Research Article

Two Kinds of New Integrable Couplings of the Negative-Order Korteweg-de Vries Equation

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Based on some known loop algebras with finite dimensions, two different negative-order integrable couplings of the negative-order Korteweg-de Vries (KdV) hierarchy of evolution equations are generated by making use of the Tu scheme, from which the corresponding negative-order integrable couplings of the negative-order KdV equations are followed to be obtained. The resulting Hamiltonian structure of one negative integrable coupling is derived from the variational identity.

1. Introduction

Since the concept on integrable couplings was proposed [1], some integrable couplings of the known integrable systems, such as the AKNS system and the KN system, were obtained. Ma [2] made use of the perturbation method to obtain the integrable couplings of the KdV equation as well as their some algebraic and geometric properties. Reference [3] employed a simple finite-dimensional Lie algebra to present a method for generating integrable couplings of integrable hierarchies of evolution equations. Later, Guo and Zhang [4] proposed the quadratic-form identity for which Hamiltonian structures of some integrable couplings were generated. Ma and Chen [5] further generalized the quadratic-form identity and completely improved it to obtain the variational identity for deducing the Hamiltonian structures of integrable couplings which is more convenient. Actually, there are lots of related references in integrable-coupling area. The adopted algebras were studied and applied in other existing literature. One of them is so-called perturbation algebra, which was even used to present 2 + 1-dimensional local bi-Hamiltonian systems by Ma and Fuchssteiner in 1996, and whose systematical presentation was presented in [6]. In addition, Ma et al. [2, 7–9] introduced different Lie algebras to obtain some new and interesting integrable couplings. By following the methods, many interesting integrable couplings

of some known integrable hierarchies were obtained, such as the works in [10–12]. However, to our best knowledge, the negative-order integrable couplings of the negative-order integrable hierarchies have not been discussed systematically. Up to now, we know that some interesting related negative-order integrable equations including the negative-order KdV equation and some associated properties were obtained, such as the results in [13–18]. But their negative-order integrable couplings have not been discussed. In the paper, we want to discuss the problem. Qiao and Fan [18] employed the Lenard sequence method to obtain the negative-order KdV equation and then represented it by its Lax pair. Enlightened by this work, we will generate the negative-order KdV hierarchy and its integrable couplings by enlarged Lie algebras and the enlarged Lax pairs. It is remarkable that the integrable couplings obtained in the paper include the linear and nonlinear integrable couplings. In the aspect of nonlinear integrable couplings, Ma and Zhu [19] proposed some semisimple Lie algebras to present a way for generating nonlinear discrete integrable couplings. Then [20–22] also reveal some nonlinear integrable couplings by constructing various Lie algebras. As reduced cases, two negative-order integrable couplings of the negative-order KdV equation are obtained. Finally, the Hamiltonian structure of one negative-order integrable coupling in the negative-order KdV hierarchy is obtained by the variational identity.

2. Two Different Negative-Order Integrable Couplings

In the section, we adopt the Tu scheme [23] which was proposed by Ma [24] and the known finite-dimensional Lie algebras to deduce two integrable couplings of the negative-order KdV hierarchy. As reduced cases, two negative-order integrable couplings of the negative-order KdV equation are also followed to present. In [25, 26], a 6-dimensional Lie algebra reads

$$G_1 = \{g_1, g_2, \dots, g_6\}, \quad (1)$$

where

$$\begin{aligned} g_1 &= \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \\ g_2 &= \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix}, \\ g_3 &= \begin{pmatrix} h_3 & 0 \\ 0 & h_3 \end{pmatrix}, \\ g_4 &= \begin{pmatrix} 0 & h_1 \\ 0 & 0 \end{pmatrix}, \\ g_5 &= \begin{pmatrix} 0 & h_2 \\ 0 & 0 \end{pmatrix}, \\ g_6 &= \begin{pmatrix} 0 & h_3 \\ 0 & 0 \end{pmatrix}; \end{aligned} \quad (2)$$

here $h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $h_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $h_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

