Research Article

Geometrical Applications of Split Octonions

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It is shown that physical signals and space-time intervals modeled on split-octonion geometry naturally exhibit properties from conventional $(3 + 1)$ -theory (e.g., number of dimensions, existence of maximal velocities, Heisenberg uncertainty, and particle generations). This paper demonstrates these properties using an explicit representation of the automorphisms on split-octonions, the noncompact form of the exceptional Lie group G_2 . This group generates specific rotations of $(3 + 4)$ -vector parts of split octonions with three extra time-like coordinates and in infinitesimal limit imitates standard Poincare transformations. In this picture translations are represented by noncompact Lorentz-type rotations towards the extra time-like coordinates. It is shown how the G_2 algebra's chirality yields an intrinsic left-right asymmetry of a certain 3-vector (spin), as well as a parity violating effect on light emitted by a moving quantum system. Elementary particles are connected with the special elements of the algebra which nullify octonionic intervals. Then the zero-norm conditions lead to free particle Lagrangians, which allow virtual trajectories also and exhibit the appearance of spatial horizons governing by mass parameters.

1. Introduction

Many properties of physical systems can be revealed from the analysis of proper mathematical structures used in descriptions of these systems. The geometry of space-time, one of the main physical characteristics of nature, can be understood as a reflection of symmetries of physical signals we receive and of the algebra used in the measurement process. Since all observable quantities we extract from single measurements are real, in geometrical applications it is possible to restrict ourselves to the field of real numbers. To have a transition from a manifold of the results of measurements to geometry, one must be able to introduce a distance between some objects and an etalon (unit element) for their comparison. In algebraic language all these physical requirements mean that to describe geometry we need a composition algebra with the unit element over the field of real numbers.

Besides usual real numbers, according to the Hurwitz theorem, there are three unique normed division algebras—complex numbers, quaternions, and octonions [1–3]. Physical applications of the widest normed algebra of octonions are relatively rare (see reviews [4–8]). One can point to

the possible impact of octonions on color symmetry [9–13]; GUTs [14–17]; representation of Clifford algebras [18–20]; quantum mechanics [21–29]; space-time symmetries [30, 31]; field theory [32–35]; quantum Hall effect [36]; Kaluza-Klein program without extra dimensions [37–42]; Strings and M-theory [43–48]; SUSY [49]; and so forth.

The essential feature of all normed composition algebras is the existence of a real unit element and a different number of hypercomplex units. The square of the unit element is always positive, while the squares of the hypercomplex basis units can be negative as well. In physical applications one mainly uses division algebras with Euclidean norms, whose hypercomplex basis elements have negative squares, similar to the ordinary complex unit i . The introduction of vector-like basis elements (with positive squares) leads to the split algebras with pseudo-Euclidean norms.

In this paper we propose parameterizing world-lines (paths) of physical objects by the elements of real split octonions [50–53]:

$$s = \omega + \lambda^n J_n + x^n j_n + ctI \quad (n = 1, 2, 3). \quad (1)$$

A pair of repeated upper and lower indices, by the standard convention, implies a summation; that is, $x^n j_n = \delta_{nm} x^n j^m$, where δ^{nm} is Kronecker's delta.

In geometric application four of the eight real parameters in (1), t and x^n , are space-time coordinates; ω and λ^n are interpreted as the phase (classical action) and the wavelengths associated with the octonionic signals [50–53].

The eight basis units in (1) are represented by one scalar (denoted by 1), the three vector-like objects J_n , the three pseudovector-like elements j_n , and one pseudoscalar-like unit I . The squares (inner products) of seven of the hypercomplex basis elements of split octonions give the unit element, 1, with the different signs:

$$\begin{aligned} J_n^2 &= 1, \\ j_n^2 &= -1, \\ I^2 &= 1. \end{aligned} \quad (2)$$

It is known that to generate a complete basis of split octonions the multiplication and distribution laws of only three vector-like elements J_n are enough [1–3]. The three pseudovector-like basis units, j_n , can be defined as the binary products,

$$j_n = \frac{1}{2} \varepsilon_{nmk} J^m J^k \quad (n, m, k = 1, 2, 3) \quad (3)$$

(ε_{nmk} is the totally antisymmetric unit tensor of rank three), and thus describes orthogonal planes spanned by two vector-like elements J_n . The seventh basis unit I (the oriented volume) is defined as the triple product of all three vector-like elements and has three equivalent representation in terms of J^n and j^n :

$$I = J_1 j_1 = J_2 j_2 = J_3 j_3. \quad (4)$$

So the algebra of all noncommuting hypercomplex basis units of split octonions has the form

$$\begin{aligned} J_n J_m &= -J_m J_n = \varepsilon_{nmk} J^k, \\ j_n j_m &= -j_m j_n = \varepsilon_{nmk} J^k, \\ j_m J_n &= -J_n j_m = \varepsilon_{nmk} J^k, \\ J_n I &= -I J_n = j_n, \\ j_n I &= -I j_n = J_n. \end{aligned} \quad (5)$$

Conjugations of octonionic basis units, which can be understood as the reflection of vector-like elements,

$$J_n^\dagger = -J_n, \quad (6)$$

reverses the order of J_n in products; that is,

$$\begin{aligned} j_n^\dagger &= \frac{1}{2} (\varepsilon_{nmk} J^m J^k)^\dagger = \frac{1}{2} \varepsilon_{nmk} J^{k\dagger} J^{m\dagger} = -j_n, \\ I^\dagger &= (J_1 J_2 J_3)^\dagger = J_3^\dagger J_2^\dagger J_1^\dagger = -I. \end{aligned} \quad (7)$$

So the conjugation of (1) gives

$$s^\dagger = \omega - \lambda_n J^n - x_n j^n - ctI. \quad (8)$$

Using (2), (5), and (8) one can find that the norm of (1) is given by

$$N^2 = ss^\dagger = s^\dagger s = \omega^2 - \lambda^2 + x^2 - c^2 t^2, \quad (9)$$

(where $\lambda^2 = \lambda_n \lambda^n$ and $x^2 = x_n x^n$), representing some kind of 8-dimensional space-time with (4+4)-signature and reducing to the classical formula of Minkowski intervals if $\omega^2 - \lambda^2 = 0$.

As for the case of ordinary Minkowski space-time, we assume that for physical events the corresponding “intervals” given by (9) are nonnegative. A second condition is that for physical signals the vector part of split octonions (1) should be time-like:

$$c^2 t^2 + \lambda_n \lambda^n > x_n x^n. \quad (10)$$

The classification of split octonions by the values of their norms is presented in Appendix A.

2. Transformations and Automorphisms

To find the geometry associated with the signals (1) we need to represent rotations by split octonions; due to the nonassociativity of the algebra the group of rotations of octonionic axes 1, J_n , j_n , and I , G_2^{NC} is not equivalent to the group of accompanied passive tensorial transformations of coordinates ω , λ_n , x_n , and t ($SO(4, 4)$). In Appendix B it is shown that by rotations in four orthogonal planes of octonionic space (one of which always includes the axis of real numbers ω) the octonion (1) always can be represented as the sum of four elements. The left multiplication, Rs , by an octonion with unit norm (Appendix A),

$$R = e^{\epsilon\theta}, \quad (11)$$

where ϵ is the (3+4)-vector defined by seven basis units J_n , j_n , and I , represents rotations by the angles θ in these four planes. The right product, sR^{-1} , will reverse the direction of rotation in the plane of ω . So we can represent a rotation, which only affects the (3+4)-vector part of s , applying the unit half-angle octonion twice and multiplying on both the left and the right (with its inverse):

$$s' = RsR^{-1} = RsR^\dagger = e^{\epsilon\theta/2} s e^{-\epsilon\theta/2}. \quad (12)$$

These maps are well defined, since the associator of the triplet (R, s, R^{-1}) vanishes. The set of rotations (12) of the seven octonionic coordinates x^n , λ^n , and t , in three planes, which do not affect the scalar part ω of (1), contains $7 \times 3 = 21$ angles of the group $SO(3, 4)$ of passive transformations of coordinates.

