned}\quad (55)$$

if the dispersion effect is equal to the nonlinear effect, then $\Delta v_S = 0$,

$$u_0 = \left(-\frac{\alpha}{3l^2} \right)^{2/5}; \quad (56)$$

if the dispersion effect is more than the nonlinear effect, $|\Delta v_D| > |\Delta v_N|$, we obtain

$$u_0 < \left(-\frac{\alpha}{3l^2} \right)^{2/5}; \quad (57)$$

if the dispersion effect is less than the nonlinear effect, $|\Delta v_D| < |\Delta v_N|$, we obtain

$$u_0 > \left(-\frac{\alpha}{3l^2} \right)^{2/5}. \quad (58)$$

From (55), we know that the whole chirp effect is related to the environment where Rossby solitary waves spread in the atmosphere and the initial amplitude; when (56) is satisfied, the Rossby solitary waves can be spread steadily. The initial amplitude is smaller, the dispersive function is stronger, the initial amplitude is bigger, and the nonlinear function is stronger. The form of Rossby solitary wave is as in Figure 1.

FIGURE 1: The form of Rossby solitary wave at $t = 1$, wave speed $c = -2$.

5. Conclusions

In this paper, we first obtain a $(2+1)$ -dimensional mKdV equation on the basis of $(1+1)$ -dimensional spectra operator. Although the $(2+1)$ -dimensional mKdV hierarchy has been obtained in this paper, we note that it is difficult to obtain all the $(2+1)$ -dimensional hierarchies by using the generalized Tu scheme and not every $(2+1)$ -dimensional hierarchy has its own Hamiltonian structure. In the final analysis, we explain the chirp effect of the Rossby solitary waves on the basis of the $(2+1)$ -dimensional mKdV equation. We discuss the relation between the nonlinear function and the dispersion and get the conclusion that the whole chirp effect is impacted by the atmosphere and ocean state of the Rossby solitary waves transmission and the initial amplitude. We can balance the whole chirp effect according to the nonlinear function and dispersion in order to let the Rossby solitary waves spread placidly. Last but definitely not least, we should try to find some more $(2+1)$ -dimensional hierarchies which have important meanings in physics and other aspects. As we all know that there are financial solitons in economics, we believe that we can apply some new $(2+1)$ -dimensional hierarchies into the economics and describe the economical phenomenon. In addition, we can also try to use other ways such as Darboux transformations and symmetry transformations to get more $(2+1)$ -dimensional hierarchies; these problems are worthy of discussing.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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