In [26], another 6-dimensional Lie algebra was given by

$$G_2 = \{f_1, f_2, \dots, f_6\}, \quad (3)$$

where

$$\begin{aligned} f_i &= g_i, \quad i = 1, 2, 3; \\ f_4 &= \begin{pmatrix} 0 & h_1 \\ 0 & h_1 \end{pmatrix}, \\ f_5 &= \begin{pmatrix} 0 & h_2 \\ 0 & h_2 \end{pmatrix}, \\ f_6 &= \begin{pmatrix} 0 & h_3 \\ 0 & h_3 \end{pmatrix}. \end{aligned} \quad (4)$$

The resulting loop algebras can be defined as $p_i(n) = p_i \otimes \lambda^n$, where $p_i \in G_1$ or $p_i \in G_2$. The loop algebras of the Lie algebras G_1 and G_2 are denoted by $\widetilde{G}_1$ and $\widetilde{G}_2$, respectively. According to the Tu scheme, we first solve a stationary 0-curvature equation

$$V_x = [U, V], \quad (5)$$

where the Lax matrices U and V belong to the loop algebra $\widetilde{G}_1$ or $\widetilde{G}_2$; U and V can be represented by the loop algebra $\widetilde{G}_1$ or $\widetilde{G}_2$:

$$U = R + \sum_{i=1}^6 u_i p_i(n), \quad (6)$$

$$V = \sum_{m=0}^{\infty} \left(\sum_{i=1}^6 V_{im} p_i(-m) \right), \quad (7)$$

where R and $p_i(n)$ belong to $\widetilde{G}_1$ or $\widetilde{G}_2$ and require $\deg(R) \geq \deg(p_i(n))$; here $\deg$ stands for the degree of the elements in the loop algebras; V_{im} are the derivative functions in x and t . Then choose a suitable modified term Δ_n so that

$$V^{(n)} = V_+^{(n)} + \Delta_n, \quad (8)$$

where

$$V_+^{(n)} = \sum_{m=0}^n \left(\sum_{i=1}^6 V_{im} p_i(n-m) \right) = \lambda^n V - V_-^{(n)} \quad (9)$$

satisfies the 0-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0, \quad (10)$$

which can derive an integrable hierarchy of evolution equations. In order to deduce negative-order integrable hierarchies, we need to change (7) to the following form:

$$V = \sum_{m=-\infty}^0 V_m \lambda^{-m} =: \sum_{m \leq 0} V_{im} p_i(-m). \quad (11)$$

Qiao and Li [17] once obtained the negative-order KdV equation through the Lenard recursive sequence

$$\begin{aligned} v_t &= w_x, \\ w_{xxx} + 4vw_x + 2v_x w &= 0 \end{aligned} \quad (12)$$

which is equivalent to a nonlinear quartic integrable system

$$uu_{xxt} - u_{xx}u_t - 2u^3u_x = 0. \quad (13)$$

The Lax pair of (12) was presented by a proposition in [18]. Furthermore, Qiao and Fan produced the Hamiltonian structure, Darboux transformations, and some exact solutions. Enlightened by their results, we will deduce the negative-order KdV hierarchy by employing the Tu scheme which can reduce to the negative-order KdV equation (12).

Denote $\widetilde{H} = \text{span}\{h_1(n), h_2(n), h_3(n)\}$ where $h_i(n) = h_i \lambda^n$, $i = 1, 2, 3$; then $\widetilde{H}$ is an obvious loop algebra. Employing the loop algebra, we introduce a pair of Lax matrices

$$\begin{aligned} U &= h_3(1) - uh_3(0) + h_2(0), \\ V &= \sum_{m=-\infty}^0 (a_m h_1(-m) + b_m h_2(-m) + c_m h_3(-m)). \end{aligned} \quad (14)$$

A solution to (5) is presented as

$$\begin{aligned} b_{m+1} &= -a_{m,x} + ub_m + c_m, \\ b_{m,x} &= -2a_m, \\ 2a_{m+1} &= c_{m,x} + 2ua_m, \quad m = 0, -1, -2, \dots, \end{aligned} \quad (15)$$

which is equivalent to the following relation:

$$Jb_{m+1} = Kb_m, \quad (16)$$

where

$$\begin{aligned} J &= \partial = \frac{\partial}{\partial x}, \\ K &= \frac{1}{4}\partial^3 + \frac{1}{2}(u\partial + \partial u). \end{aligned} \quad (17)$$