To represent the active rotations in the space of s , which preserve the norm (9) and multiplicative structure of octonions (5) as well, we would need the transformations to be automorphisms (Appendix C). An automorphism U of any two octonions, O_1 and O_2 , gives the equation

$$(UO_1U^{-1})(UO_2U^{-1}) = UO_1O_2U^{-1}, \quad (13)$$

which only holds in general if multiplication of U , O_1 , and O_2 is associative. Combinations of the rotations (12) around different octonionic axes are not unique. This means that not all transformations of $SO(3, 4)$ form a group and can be considered as real rotations. Only the transformations that have a realization as associative multiplications should be considered. Also note that an automorphism of split octonions can be generated only by the octonions with positive norms (Appendix A), since the set of split octonions with negative norms do not form a group (it is not closed under multiplication). So to model space-time symmetries we need to use the group of automorphisms of split octonions with positive norms.

It is known that associative transformations of split octonions can be done by the specific simultaneously rotations in two (and not in three as for $SO(3, 4)$) orthogonal octonionic planes (Appendix C). These rotations form a subgroup of $SO(3, 4)$ with $2 \times 7 = 14$ parameters, known as the automorphism group of split octonions, G_2^{NC} . Generators of this real noncompact form of Cartan's smallest exceptional Lie group G_2 is presented in Appendix D.

Infinitesimal transformations of coordinates which accompany the group of active transformations of octonionic basis units, G_2^{NC} , and preserve the diagonal quadratic form, $x^2 - \lambda^2 - dt^2 \geq 0$, can be written in the form (Appendix C)

$$\begin{aligned} x'_n &= x_n - \varepsilon_{nmk} \alpha^m x^k - \theta_n ct \\ &+ \frac{1}{2} (|\varepsilon_{nmk}| \phi^m + \varepsilon_{nmk} \theta^m) \lambda^k \\ &+ \left(\varphi_n - \frac{1}{3} \sum_m \varphi_m \right) \lambda_n, \\ ct' &= ct - \beta_n \lambda^n - \theta_n x^n, \\ \lambda'_n &= \lambda_n - \varepsilon_{nmk} (\alpha^m - \beta^m) \lambda^k + \beta_n ct \\ &+ \frac{1}{2} (|\varepsilon_{nmk}| \phi^m - \varepsilon_{nmk} \theta^m) x^k \\ &+ \left(\varphi_n - \frac{1}{3} \sum_m \varphi_m \right) x_n, \end{aligned} \quad (14)$$

with no summing over n in the last terms of x'_n and λ'_n . From the five 3-angles in (14), α^m , β^m , ϕ^m , θ^m , and φ^m , only 14 are independent because of the condition

$$\sum_n \left(\varphi_n - \frac{1}{3} \sum_m \varphi_m \right) = 0. \quad (15)$$

The Lorentz-type transformations (14) of the 7-dimensional space with four time-like coordinates should describe space-time symmetries if the split octonions form relevant algebraic structures for microphysics. The transformations (14) formally can be divided into three distinct classes (Appendix C): Euclidean rotations of the spatial, x_n , and time-like (t and λ_n) coordinates by the compact 3-angle

α^n and β^n , respectively; boosts, mixing of spatial and time-like coordinates by the two hyperbolic 3-angle θ^n and ϕ^n ; and diagonal boosts of the spatial coordinates, x_n , and corresponding time-like parameters, λ_n , by the hyperbolic angles φ^n .

We notice that if we consider rotations by the angles α^n , β^n , and θ^n , that is, assuming that

$$\phi^m = \varphi^m = 0, \quad (16)$$

the passive G_2^{NC} -transformations (14) of only ordinary space-time coordinates, x_n and t , will imitate the ordinary infinitesimal Poincaré transformations of $(3 + 1)$ -Minkowski space:

$$\begin{aligned} x'_n &= x_n - \varepsilon_{nmk} \alpha^m x^k - \theta_n ct + a_n, \\ ct' &= ct - \theta_n x^n + a_0. \end{aligned} \quad (17)$$

Here the space-time translations,

$$\begin{aligned} a_n &= \frac{1}{2} \varepsilon_{nmk} \theta^m \lambda^k, \\ a_0 &= -\beta_n \lambda^n, \end{aligned} \quad (18)$$

are generated by the Lorentz-type rotations toward the time-like directions λ_n . So in the language of octonionic geometry any motion in ordinary space-time is generated by $\lambda^n \sim p^n/p^2$. Time translations a_0 are smooth, since β_n are compact angles. However, the angles θ^m are hyperbolic and for any active spatial translation a_n there exists a horizon (analogues to the Rindler horizon), which is equivalent to the introduction of some mass scale or inertia (see Section 6).

For completeness note that there exists a second well-known representation of G_2^{NC} as the symmetry group of a ball rolling on a larger fixed ball without slipping or twisting, when the ratio of the balls radii is $1/3$ [54–58]. Understanding the exceptional Lie groups as the symmetry groups of naturally occurring objects is a long-standing program in mathematics. Symmetries of rolling balls are visualized in (14) by the presence of two 3-vector, x_n and λ_n , one of which is time-like. The fourth time-like coordinate (ordinary time t), which also is affected by (14), breaks the symmetry between “balls” and the factor $1/3$ of the ratio of their radii corresponds to the existence of the three extra time-like coordinates λ_n .

3. Boosts, Heisenberg Uncertainty, and Chirality

In the algebra of split octonions there exist only three space-like parameters, x_n , and three independent compact rotations around corresponding axes, which explains why physical space described by octonionic signals has three spatial dimensions. Let us analyze new features of the G_2^{NC} -transformations (14) in comparison with standard Lorentz's formulas for $(3 + 1)$ -Minkowski space.

Euclidean rotations around one of the space-like axes, x_1 , the automorphism (C.1), correspond to the following passive infinitesimal transformations of coordinates:

$$\begin{aligned} x'_1 &= x_1, \\ x'_2 &= x_2 + \alpha_1 x_3, \\ x'_3 &= x_3 - \alpha_1 x_2, \\ ct' &= ct - \beta_1 \lambda_1, \\ \lambda'_1 &= \lambda_1 + \beta_1 ct, \\ \lambda'_2 &= \lambda_2 + (\alpha_1 - \beta_1) \lambda_3, \\ \lambda'_3 &= \lambda_3 - (\alpha_1 - \beta_1) \lambda_2. \end{aligned} \quad (19)$$

When $\alpha_1 = \beta_1$ this reduces to the standard rotation by the Euler angle α_1 , which also causes translations of time, t , due to mixing with the extra time-like parameter λ_1 . In general, any active Euclidean 3-rotation of the spatial coordinates x^n changes the time parameter by the amount of the corresponding λ_n , which can be understood as passing of time in our world.