Equation (16) is called the Lenard sequence [18]. Denote

$$\begin{aligned} V^{(-n)} &= \lambda^{-n}V - V_-^{(-n)} = \sum_{m=-n+1}^0 V_m \lambda^{-n-m}, \\ n &= 1, 2, 3, \dots \end{aligned} \quad (18)$$

Equation (5) can be decomposed into

$$-(V_+^{(-n)})_x + [U, V_+^{(-n)}] = (V_-^{(-n)})_x - [U, V_-^{(-n)}]. \quad (19)$$

With the help of (14) and (19), we get that

$$-(V_+^{(-n)})_x + [U, V_+^{(-n)}] = -2a_{1-n}h_3(0) + b_{1-n}h_1(0). \quad (20)$$

Take

$$\begin{aligned} \Delta_n &= -b_{1-n}h_1(0), \\ V^{(-n)} &= V_+^{(-n)} + \Delta_n; \end{aligned} \quad (21)$$

we have that

$$\begin{aligned} -V_x^{(-n)} + [U, V^{(-n)}] &= [(b_{1-n})_x - 2a_{1-n}]h_3(0) \\ &= 2(b_{1-n})_x h_3(0). \end{aligned} \quad (22)$$

Therefore, the 0-curvature equation

$$U_t - V_x^{(-n)} + [U, V^{(-n)}] = 0 \quad (23)$$

admits the following integrable hierarchy:

$$u_{t_{n-1}} = 2(b_{1-n})_x, \quad n = 1, 2, 3, \dots, \quad (24)$$

which is called the negative-order KdV hierarchy. Obviously, we have

$$\begin{aligned} u_{t_0} &= 2b_{0,x}, \\ u_{t_1} &= 2b_{-1,x}, \\ u_{t_2} &= 2b_{-2,x}, \\ &\vdots \end{aligned} \quad (25)$$

When $m = 0$, (16) gives $b_0 = 0$, $b_{1,x} = 0$; here let $b_1 = 0$.

When $m = -1$, (16) reduces to

$$b_{0,x} = \frac{1}{4}b_{-1,xxx} + \frac{1}{2}ub_{-1,x} + \frac{1}{2}(ub_{-1})_x. \quad (26)$$

Assume $b_{-1} = w$; then (26) becomes

$$w_{xxx} + 4uw_x + 2u_xw = 0. \quad (27)$$

Combined with (25), we obtain the negative-order KdV equation when $t_1 = t$:

$$\begin{aligned} u_t &= 2w_x, \\ w_{xxx} + 4uw_x + 2u_xw &= 0, \end{aligned} \quad (28)$$

which is the same as (12) except for a constant 2 in the first equation. Qiao and Fan [18] established a Lax pair of the negative-order KdV equation by employing the Lenard sequence method.

In what follows, we can directly single out the Lax pair of (28). It is easy to see that

$$\begin{aligned} V^{(-2)} &= \sum_{m=-1}^0 (a_m h_1(-2-m) + b_m h_2(-2-m) \\ &\quad + c_m h_3(-2-m)) - b_{-1} h_3(0) \\ &= a_0 h_1(-2) + b_0 h_2(-2) + c_0 h_3(-2) \\ &\quad + a_{-1} h_1(-1) + b_{-1} h_2(-1) + c_{-1} h_3(-1) - b_{-1} h_3(0) \\ &= \begin{pmatrix} q_{-1} \lambda^{-1} & b_{-1} \lambda^{-1} \\ c_{-1} \lambda^{-1} - b_{-1} & -a_{-1} \lambda^{-1} \end{pmatrix}. \end{aligned} \quad (29)$$

Due to $b_0 = 0$, we can get $a_0 = 0$. In terms of (15), we have $a_{-1} = -(1/2)w_x$, $c_{-1} = -(1/2)w_{xx} - uw$. Thus, one infers that

$$V^{(-2)} = \begin{pmatrix} -\frac{1}{2}w_x \lambda^{-1} & w \lambda^{-1} \\ -\left(\frac{1}{2}w_{xx} + uw\right) \lambda^{-1} + w & \frac{1}{2}w_x \lambda^{-1} \end{pmatrix}. \quad (30)$$

Hence, the compatibility condition of the Lax pair

$$\begin{aligned} \varphi_x &= U\varphi, \quad U = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \\ \varphi_t &= V^{(-2)}\varphi, \end{aligned} \quad (31)$$

gives rise to the negative-order KdV equation (28); that is, the 0-curvature equation

$$U_t - V_x^{(-2)} + [U, V^{(-2)}] = 0 \quad (32)$$

admits (28). The Hamiltonian structure of (28) can be given similar to that in [18]; here we do not repeat it again.

3. Two Negative-Order Integrable Couplings

We first make use of the loop algebra $\widetilde{G}_1$ to deduce the first negative-order integrable couplings of the negative-order KdV hierarchy and then employ the second loop algebra $\widetilde{G}_2$ to derive the second negative-order integrable couplings.

3.1. *The First Negative-Order Integrable Coupling.* Set

$$\begin{aligned} U &= g_3(1) - ug_3(0) + g_2(0) + rg_6(0), \\ V &= \sum_{m=-\infty}^0 \left(\sum_{i=1}^6 V_{im} g_i(-m) \right); \end{aligned} \quad (33)$$

then (5) gives the following recursive relations among V_{im} ($i = 1, 2, \dots, 6$):

$$\begin{aligned} (V_{1m})_x &= -V_{2,m+1} + uv_{2m} + V_{3m}, \\ (V_{2m})_x &= -2V_{1m}, \\ (V_{3m})_x &= 2V_{1,m+1} - 2uV_{1m}, \quad m = 0, -1, -2, \dots; \\ (V_{4m})_x &= -V_{5,m+1} + uV_{5m} + V_{6m} - rV_{2m}, \\ (V_{5m})_x &= -2V_{4m}, \\ (V_{6m})_x &= 2V_{4,m+1} - 2uV_{4m} + 2rV_{1m}, \quad m = 0, -1, -2, \dots; \end{aligned} \quad (34)$$

here (34) is just the same as (15). Denote

$$\begin{aligned} V^{(-n)} &= \sum_{m=-n+1}^0 \left(\sum_{i=1}^6 V_{im} g_i(-n-m) \right) - V_{2,1-n} g_3(0) \\ &\quad - V_{5,1-n} g_4(0); \end{aligned} \quad (36)$$

after tedious calculation, one infers that

$$\begin{aligned} -V_x^{(-n)} + [U, V^{(-n)}] &= [(V_{2,1-n})_x - 2V_{1,1-n}] g_3(0) \\ &\quad + [(V_{5,1-n})_x - 2V_{4,1-n}] g_6(0). \end{aligned} \quad (37)$$

Therefore, the 0-curvature equation (10) is equivalent to

$$\begin{aligned} u_{t_{n-1}} &= (V_{2,1-n})_x - 2V_{1,1-n} = 2(V_{2,1-n})_x, \\ r_{t_{n-1}} &= -(V_{5,1-n})_x + 2V_{4,1-n} = -2(V_{5,1-n})_x, \end{aligned} \quad (38)$$

which is the negative-order integrable couplings of the negative-order KdV hierarchy. In what follows, we want to consider its reduction. First of all, we have a recursive relation from (35):

$$\begin{aligned} (V_{5,m+1})_x &= \frac{1}{4} (V_{5m})_{xxx} \\ &\quad + \frac{1}{2} [u(V_{5m})_x + (uV_{5m})_x - r(V_{2m})_x - (rV_{2m})_x]. \end{aligned} \quad (39)$$