Analysis of boosts of G_2^{NC} can help us in the physical interpretation of the extra time-like parameters λ_n . Consider the automorphisms (C.2) by the hyperbolic angles θ_1 and ϕ_1 :

$$\begin{aligned} x'_1 &= x_1 - \theta_1 ct, \\ x'_2 &= x_2 + \frac{1}{2} (\phi_1 - \theta_1) \lambda_3, \\ x'_3 &= x_3 + \frac{1}{2} (\phi_1 + \theta_1) \lambda_2, \\ ct' &= ct - \theta_1 x_1, \\ \lambda'_1 &= \lambda_1, \\ \lambda'_2 &= \lambda_2 + \frac{1}{2} (\phi_1 + \theta_1) x_3, \\ \lambda'_3 &= \lambda_3 + \frac{1}{2} (\phi_1 - \theta_1) x_2. \end{aligned} \quad (20)$$

The case $\phi_1 = \theta_1$ corresponds to ordinary boost in (t, x) -planes—transitions to the reference frame moving with the velocity θ_1 along the axis x_1 . Then if we consider the motion of the origin of the moving system,

$$x' = y' = z' = 0, \quad (21)$$

from the first line in (20) we find that

$$\begin{aligned} \theta_1 &= \frac{x_1}{ct}, \\ \lambda_3 &= \frac{x_2}{\theta_1}, \\ \lambda_2 &= -\frac{x_3}{\theta_1}; \end{aligned} \quad (22)$$

that is, the quantities λ_2 and λ_3 are inversely proportional to the velocity, x_1/t . In the space of split octonions (1) there are two classes of time-like parameters (t and λ_n) and two different light-cones. So, in addition to c , there must exist the second fundamental constant, which can be extracted from λ_n . In quantum mechanics the quantity with the dimension of length, which is proportional to a fundamental physical constant and is inversely proportional to velocity (or momentum, p_n), is called the wavelength. So it is natural to assume that

$$\lambda^n = \hbar \frac{p^n}{p^2} \sim \frac{\hbar}{p_n}, \quad (23)$$

where $\hbar$ is the Planck constant and p^n is the 3-momentum associated with the octonionic signal. So in our approach the two fundamental physical constants, c and $\hbar$, have geometrical origin and correspond to two kinds of light-cone signals in the space of split octonions (9).

When $\phi_1 = 0$ from (20) we find that the ratios,

$$\begin{aligned} \frac{\Delta x_2}{\Delta \lambda_2} &= -\frac{\lambda_3}{x_3}, \\ \frac{\Delta x_3}{\Delta \lambda_3} &= -\frac{\lambda_2}{x_2}, \end{aligned} \quad (24)$$

(where $\Delta x_n = x'_n - x_n$ and $\Delta \lambda_n = \lambda'_n - \lambda_n$), are unchanged under infinitesimal transformations (20). Similar relations can be obtained for the boosts along two other space-like axes. From the invariance of octonionic norms (B.14) we know that

$$|x_n| \geq |\lambda_n|. \quad (25)$$

Then inserting (23) and (25) into (24) we can conclude that the uncertainty relations,

$$\Delta x_n \Delta p_n \geq \hbar, \quad (26)$$

in our model have the same geometrical meaning as the existence of the maximal velocity, c [50–53, 59].

From (20) we also notice that in the planes orthogonal to $v_1 = x_1/t$ the pair x'_2, λ'_3 increases, while x'_3, λ'_2 decreases, and for the case (21) we have

$$\frac{x_2}{\lambda_3} = -\frac{x_3}{\lambda_2} = \theta_1. \quad (27)$$

Using similar relations for the boosts along other two spatial directions we conclude that there exists some 3-vector (spin):

$$\sigma_n = \frac{1}{\hbar} \varepsilon_{nmk} x^m p^k \sim \varepsilon_{nmk} \frac{x^m}{\lambda_k}, \quad (28)$$

which characterizes relative rotations of x_n and λ_n in the direction orthogonal to the velocity planes. One can speculate on the connection of the left-right asymmetry in relative rotations of x_n and λ_n in (20) with the right-handed neutrino problem for massless case.

4. Parity Violation

For the reference frame (21), for the finite angles $\phi_1 = \theta_1$, from (20) one can obtain the standard relativistic expressions:

$$\begin{aligned}\tanh \theta_1 &= \frac{\mathcal{V}}{c}, \\ \cosh \theta_1 &= \frac{1}{\sqrt{1 - \mathcal{V}^2/c^2}}, \\ \sinh \theta_1 &= \frac{\mathcal{V}/c}{\sqrt{1 - \mathcal{V}^2/c^2}},\end{aligned}\quad (29)$$

where $\mathcal{V}$ is the velocity of moving system along x_1 . Then from (20) there follows the generalized rule of velocity addition:

$$\begin{aligned}v'_1 &= \frac{v_1 - c \tanh \theta_1}{1 - \tanh \theta_1 (v_1/c)}, \\ v'_2 &= \frac{v_2 - \tanh \theta_1 \dot{\lambda}_3}{1 - \tanh \theta_1 (v_1/c)}, \\ v'_3 &= \frac{v_3 + \tanh \theta_1 \dot{\lambda}_2}{1 - \tanh \theta_1 (v_1/c)}.\end{aligned}\quad (30)$$

We see that the standard expressions are altered only in the planes orthogonal to $\mathcal{V}$ and that what is important are the terms with different sign. One consequence of this fact is that the formula for the aberration of light will be modified.

Consider photons moving in the (x_1, x_2) -plane:

$$\begin{aligned}v_1 &= c \cos \gamma_{12}, \\ v_2 &= c \sin \gamma_{12}, \\ v'_1 &= c \cos \gamma'_{12}, \\ v'_2 &= c \sin \gamma'_{12},\end{aligned}\quad (31)$$

where γ_{12} and γ'_{12} are angles between v_1 and v'_1 and the axes x_1 and x'_1 , respectively. Using (30), for the case $\mathcal{V}/c \ll 1$, we have

$$\begin{aligned}\sin \gamma'_{12} &= \frac{\sin \gamma_{12} - (\mathcal{V}/c^2) \dot{\lambda}_3}{1 - (\mathcal{V}/c) \cos \gamma_{12}}, \\ \cos \gamma'_{12} &= \frac{\cos \gamma_{12} - \mathcal{V}/c}{1 - (\mathcal{V}/c) \cos \gamma_{12}},\end{aligned}\quad (32)$$

where $\dot{\lambda}_3$ is the rate of Doppler's shift along an axis, x_3 , orthogonal to the photon's trajectory. Then angle of aberration in the (x_1, x_2) -plane give the value

$$\Delta \gamma_{12} = \gamma'_{12} - \gamma_{12} = \frac{\mathcal{V}}{c} \sin \gamma_{12} - \frac{\mathcal{V}}{c^2} \dot{\lambda}_3. \quad (33)$$

Analogous to (33) the formula for the aberration angle in the (x_1, x_3) -plane has the form

$$\Delta \gamma_{13} = \frac{\mathcal{V}}{c} \sin \gamma_{13} + \frac{\mathcal{V}}{c^2} \dot{\lambda}_2, \quad (34)$$

where $\dot{\lambda}_2$ is the rate of the Doppler shift towards the orthogonal to the (x_1, x_3) -plane direction. This spatial asymmetry of aberration, which distinguishes the left and right coordinate systems, may be detectable by precise quantum measurements.

5. Spin and Hypercharge

Now consider the last class of automorphisms (Appendix C):

$$\begin{aligned}x'_n &= x_n + \left(\varphi_n - \frac{1}{3} \sum_m \varphi_m \right) \lambda_n, \\ t' &= t, \\ \lambda'_n &= \lambda_n + \left(\varphi_n - \frac{1}{3} \sum_m \varphi_m \right) x_n\end{aligned}\quad (35)$$

with no summing by n . These transformations represent rotations of the three pairs of space-like and time-like coordinates, (x_1, λ_1) , (x_2, λ_2) , and (x_3, λ_3) , into each other. We have the three planes, $(x_1 - \lambda_1)$, $(x_2 - \lambda_2)$, and $(x_3 - \lambda_3)$, that undergo rotations through hyperbolic angles, φ_1 , φ_2 , and φ_3 (of the three only two are independent), which are the only hyper-planes in the space of split octonions that are affected by one and not two 3-angles of automorphisms. So it is natural to define the Abelian subalgebra of G_2^{NC} by generators of two independent rotations in these planes. It is known that the rank of the Cartan subalgebra of G_2^{NC} is the same as of the group $SU(3)$ [9–11, 60]. Indeed, in terms of two parameters, K_1 and K_2 , which are related to the angles φ_n and (C.4) as

$$\begin{aligned}K_1 &= k_1 = \frac{1}{3} (2\varphi_1 - \varphi_2 - \varphi_3), \\ K_2 &= \frac{\sqrt{3}}{2} (k_1 + k_1) = -\frac{1}{2\sqrt{3}} (2\varphi_3 - \varphi_1 - \varphi_2),\end{aligned}\quad (36)$$

the transformations (35) can be written more concisely:

$$\begin{pmatrix} \lambda'_1 + Ix'_1 \\ \lambda'_2 + Ix'_2 \\ \lambda'_3 + Ix'_3 \end{pmatrix} = e^{(K_1 \Lambda_3 + K_2 \Lambda_8)I} \begin{pmatrix} \lambda_1 + Ix_1 \\ \lambda_2 + Ix_2 \\ \lambda_3 + Ix_3 \end{pmatrix}, \quad (37)$$

where I is the vector-like octonionic basis unit ($I^2 = 1$) and Λ_3 and Λ_8 are the standard 3×3 Gell-Mann matrices:

$$\begin{aligned}\Lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \Lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.\end{aligned}\quad (38)$$

Then, using analogies with $SU(3)$, one can classify irreducible representations of the space-time group of our model, G_2^{NC} , by the two fundamental simple roots (K_1 and K_2)

corresponding to spin and hypercharge. It is known that all quarks, antiquarks, and mesons can be imbedded in the adjoint representation of G_2^{NC} [9–11]. So the symmetry (35), in addition to the uncertainty relations, probably shows the existence of three generations of objects which are necessary to extract three spatial coordinates from any octonionic signal s . To clarify this point note that (9) can be viewed as some kind of space-time interval with four time-like dimensions. The ordinary time parameter, t , corresponds to the distinguished octonionic basis unit, I , while the other three time-like parameters, λ_n , have a natural interpretation as wavelengths; that is, they do not relate to time as conventionally understood. It is known that a unique physical system in multidimensional geometry generates a large variety of “shadows” in $(3 + 1)$ -subspace as different dynamical systems (in terms of different Hamiltonians) [61–70]. The information of multidimensional structures, which is retained by these images of the initial system, takes the form of hidden symmetries. For the case of fundamental physical signals with three extra time-like dimensions, in addition to massless particles (which are not affected by extra times), one can observe three generations of particles with different mass (corresponding to rotations of t in one, two, or all three extra time-like planes); see (E.7). Note the factor $1/3$ in front of ω (action) in the first equation (E.7) that appears due to the existence of the three extra time-like parameters, λ_n . Then three independent (ω, λ_n) -rotations in 8-dimensional octonionic space will give the appearance of three generations of particles in ordinary $(3 + 1)$ -dimensions. Such a possibility is not ruled out by problems with ghosts and with unitarity, since split octonions with positive norms form the division algebra.

6. Free Particle Lagrangians

It is known that in split algebras there can be constructed special elements with zero norms, which are called zero divisors [1]. These zero-norm objects are important structures in physical applications [71]. Zero divisors (light-cone operators) of split octonions could serve as the unit signals and thus may describe elementary particles. Then the number of primitive idempotents (eight) and nilpotents (twelve) numerates the types of fundamental particles (Appendix E), bosons, and fermions, respectively. Indeed, the properties that the product of two projection operators reduces to the same idempotent (E.2), while the product of two Grassmann numbers is zero (E.4) naturally explain the validity of Bose and Fermi statistics for the corresponding elementary particles.

The zero-norm condition, $N^2 = 0$, for the paths (9) of elementary particles, be they massless or massive, can be written in the form

$$ds^2 = ds ds^\dagger = d\omega^2 - c^2 dt^2 \left(1 - \frac{v^2}{c^2} + \frac{\dot{\lambda}^2}{c^2} \right) = 0. \quad (39)$$

For photons, $|v| = c$, the condition (39) is equivalent to the Eikonal equation:

$$\frac{d\omega}{d\lambda_n} \frac{d\omega}{d\lambda^n} = 1, \quad (40)$$

where the wavelengths, $\lambda_n \sim \hbar/p_n$, serve as the parameters of trajectory. The relation (40) justifies the interpretation of ω as the phase function (frequency) corresponding to the classical action of particles.

For massive particles the time coordinate, t , is good parameter to describe motion and from (39) we find that the scalar part of a split octonionic signal, ω , which is unchanged under automorphisms of G_2^{NC} , is the conserved function:

$$\frac{d\omega}{dt} = 0. \quad (41)$$

Then (39) can be written as

$$\omega = \frac{A}{mc} = -c \int dt \sqrt{1 - \frac{v^2}{c^2} + \frac{\dot{\lambda}^2}{c^2}}, \quad (42)$$

where A is the classical action of the particle with the mass m . So using (23) the one-particle Lagrangian,

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2} + \hbar^2 \frac{\dot{p}^2}{c^2 p^4}}, \quad (43)$$

contains an extra “quantum” term, which may be relevant for the relativistic velocities, $|v| \sim c$, or for the case of large forces:

$$\dot{p}_n \sim \frac{cp^2}{\hbar}. \quad (44)$$

From (43) we see that on small distance scales, $\sim \hbar/|p|$, a particle can even exceed the speed of light (become virtual), since the new quantum term in (43) is positive. The condition (39) and the invariance of the classical action (42) under G_2^{NC} hold only for real trajectories. However, there exists the larger group of invariances of the interval (9), $SO(4, 4)$, the group of passive tensorial transformations of all eight octonionic coordinates, which in general mixes space-like and time-like subspaces and thus introduces virtual trajectories of particles.