When $m = 0$, we take $V_{2,0} = V_{2,1} = V_{5,0} = V_{5,1} = 0$. When $m = -1$, we have in terms of (39) that

$$\begin{aligned} (V_{5,0})_x &= \frac{1}{4} (V_{5,-1})_{xxx} \\ &\quad + \frac{1}{2} [u(V_{5,-1})_x + (uV_{5,-1})_x \\ &\quad - r(V_{2,-1})_x - (rV_{2,-1})_x]. \end{aligned} \quad (40)$$

Let $V_{5,-1} = -q$; we find from (40) that

$$q_{xxx} + 2uq_x + 2(qu)_x + 2rw_x + 2(rw)_x = 0. \quad (41)$$

Assume $t_1 = t$; since $r_t = -2q_x$, thus we have

$$\begin{aligned} w_{xxx} + 4uw_x + 2u_x w, \quad u_t &= 2w_x, \\ q_{xxx} + 2uq_x + 2(qu)_x + 2rw_x + 2(rw)_x &= 0, \\ r_t &= -2q_x. \end{aligned} \quad (42)$$

The first equation in (42) is the negative-order KdV equation, while the second equation is just coupled part of integrable couplings. Therefore, (42) is an integrable coupling of the negative-order KdV equation.

3.2. *The Second Negative-Order Integrable Coupling.* Set

$$\begin{aligned} U &= f_3(1) - uf_3(0) + f_2(0) + vf_6(0), \\ V &= \sum_{m \leq 0} \left(\sum_{i=1}^6 V_{im} f_i(-m) \right). \end{aligned} \quad (43)$$

Similar to the previous case, we can get (34) and the following equations which are different from (35):

$$\begin{aligned} (V_{4m})_x &= -V_{5,m+1} + (u-v)V_{5m} + V_{6m} - vV_{2m}, \\ (V_{5m})_x &= -2V_{4m}, \\ (V_{6m})_x &= 2V_{4,m+1} + (2v-2u)V_{4m} + 2vV_{1m}, \\ m &= 0, -1, -2, \dots, \end{aligned} \quad (44)$$

from which we have

$$\begin{aligned} (V_{5,m+1})_x &= \frac{1}{4} (V_{5m})_{xxx} \\ &\quad + \frac{1}{2} [(u-v)(V_{5m})_x + ((u-v)V_{5m})_x \\ &\quad - (V_{2m})_x - (vV_{2m})_x]. \end{aligned} \quad (45)$$

Take $V_{5,0} = V_{2,0} = V_{2,1} = 0$; we have

$$\begin{aligned} (V_{5,0})_x &= \frac{1}{4} (V_{5,-1})_{xxx} \\ &\quad + \frac{1}{2} [(u-v)(V_{5,-1})_x + ((u-v)V_{5,-1})_x \\ &\quad - (V_{2,-1})_x - (vV_{2,-1})_x]. \end{aligned} \quad (46)$$

Set $V_{5,-1} = -s$; one gets from (46) that

$$s_{xxx} + 2(u-v)s_x + (us - vs)_x + w_x + (vw)_x = 0. \quad (47)$$

Denote

$$\begin{aligned} V^{(-n)} &= \sum_{m=-n+1}^0 \left(\sum_{i=1}^6 V_{im} f_i(-n-m) \right) - V_{2,1-n} f_3(0) \\ &\quad - V_{5,1-n} f_6(0); \end{aligned} \quad (48)$$

one infers that

$$\begin{aligned} -V_x^{(-n)} + [U, V^{(-n)}] &= [(V_{2,1-n})_x - 2V_{1,1-n}] f_3(0) \\ &+ [(V_{5,1-n})_x - 2V_{4,1-n}] f_6(0). \end{aligned} \quad (49)$$

Thus, the 0-curvature equation

$$U_t - V_x^{(-n)} + [U, V^{(-n)}] = 0 \quad (50)$$

gives that

$$\begin{aligned} u_{t_{n-1}} &= 2(V_{2,1-n})_x, \\ v_{t_{n-1}} &= -2(V_{5,1-n})_x, \quad n = 1, 2, \dots \end{aligned} \quad (51)$$