Applying the condition (10) (that for physical signals the vector part of a split octonion should be time-like) to (43) it follows that there should exist some maximal force:

$$|\dot{p}| \leq |v| \frac{p^2}{\hbar} = m^2 \frac{c^3}{\hbar}, \quad (45)$$

where m denotes the maximal possible value for the mass of particle associated with physical signals. Using estimations of [72, 73] for the maximum force, $F_{\max} = c^4/4G$, where G is the gravitational constant; from (45) we find that the maximal mass of particles associated with any octonionic signal is the Planck mass:

$$m^2 \sim \frac{\hbar c}{G}. \quad (46)$$

7. Conclusion

To conclude, in this paper, it was analyzed consequences of describing physical signals in terms of split octonions over the field of real numbers. Eight real parameters of

split octonions were related to space-time coordinates, the phase function (classical action), and wavelengths characterizing physical signals. The $(4 + 4)$ -norms of the real split octonions are invariant under the group $SO(4, 4)$ of tensorial transformations of parameters (passive transformations of coordinates), while the representation of rotations by octonions themselves (i.e., the active transformations of octonionic basis units) corresponds to the real noncompact form of Cartan's exceptional Lie group G_2^{NC} (the subgroup of $SO(4, 4)$), which under certain conditions imitate the Poincaré transformations in ordinary space-time. Within this picture, in front of time-like coordinates in the expression of pseudo-Euclidean octonionic intervals, there naturally appear two fundamental physical parameters, the light speed and Planck's constant. From the requirement of positive definiteness of norms under G_2^{NC} -transformations, together with the introduction of the maximal velocity, c , there follow conditions which are equivalent to uncertainty relations in quantum physics. By examining of G_2^{NC} -automorphisms it is explicitly shown that there exists a 3-vector (spin) that is fully chiral, which corresponds to the observed chirality of neutrinos in nature. We examined also the G_2^{NC} -effects on a quantum system moving at constant speed with respect to the observer, which emits light in various planes relative to the direction of motion, and new parity-violating effect on aberration and Doppler shift was derived. It was shown how a particular class of G_2^{NC} -automorphisms of space-time coordinates can be expressed in concise form using two simple roots of the $SU(3)$ Lie algebra. Due to their conventional association with spin and hypercharge in physics, we argued that our split-octonion geometry inherently modeled three generations of fundamental particles. In our approach elementary particles are interpreted as light-cone operators; that is, they are connected with the special elements of the algebra which nullify octonionic intervals. Then the zero-norm conditions in G_2^{NC} -geometry lead to corresponding free particle Lagrangians, which allow virtual particle trajectories (exceeding the speed of light, in certain cases) and exhibit the existence of spatial horizons governing by mass parameters.

Appendices

A. Classification of Split Octonions

The algebra of split octonions is a noncommutative, nonassociative, nondivision ring. Any element (1) of the ring can be represented as

$$\begin{aligned} s &= \omega + V, \\ s^\dagger &= \omega - V, \end{aligned} \quad (\text{A.1})$$

where the symbols ω and

$$V = \lambda^n J_n + x^n j_n + ctI \quad (n = 1, 2, 3) \quad (\text{A.2})$$

are called the scalar and vector parts of octonion, respectively. Similar to the case of split quaternions [59, 74], there exist

three classes of split octonions having positive, negative, or zero norm:

$$N^2 = ss^\dagger = s^\dagger s = \omega^2 - V^2, \quad (\text{A.3})$$

where

$$V^2 = -x^2 + c^2 t^2 + \lambda^2. \quad (\text{A.4})$$

In addition one can distinguish split octonions with time-like, space-like, and light-like vector parts (A.2):

$$\begin{aligned} V^2 &< 0, & (\text{space-like}) \\ V^2 &> 0, & (\text{time-like}) \\ V^2 &= 0, & (\text{light-like}). \end{aligned} \quad (\text{A.5})$$

Split octonions with nonzero norms, $\omega^2 \neq V^2$, have multiplicative inverses,

$$s^{-1} = \frac{s^\dagger}{N^2}, \quad (\text{A.6})$$

with the property

$$ss^{-1} = s^{-1}s = 1, \quad (\text{A.7})$$

while octonions with zero norm, $\omega^2 = V^2$, have no inverses.

One can construct also the polar form of a split octonion with nonzero norm:

$$s = Ne^{\epsilon\theta}, \quad (\text{A.8})$$

where ϵ is a unit 7-vector ($\epsilon^2 = \pm 1$). For $N \neq 0$ the quantity

$$R = \frac{s}{N} \quad (\text{A.9})$$

is called unit octonion, which is useful to represent rotations of $(3 + 4)$ -vector spaces (A.2).

Depending on the values of (A.3) and (A.4) split octonions have three different representations:

(i) Every split octonion with negative norm, $\omega^2 < V^2$, can be written in the form

$$s = N (\sinh \theta + \epsilon \cosh \theta), \quad (\text{A.10})$$

where

$$\begin{aligned} \sinh \theta &= \frac{\omega}{N}, \\ \cosh \theta &= \frac{|V|}{N}, \\ \epsilon &= \frac{\lambda^n J_n + x^n j_n + ctI}{|V|}, \end{aligned} \quad (\text{A.11})$$

and ϵ is a unit time-like 7-vector.

(ii) Every split octonion with positive norm, $\omega^2 > V^2$, and with time-like vector part, $V^2 > 0$, can be written in the form

$$s = N (\cosh \theta + \epsilon \sinh \theta), \quad (\text{A.12})$$

where

$$\begin{aligned}\cosh \theta &= \frac{\omega}{N}, \\ \sinh \theta &= \frac{|V|}{N}, \\ \epsilon &= \frac{\lambda^n J_n + x^n j_n + ctI}{|V|},\end{aligned}\quad (\text{A.13})$$

so ϵ again is a unit time-like 7-vector.

(iii) Every split octonion with positive norm, $\omega^2 > V^2$, and with space-like vector part, $V^2 < 0$, can be written in the form

$$s = N (\cos \theta + \epsilon \sin \theta), \quad (\text{A.14})$$

where

$$\begin{aligned}\cos \theta &= \frac{\omega}{N}, \\ \sin \theta &= \frac{|V|}{N}, \\ \epsilon &= \frac{\lambda^n J_n + x^n j_n + ctI}{|V|},\end{aligned}\quad (\text{A.15})$$

thus ϵ now is a unit space-like 7-vector.

The norm of a split octonion (A.3) contains two types of “light-cones,” $N = 0$ and $V = 0$, and we need two constants that characterize the maximum spread angles of these cones. The first fundamental parameter, the speed of light c , appears in standard Lorentz boosts for the time-like signals, $V^2 > 0$. The second fundamental constant, $\hbar$, corresponds to maximal rotations in (ω, V) -planes, when physical events have $N^2 \geq 0$ and still can be described by the actions $\sim \omega$.

B. One-Side Products

The 8-dimensional octonionic space (1) always can be represented as the sum of four elements by rotations in four orthogonal planes (see (B.5), (B.10), and (B.14) below), one of which always includes the axis of real numbers ω [75]. Then it can be shown that multiplication of one octonion by another octonion of unit norm (A.9) represents some kind of rotation in these four planes. In the algebra of split octonions there exist seven basis units, J_n , j_n , and I , which define the 7-vector ϵ in the formulas for one-sided (e.g., the left) products:

$$Rs = e^{\epsilon\theta/2} s. \quad (\text{B.1})$$

So totally we have $4 \times 7 = 28$ parameters of the group $SO(4, 4)$, which preserve the norm (9) [75]. Octonionic hypercomplex basis units have different algebraic properties and lead to the three distinct classes of the rotation operator R :

(i) The three pseudovector-like basis elements, j_n , behave as the ordinary complex unit ($j_n^2 = -1$) and rotations around the corresponding spatial axes, x_n , can be done by the unit octonions with $N^2 > 0$ having space-like vector part:

$$e^{j_n \alpha_n} = \cos \alpha_n + j_n \sin \alpha_n, \quad (\text{B.2})$$

where α_n are three real compact angles. To show this explicitly let us consider the rotation operator R in (B.1) when $\epsilon = j_1$. In the octonionic algebra (5) the basis unit j_1 has three different representations by other basis elements that form an associative triplets with j_1 :

$$j_1 = J_2 J_3 = j_2 j_3 = J_1 I \quad (\text{B.3})$$

or, equivalently,

$$\begin{aligned}j_1 J_3 &= J_2, \\ j_1 j_2 &= j_3, \\ j_1 I &= J_1.\end{aligned}\quad (\text{B.4})$$

Then it is possible to “rotate out” four octonionic axes and rewrite (1) in the equivalent form

$$\begin{aligned}s &= \sqrt{\omega^2 + x_1^2} e^{j_1 \alpha_\omega} + \sqrt{\lambda_2^2 + \lambda_3^2} e^{j_1 \alpha_\lambda} J_3 \\ &\quad + \sqrt{x_2^2 + x_3^2} e^{j_1 \alpha_x} j_2 + \sqrt{t^2 + \lambda_1^2} e^{j_1 \alpha_t} I,\end{aligned}\quad (\text{B.5})$$

where four angles are given by

$$\begin{aligned}\cos \alpha_\omega &= \frac{\omega}{\sqrt{\omega^2 + x_1^2}}, \\ \cos \alpha_\lambda &= \frac{\lambda_3}{\sqrt{\lambda_2^2 + \lambda_3^2}}, \\ \cos \alpha_x &= \frac{x_2}{\sqrt{x_2^2 + x_3^2}}, \\ \cos \alpha_t &= \frac{ct}{\sqrt{t^2 + \lambda_1^2}}.\end{aligned}\quad (\text{B.6})$$

Now it is obvious that when $R = e^{j_1 \alpha_1}$ and s is represented by (B.5), the product (B.1) leads to simultaneous rotations in the four planes, (ω, x_1) , (λ_2, λ_3) , (x_2, x_3) , and (λ_1, t) , by the same compact angle α_1 . Similar results can be obtained for the rotations by the other two pseudovector-like units, j_2 and j_3 .

(ii) For the three vector-like basis elements, J_n , with the positive squares, $J_n^2 = 1$, we need the unit octonions with $N^2 > 0$ having time-like vector part:

$$e^{J_n \theta_n} = \cosh \theta_n + J_n \sinh \theta_n, \quad (\text{B.7})$$

where θ_n are three real hyperbolic angles. Let us consider the example when $\epsilon = J_1$ in (B.1). The equivalent representations of J_1 in the algebra (5) are

$$J_1 = J_2 j_3 = -J_3 j_2 = j_1 I \quad (\text{B.8})$$

or

$$\begin{aligned}J_1 J_2 &= -j_3, \\ J_1 j_2 &= J_3, \\ J_1 I &= j_1.\end{aligned}\quad (\text{B.9})$$

Then (1) can be rewritten as

$$s = \sqrt{\omega^2 - \lambda_1^2} e^{J_1 \theta_\omega} + \sqrt{x_3^2 - \lambda_2^2} e^{-J_1 \theta_\lambda} J_2 + \sqrt{x_2^2 - \lambda_3^2} e^{J_1 \theta_x} j_2 + \sqrt{x_1^2 - t^2} e^{J_1 \theta_t} I, \quad (\text{B.10})$$

where the angles are given by

$$\begin{aligned} \cosh \theta_\omega &= \frac{\omega}{\sqrt{\omega^2 - x_1^2}}, \\ \cosh \theta_\lambda &= \frac{\lambda_2}{\sqrt{x_3^2 - \lambda_2^2}}, \\ \cosh \theta_x &= \frac{x_2}{\sqrt{x_2^2 - \lambda_3^2}}, \\ \cosh \alpha_t &= \frac{t}{\sqrt{x_1^2 - t^2}}. \end{aligned} \quad (\text{B.11})$$

So when $R = e^{J_1 \theta_1}$, the product (B.1) corresponds to the simultaneous rotations in the planes (ω, λ_1) , (λ_2, x_3) , (x_2, λ_3) , and (x_1, t) by the hyperbolic angle θ_1 . We have similar results for the rotations by the other two vector-like units, J_2 and J_3 .

(iii) The last vector-like basis element, I , also has the positive square, $I^2 = 1$, and to represent corresponding transformations we need the unit octonion with $N^2 > 0$ having time-like vector part:

$$e^{I\sigma} = \cosh \sigma + I \sinh \sigma, \quad (\text{B.12})$$

where σ is the real hyperbolic angle. When in (B.1) we put $\epsilon = I$, we can use the expressions (4) of the equivalent representations of I or

$$\begin{aligned} Ij_1 &= -J_1, \\ Ij_2 &= -J_2, \\ Ij_3 &= -J_3. \end{aligned} \quad (\text{B.13})$$

Then the split octonion (1) can be rewritten as

$$s = \sqrt{\omega^2 - t^2} e^{I\sigma_\omega} + \sqrt{x_1^2 - \lambda_1^2} e^{-I\sigma_1} j_1 + \sqrt{x_2^2 - \lambda_2^2} e^{-I\sigma_2} j_2 + \sqrt{x_3^2 - \lambda_3^2} e^{-I\sigma_3} j_3, \quad (\text{B.14})$$

where four hyperbolic angles are given by

$$\begin{aligned} \cosh \sigma_\omega &= \frac{\omega}{\sqrt{\omega^2 - t^2}}, \\ \cosh \sigma_1 &= \frac{x_1}{\sqrt{x_1^2 - \lambda_1^2}}, \\ \cosh \sigma_2 &= \frac{x_2}{\sqrt{x_2^2 - \lambda_2^2}}, \\ \cosh \sigma_3 &= \frac{x_3}{\sqrt{x_3^2 - \lambda_3^2}}. \end{aligned} \quad (\text{B.15})$$

So the left product (B.1), when $R = e^{I\sigma}$, corresponds to the simultaneous rotations in the planes (ω, t) , (x_1, λ_1) , (x_2, λ_2) , and (x_3, λ_3) by the hyperbolic angle σ .

So the one-side products (B.1) of an octonion s by each of the seven operators (B.2), (B.7), and (B.12) yield simultaneous rotations in four mutually orthogonal planes of the octonionic $(4 + 4)$ -space by the same angles [75]. One of these four planes is formed by the unit element, 1, and the hypercomplex basis unit that defines the axis of rotation. The remaining orthogonal planes are given by three pairs of other basis elements which form associative triplets with the chosen basis unit.

The decomposition of a split octonion in the form (B.5) is valid only if its norm (9) is positive. In contrast with uniform rotations around j_n , we have limited rotations, (B.10) and (B.14), around J_n and I . For these cases we should require also positiveness of the 2 norms of each of the four planes of rotations. For example, for (B.14), the expressions of 2 norms of four orthogonal planes are $\sqrt{\omega^2 - t^2}$, $\sqrt{x_1^2 - \lambda_1^2}$, $\sqrt{x_2^2 - \lambda_2^2}$, and $\sqrt{x_3^2 - \lambda_3^2}$. These hyperbolic properties are the result of the existence of zero divisors in split algebras (Appendix E).