We call (51) a negative-order integrable coupling of the negative-order KdV hierarchy. A reduced case of (51) presents that when $n = 1, 2$,

$$\begin{aligned} w_{xxx} + 4uw_x + 2u_x w &= 0, \quad u_t = 2w_x, \\ s_{xxx} + 2(u-v)s_x + (us-vs)_x + w_x + (vw)_x &= 0, \\ v_t &= -2s_x, \end{aligned} \quad (52)$$

which is the second negative-order integrable coupling of the negative-order KdV equation. As comparison, we rewrite the second equation in (42) as follows:

$$r_{xxt} + 4ur_t - 8w_x r - 4wr_x + 2u_x \int^x r_t dx = 0, \quad (53)$$

which is linear with respect to the dependent variable r . The second coupled part in (52) presents again

$$\begin{aligned} v_{xxt} + 3(u-v)v_t - 2w_x - 2wv_x - 2w_x v \\ + (u-v)_x \int^x v_t dx = 0, \end{aligned} \quad (54)$$

which is nonlinear in the variable v . Therefore, we obtain the linear integrable coupling of the negative-order KdV equation (28):

$$\begin{aligned} u_t = 2w_x, \quad w_{xxx} + 4uw_x + 2u_x w &= 0, \\ r_{xxt} + 4ur_t - 8w_x r - 4wr_x + 2u_x \int^x r_t dx &= 0, \end{aligned} \quad (55)$$

and the nonlinear integrable coupling:

$$\begin{aligned} u_t = 2w_x, \quad w_{xxx} + 4uw_x + 2u_x w &= 0, \\ v_{xxt} + 3(u-v)v_t - 2w_x - 2wv_x - 2w_x v \\ + (u-v)_x \int^x v_t dx &= 0. \end{aligned} \quad (56)$$

Remark 1. As previous statement, Qiao and Fan [18] have obtained the Hamiltonian structure of the negative-order KdV equation by using the trace identity proposed by Tu [23]. In what follows, we will deduce the Hamiltonian structure of the whole negative-order integrable coupling (51) by employing the variational identity.

3.3. The Hamiltonian Structure of the Nonlinear Integrable Couplings. Rewrite (43) as the forms of square matrices

$$U = \begin{pmatrix} h_3(1) - uh_3(0) + h_2(0) & & & \\ 0 & & & \\ & h_3(1) - uh_3(0) + h_2(0) + vh_3(0) & & \\ & & & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda - u & 0 & v & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda - u + v & 0 \end{pmatrix}, \quad (57)$$

$$V = \begin{pmatrix} \sum_{i=1}^3 V_i h_i & \sum_{i=4}^6 V_i h_{i-3} \\ 0 & \sum_{i=1}^3 V_i h_i + \sum_{i=4}^6 V_i h_{i-3} \end{pmatrix} = \begin{pmatrix} V_1 & V_2 & V_4 & V_5 \\ V_3 & -V_1 & V_6 & -V_4 \\ 0 & 0 & V_1 + V_4 & V_2 + V_5 \\ 0 & 0 & V_3 + V_6 & -V_1 - V_4 \end{pmatrix}, \quad (58)$$

where $V_i = \sum_{m=-\infty}^{-1} V_{im} \lambda^{-m}$, $i = 1, 2, \dots, 6$.

In order to employ the variational identity, we should establish a Lie algebra with column-vector elements. For that sake, we consider the linear space R^6 equipped with a kind of commutator as follows:

$$[a, b] = \begin{pmatrix} [a, b]_1 \\ [a, b]_2 \end{pmatrix}, \quad (59)$$

where

$$\begin{aligned} a &= \sum_{i=1}^6 a_i f_i, \\ b &= \sum_{i=1}^6 b_i f_i \in \tilde{G}_2, \end{aligned}$$

$$[a, b]_1 = (a_2 b_3 - a_3 b_2, 2a_1 b_2 - 2a_2 b_1, 2a_3 b_1 - 2a_1 b_3)^T,$$

$$[a, b]_2 = (a_2 b_6 - a_6 b_2 + a_5 b_3 - a_3 b_5 + a_5 b_6 - a_6 b_5,$$

$$2a_1 b_5 - 2a_5 b_1 + 2a_4 b_2 - 2a_2 b_4 + 2a_4 b_5 - 2a_5 b_4,$$

$$2a_3 b_4 - 2a_4 b_3 + 2a_6 b_1 - 2a_1 b_6 + 2a_6 b_4 - 2a_4 b_6)^T. \quad (60)$$