C. Automorphisms and G_2

From the expressions of a hypercomplex unit of split octonions by two other basis elements, which form associative triplets with the selected unit, one can find orthogonal to it planes of rotations. An automorphism (13) for each basis unit introduces two angles of rotation in these planes. So there are seven independent automorphisms each by two angles that correspond to $2 \times 7 = 14$ generators of the algebra G_2^{NC} . For example, the automorphism by the two real compact angles, α_1 and β_1 , which leaves unchanged the basis unit $j_1 = J_2 J_3 = j_2 j_3 = J_1 I$, has the form

$$\begin{aligned} j'_1 &= j_1, \\ j'_2 &= j_2 \cos \alpha_1 + j_3 \sin \alpha_1, \\ j'_3 &= j'_1 j'_2 = j_3 \cos \alpha_1 - j_2 \sin \alpha_1, \\ J'_1 &= J_1 \cos \beta_1 + I \sin \beta_1, \\ I' &= J'_1 j'_1 = I \cos \beta_1 - J_1 \sin \beta_1, \\ J'_2 &= j'_2 I' = J_2 \cos(\alpha_1 - \beta_1) + J_3 \sin(\alpha_1 - \beta_1), \\ j'_3 &= j'_3 I' = J_3 \cos(\alpha_1 - \beta_1) - J_2 \sin(\alpha_1 - \beta_1). \end{aligned} \quad (\text{C.1})$$

It is known that automorphisms in the octonionic algebra are completely defined by the images of three basis elements that do not form associative subalgebras (j_1 , j_2 , and J_1 in the case of (C.1)). By the definition (13) an automorphism does not affect the scalar part of s , while the images of the other hypercomplex elements (j_3 , I , J_2 , and J_3 in (C.1)) are determined using the algebra (5). Then it is obvious that the transformed basis J'_n, j'_n, I' satisfy the same multiplication rules as J_n, j_n, I .

Similar to (C.1) there exist two automorphisms with fixed $j_2 = J_3 J_1 = j_3 j_1 = J_2 I$ and $j_3 = J_1 J_2 = j_1 j_2 = J_3 I$ axes, each generating the two compact angles, α_2, β_2 and α_3, β_3 . Three different representations of a basis unit indicate the planes of corresponding automorphisms and the positive directions of rotations.

One can define also hyperbolic rotations (automorphisms) around the three vector-like units J_n by the angles θ_n and ϕ_n . For example, similar to the transformation (C.1), automorphism around the axis $J_1 = j_3 J_2 = J_3 j_2 = j_1 I$ is

$$\begin{aligned} J'_1 &= J_1, \\ J'_2 &= \frac{J_2 \cosh(\phi_1 + \theta_1)}{2} + \frac{j_3 \sinh(\phi_1 + \theta_1)}{2}, \\ j'_3 &= J'_1 J'_2 = \frac{j_3 \cosh(\theta_1 + \phi_1)}{2} + \frac{J_2 \sinh(\theta_1 + \phi_1)}{2}, \\ I' &= I \cosh \theta_1 - j_1 \sinh \theta_1, \\ j'_1 &= J'_1 I' = j_1 \cosh \theta_1 - I \sinh \theta_1, \\ j'_2 &= J'_2 I' = \frac{j_2 \cosh(\phi_1 - \theta_1)}{2} + \frac{J_3 \sinh(\phi_1 - \theta_1)}{2}, \\ J'_3 &= j'_3 I' = \frac{J_3 \cosh(\phi_1 - \theta_1)}{2} + \frac{j_2 \sinh(\phi_1 - \theta_1)}{2}, \end{aligned} \quad (C.2)$$

where we chose the representation with half-angle rotations in order to write the transformations of λ_n and x_n in symmetric form. The automorphisms with the fixed $J_2 = j_1 J_3 = J_1 j_3 = j_2 I$ and $J_3 = j_2 J_1 = J_2 j_1 = j_3 I$ axes generate hyperbolic angles, θ_2, ϕ_2 and θ_3, ϕ_3 , respectively.

Finally for the rotations around the time direction, $I = J_1 j_1 = J_2 j_2 = J_3 j_3$, we have only two hyperbolic angles, k_1 and k_2 :

$$\begin{aligned} I' &= I, \\ j'_1 &= j_1 \cosh k_1 + J_1 \sinh k_1, \\ j'_2 &= j_2 \cosh k_2 + J_2 \sinh k_2, \\ J'_1 &= j'_1 I' = J_1 \cosh k_1 + j_1 \sinh k_1, \\ J'_2 &= j'_2 I' = J_2 \cosh k_2 + j_2 \sinh k_2, \\ j'_3 &= j'_1 j'_2 = j_3 \cosh(k_1 + k_2) - J_3 \sinh(k_1 + k_2), \\ J'_3 &= j'_3 I' = J_3 \cosh(k_1 + k_2) - j_3 \sinh(k_1 + k_2). \end{aligned} \quad (C.3)$$

In order to present the transformations of λ_n and x_n in (14) in the symmetric form, it is convenient to introduce instead of k_1 and k_2 the three hyperbolic angles φ_n :

$$\begin{aligned} k_1 &= \varphi_1 - \frac{1}{3} \sum_n \varphi_n, \\ k_2 &= \varphi_2 - \frac{1}{3} \sum_n \varphi_n, \\ -(k_1 + k_2) &= \varphi_3 - \frac{1}{3} \sum_n \varphi_n. \end{aligned} \quad (C.4)$$

D. Generators of G_2

The group G_2^{NC} first was studied by Cartan in his thesis [76, 77]. He considered $(3 + 4)$ -space with the seven coordinates, (y^n, t, z^n) , and linear operators of their transformations:

$$\begin{aligned} X_{mn} &= -z_n \frac{\partial}{\partial z_n} + y_n \frac{\partial}{\partial y_n} \\ &\quad + \frac{1}{3} \sum_{m=1}^3 \left(z_m \frac{\partial}{\partial z_m} - y_m \frac{\partial}{\partial y_m} \right), \\ &\quad \text{(no summing over } n) \\ X_{n0} &= -2t \frac{\partial}{\partial z_n} + y_n \frac{\partial}{\partial t} \\ &\quad + \frac{1}{2} \varepsilon_{nmk} \left(z^m \frac{\partial}{\partial y_k} - z^k \frac{\partial}{\partial y_m} \right), \\ X_{0n} &= -2t \frac{\partial}{\partial y_n} + z_n \frac{\partial}{\partial t} \\ &\quad + \frac{1}{2} \varepsilon_{nmk} \left(y^m \frac{\partial}{\partial z_k} - y^k \frac{\partial}{\partial z_m} \right), \\ X_{mm} &= -z_m \frac{\partial}{\partial z_n} + y_n \frac{\partial}{\partial y_m}, \quad (n \neq m), \end{aligned} \quad (D.1)$$

which preserves the quadratic form $dt^2 + z_n y^n$. Totally there are 15 generators in (D.1) since $n, m = 1, 2, 3$, but the operators X_{mn} are linearly dependent:

$$X_{11} + X_{22} + X_{33} = 0, \quad (D.2)$$

which explains why G_2^{NC} is 14- and not 15-dimensional.