The commutator (59) can be written as

$$[a, b] = a^T R(b), \quad (61)$$

where

$$\begin{aligned}
 a^T &= (a_1, a_2, \dots, a_6), \\
 R(b) &= \begin{pmatrix} R_{11}(b) & R_{12}(b) \\ 0 & R_{22}(b) \end{pmatrix}, \\
 R_{11}(b) &= \begin{pmatrix} 0 & 2b_2 & -2b_3 \\ b_3 & -2b_1 & 0 \\ -b_2 & 0 & 2b_1 \end{pmatrix}, \\
 R_{12}(b) &= \begin{pmatrix} 0 & 2b_5 & -2b_6 \\ b_6 & -2b_4 & 0 \\ b_5 & 0 & 2b_4 \end{pmatrix}, \\
 R_{22}(b) &= \begin{pmatrix} 0 & 2b_2 + 2b_5 & -2b_3 - 2b_6 \\ b_3 + b_6 & -2b_1 - 2b_4 & 0 \\ -b_2 - b_5 & 0 & 2b_1 + 2b_4 \end{pmatrix}.
 \end{aligned} \tag{62}$$

A direct verification indicates that the linear space R^6 becomes a Lie algebra with the commutator (59). It is obvious that the linear map

$$\begin{aligned}
 \delta : G_2 &\longrightarrow R^6, \\
 A &\longrightarrow (a_1, a_2, \dots, a_6)^T
 \end{aligned} \tag{63}$$

is an isomorphism between the Lie algebras G_2 and R^6 , where the matrix $A = V$ is presented in (58). Therefore, we can rewrite the Lax matrices (57) and (58) as follows:

$$\begin{aligned}
 U &= (0, 1, \lambda - u, 0, 0, v)^T, \\
 V &= (V_1, V_2, \dots, V_6)^T,
 \end{aligned} \tag{64}$$

where $V_i = \sum_{m=-\infty}^{-1} V_{im} \lambda^{-m}$, $i = 1, 2, \dots, 6$.

The compatibility condition of (64) under the frame of the Lie algebra R^6 is equivalent to that of (43) under the Lie algebra G_2 .

According to the scheme for generating Hamiltonian structure of integrable couplings by employing the variational identity, a constant symmetric matrix F should be obtained which satisfies the matrix equation

$$R(b)F = -F(R(b))^T, \quad F^T = F. \tag{65}$$

A direct verification shows that

$$F = \begin{pmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & \eta_2 \\ 0 & \eta_2 & 0 & 0 & \eta_2 & 0 \end{pmatrix} \tag{66}$$

satisfies (65), where $\eta_1 \neq \eta_2$, constants. Define a linear functional

$$\{a, b\} = a^T F b. \tag{67}$$

With the help of (64), (66), and (67), we get that

$$\begin{aligned}
 \left\{ V, \frac{\partial U}{\partial u} \right\} &= -\eta_1 V_2 - \eta_2 V_5, \\
 \left\{ V, \frac{\partial U}{\partial v} \right\} &= \eta_2 (V_2 + V_5), \\
 \left\{ V, \frac{\partial U}{\partial \lambda} \right\} &= \eta_1 V_3 + \eta_2 V_6.
 \end{aligned} \tag{68}$$

Substituting the above consequences into the variational identity reads

$$\begin{aligned}
 \frac{\delta}{\delta \tilde{u}} \int^x (\eta_1 V_3 + \eta_2 V_6) dx \\
 = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{pmatrix} -\eta_1 V_2 - \eta_2 V_5 \\ \eta_2 (V_2 + V_5) \end{pmatrix},
 \end{aligned} \tag{69}$$

where $\delta/\delta \tilde{u} = (\delta/\delta u, \delta/\delta v)^T$.