By transition to our coordinates, x_n and λ_n ,

$$\begin{aligned} y_n &= \lambda_n + x_n, \\ \frac{\partial}{\partial y_n} &= \frac{1}{2} \left(\frac{\partial}{\partial \lambda_n} + \frac{\partial}{\partial x_n} \right), \\ z_n &= \lambda_n - x_n, \\ \frac{\partial}{\partial z_n} &= \frac{1}{2} \left(\frac{\partial}{\partial \lambda_n} - \frac{\partial}{\partial x_n} \right); \end{aligned} \quad (D.3)$$

from (D.1) we can find the operators of the transformation of the vector part V of (1) which preserves the diagonal quadratic form, $V^2 = \lambda_n \lambda^n + dt^2 - x_n x^n$:

$$\begin{aligned} X_{mn} &= \\ &= x_n \frac{\partial}{\partial x_n} + \lambda_n \frac{\partial}{\partial \lambda_n} - \frac{1}{3} \sum_{m=1}^3 \left(x_m \frac{\partial}{\partial x_m} + \lambda_m \frac{\partial}{\partial \lambda_m} \right), \end{aligned}$$

X_{n0}

$$\begin{aligned}
&= \left(\lambda_n \frac{\partial}{\partial t} - t \frac{\partial}{\partial \lambda_n} \right) \\
&+ \left(x_n \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_n} \right) + \frac{1}{4} \varepsilon_{nmk} \left(\lambda^m \frac{\partial}{\partial \lambda_k} - \lambda^k \frac{\partial}{\partial \lambda_m} \right) \\
&- \frac{1}{4} \varepsilon_{nmk} \left(x^m \frac{\partial}{\partial x_k} - x^k \frac{\partial}{\partial x_m} \right) \\
&+ \frac{1}{4} \varepsilon_{nmk} \left(\lambda^m \frac{\partial}{\partial x_k} + x^k \frac{\partial}{\partial \lambda_m} \right) \\
&- \frac{1}{4} \varepsilon_{nmk} \left(x^m \frac{\partial}{\partial \lambda_k} + \lambda^k \frac{\partial}{\partial x_m} \right),
\end{aligned}$$

 X_{0n}

$$\begin{aligned}
&= \left(\lambda_n \frac{\partial}{\partial t} - t \frac{\partial}{\partial \lambda_n} \right) - \left(x_n \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_n} \right) \\
&+ \frac{1}{4} \varepsilon_{nmk} \left(\lambda^m \frac{\partial}{\partial \lambda_k} - \lambda^k \frac{\partial}{\partial \lambda_m} \right) \\
&- \frac{1}{4} \varepsilon_{nmk} \left(x^m \frac{\partial}{\partial x_k} - x^k \frac{\partial}{\partial x_m} \right) \\
&- \frac{1}{4} \varepsilon_{nmk} \left(\lambda^m \frac{\partial}{\partial x_k} + x^k \frac{\partial}{\partial \lambda_m} \right) \\
&+ \frac{1}{4} \varepsilon_{nmk} \left(x^m \frac{\partial}{\partial \lambda_k} + \lambda^k \frac{\partial}{\partial x_m} \right),
\end{aligned}$$

 X_{nm}

$$\begin{aligned}
&= -\frac{1}{2} \left(\lambda_n \frac{\partial}{\partial \lambda_m} - \lambda_m \frac{\partial}{\partial \lambda_n} \right) \\
&- \frac{1}{2} \left(x_n \frac{\partial}{\partial x_m} - x_m \frac{\partial}{\partial x_n} \right) \\
&+ \frac{1}{2} \left(x_n \frac{\partial}{\partial \lambda_m} + \lambda_m \frac{\partial}{\partial x_n} \right) \\
&+ \frac{1}{2} \left(\lambda_n \frac{\partial}{\partial x_m} + x_m \frac{\partial}{\partial \lambda_n} \right) \quad (n \neq m).
\end{aligned}$$

(D.4)

E. Zero Divisors

In the algebra of split octonions two types of zero divisors, idempotent elements (projection operators) and nilpotent elements (Grassmann numbers), can be constructed [1, 71].

There exist four noncommuting (totally eight) primitive idempotents:

$$\begin{aligned}
D_n^{\pm(I)} &= \frac{1}{2} (1 \pm J_n), \\
D^{\pm(I)} &= \frac{1}{2} (1 \pm I),
\end{aligned} \tag{E.1}$$

$(n = 1, 2, 3)$

which obey the relations

$$\begin{aligned}
D_n^{\pm(I)} D_n^{\pm(I)} &= D_n^{\pm(I)}, \\
D_n^{+(I)} D_n^{- (I)} &= 0, \\
D^{\pm(I)} D^{\pm(I)} &= D^{\pm(I)}, \\
D^{+(I)} D^{- (I)} &= 0.
\end{aligned} \tag{E.2}$$

We have also the four noncommuting classes (totally twelve) of primitive nilpotents:

$$\begin{aligned}
G_n^{\pm(I)} &= \frac{1}{2} (J_n \pm j_n), \\
G_n^{\pm(I)} &= \frac{1}{2} (I \pm j_n),
\end{aligned} \tag{E.3}$$

$(n = 1, 2, 3)$

with the properties

$$\begin{aligned}
G_n^{\pm(I)} G_n^{\pm(I)} &= 0, \\
G_n^{\pm(I)} G_n^{\mp(I)} &= D^{\mp(I)}, \\
G_n^{\pm(I)} G_n^{\pm(I)} &= 0, \\
G_n^{\pm(I)} G_n^{\mp(I)} &= D_n^{\pm(I)}.
\end{aligned} \tag{E.4}$$

From these relations we see that separately nilpotents are Grassmann numbers, but different G_n -s do not commute with each other, in contrast to the projection operators (E.2). Instead the quantities $G_n^{\pm}$ are elements of the so-called algebra of Fermi operators with the anticommutators:

$$\begin{aligned}
G^{\pm(I)} G^{\mp(I)} + G^{\mp(I)} G^{\pm(I)} &= 1, \\
G^{\pm(I)} G^{\mp(I)} + G^{\mp(I)} G^{\pm(I)} &= 1.
\end{aligned} \tag{E.5}$$

The algebra of Fermi operators is some syntheses of the Grassmann and Clifford algebras.

For the completeness we note that the idempotents and nilpotents obey the following algebra:

$$\begin{aligned}
D_n^{\pm(I)} G_n^{\pm(I)} &= G_n^{\pm(I)}, \\
D_n^{\pm(I)} G_n^{\mp(I)} &= 0, \\
D^{\pm(I)} G_n^{\pm(I)} &= 0, \\
D^{\pm(I)} G_n^{\mp(I)} &= G_n^{\mp(I)}.
\end{aligned} \tag{E.6}$$

Using commuting zero divisors any octonion (1) can be written in the two equivalent forms:

$$\begin{aligned}
s &= D_n^{+(I)} \left(\frac{1}{3} \omega + \lambda^n \right) + G_n^{+(I)} (ct + x^n) \\
&+ D_n^{-(I)} \left(\frac{1}{3} \omega - \lambda^n \right) + G_n^{-(I)} (ct - x^n),
\end{aligned}$$

$$s = D^{+(I)} (\omega + ct) + G_n^{+(I)} (\lambda^n + x^n) + D^{-(I)} (\omega - ct) + G_n^{-(I)} (\lambda^n - x^n). \quad (\text{E.7})$$

The norm of a split octonion (A.4) contains two types of “light-cones” and we can introduce two types of critical signals (elementary particles) visualized in the decompositions (E.7).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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