Inserting $V_j = \sum_{m=-\infty}^{-1} V_{jm} \lambda^{-m}$, $j = 2, 3, 5, 6$, into (69) gives

$$\begin{aligned}
 \frac{\delta}{\delta \tilde{u}} \int^x \left(\eta_1 \sum_{m=-\infty}^{-1} V_{3m} \lambda^{-m} + \eta_2 \sum_{m=-\infty}^{-1} V_{6m} \lambda^{-m} \right) dx \\
 = (-m + \gamma) \begin{pmatrix} -\sum_{m=-\infty}^{-1} (\eta_1 V_{2m} + \eta_2 V_{5m}) \lambda^{-m-1} \\ \eta_2 \sum_{m=-\infty}^{-1} (V_{2m} + V_{5m}) \lambda^{-m-1} \end{pmatrix} \\
 = (1 + \gamma - m) \begin{pmatrix} -\sum_{m=-\infty}^{-1} (\eta_1 V_{2,m-1} + \eta_2 V_{5,m-1}) \lambda^{-m} \\ \sum_{m=-\infty}^{-1} (V_{2,m-1} + V_{5,m-1}) \lambda^{-m} \end{pmatrix} \\
 + (1 + \gamma) \begin{pmatrix} -\eta_1 V_{2,-1} - \eta_2 V_{5,-1} \\ \eta_2 (V_{2,-1} + V_{5,-1}) \end{pmatrix}, \quad m = -1, -2, \dots
 \end{aligned} \tag{70}$$

If taking $V_{2,-1} + V_{5,-1} \neq 0$, we have $\gamma = -1$. Hence, we get from (70) that

$$\begin{pmatrix} -\eta_1 V_{2,m-1} - \eta_2 V_{5,m-1} \\ \eta_2 (V_{2,m-1} + V_{5,m-1}) \end{pmatrix} = \frac{\delta H_m}{\delta \tilde{u}}, \tag{71}$$

where $H_m = -\int^x ((\eta_1 V_{3m} + \eta_2 V_{6m})/m) dx$ are the conserved densities of the negative-order integrable coupling (51).

Therefore, (51) can be written as the Hamiltonian structure form

$$\begin{aligned}
 \tilde{u}_{t_{n-1}} &= \begin{pmatrix} 2\partial_x (V_{2,1-n}) \\ -2\partial_x (V_{5,1-n}) \end{pmatrix} \\
 &= \begin{pmatrix} \frac{2\partial}{\eta_2 - \eta_1} & \frac{2\partial}{2\eta_2\partial} \\ \frac{2\partial}{\eta_2 - \eta_1} & \frac{2\partial}{\eta_2(\eta_2 - \eta_1)} \end{pmatrix} \begin{pmatrix} -\eta_1 V_{2,1-n} - \eta_2 V_{5,1-n} \\ \eta_2 (V_{2,1-n} + V_{5,1-n}) \end{pmatrix} \\
 &= J \begin{pmatrix} -\eta_1 V_{2,1-n} - \eta_2 V_{5,1-n} \\ \eta_2 (V_{2,1-n} + V_{5,1-n}) \end{pmatrix} \\
 &= J \frac{\delta H_{1-n}}{\delta \tilde{u}}, \quad n = 1, 2, \dots,
 \end{aligned} \tag{72}$$

where J is a Hamiltonian operator.

Remark 2. Two negative-order integrable couplings of the negative-order KdV equation were obtained. One of them is linear; the other is nonlinear. We can only generate the Hamiltonian structure of the nonlinear integrable coupling (51) by employing the variational identity; however, up to now, we have not found a suitable method to deduce the Hamiltonian structure of the linear negative-order integrable coupling (38). May its Hamiltonian structure not exist? which is worth of further discussing in forthcoming days. Besides, we observe that the recurrence relations in negative-order integrable couplings are all nonlocal. In addition, [27] presented some higher dimensional Lie algebras and were devoted to discussion of 2 + 1-dimensional integrable systems, which may be used to produce more interesting negative-order integrable hierarchies.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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