Generalized Bilinear Differential Operators Application in a (3+1)-Dimensional Generalized Shallow Water Equation

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1. Introduction

The studies of exact solutions of nonlinear partial differential equations (NPDEs) have received considerable attention in connection with the important problems that arise in scientific applications. Many powerful methods have been proposed to obtain exact solutions of (NPDEs); a series of methods have been proposed, such as Painlevé test [1], Bäcklund transformation method [2, 3], Darboux transformation [4], inverse scattering transformation method [5], Lie group method [6, 7], Hamiltonian method [8, 9], and the Hirota method [10, 11].

In order to seek the periodic solutions of nonlinear evolution equations, Porubov and Parker proposed Weierstrass elliptic function expansion method [12]; Liu et al. proposed Jacobi elliptic sine function expansion methods [13, 14] and obtained some exact periodic solutions of some nonlinear evolution equations. They pointed out that their method can be applied to solve the nonlinear evolution equations in which the odd- and even-order derivative terms do not coexist. Zhang [15] developed Jacobi elliptic function expansion method to solve some nonlinear evolution equations in which the odd- and even-order derivative term coexist and obtained some exact periodic solutions of the equations. The bilinear method developed by Hirota have proved to be particularly powerful in obtaining the soliton solutions, quasiperiodic wave solutions, and periodic wave solutions [16, 17]. As we all know, once the bilinear forms of nonlinear differential equations are obtained, the multisoliton solutions, the bilinear Bäcklund transformation, and Lax pairs of NPDEs can be constructed easily. It is clear that the key of Hirota direct method is finding the bilinear forms of the given differential equations by the Hirota differential D-operators. However, Hirota bilinear equations are special and there are many other bilinear differential equations which are not written in the Hirota bilinear form.

In fact, solving nonlinear equations (especially nonlinear partial differential equations) is very difficult, and there is no unified method. The present methods can only be applied to a certain equation or some equations. So the work of continuing to find some effective method of solving nonlinear equations is important and meaningful. Recently, Ma put forward generalized bilinear differential operators named $D_f$-operators in [18], which are used to create bilinear differential equations. Furthermore, different symbols are also used to furnish relations with Bell polynomials in [19] and even for trilinear equations in [20]. In this paper, we would like to explore how to construct the bilinear forms...
with $D_p$-operators and how to obtain the exact solutions of nonlinear equation with the help of $D_p$ bilinear operators method.

The paper is structured as follows. In Section 2, we will give a brief introduction about the bilinear $D_p$-operators. In Section 3, we explore the relations between multivariate binary Bell polynomials and the $D_p$-operators. The $D_p$ bilinear forms of some nonlinear evolutions are given quickly and easily from the relations. In Section 4, we will use the relation in Section 2 to seek the bilinear form with $D_p$-operators of the $(3+1)$-dimensional generalized shallow water equation and then take advantage of the $D_p$-operators and the Riemann theta function [21, 22] to obtain its exact periodic wave solution which can be reduced to the soliton solution via asymptotic analysis.

## 2. Bilinear $D_p$-Operators

It is known to us that Hirota bilinear $D$-operators play a significant role in Hirota direct method. The $D$-operators are defined as follows:

$$D^m f \cdot g = \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)^m \left( \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right)^k \left[ f(x_1, y_1) g(x_2, y_2) \right],$$

where the right-hand side is computed in

$$x_1 = x_2 = x, \quad y_1 = y_2 = y.$$  

According to the definition of Hirota bilinear $D$-operators, we have

$$D_x f \cdot g = f_x g - f g_x,$$

$$D^2_x f \cdot g = f_{xx} g - 2 f_x g_x + f g_{xx},$$

$$D_{x} D_{t} f \cdot g = f_{xt} g - f_{x} g_t - f_x g_x + f g_{xt},$$

$$D^2_{x} f \cdot g = f_{xxx} g - 3 f_{xx} g_x + 3 f_x g_{xx} - f g_{xxx},$$

$$D_{x} D^2_{x} f \cdot g = f_{xxt} g - f_{xt} g_t - f_{x} g_{xt} + f_x g_{xt} + 3 f_{x} g_{xx} - 3 f_x g_{xt} + f g_{xxt}.$$  

Based on the Hirota $D$-operators, Professor Ma put forward a kind of bilinear $D_p$-operators in [18]:

$$D^m_{p,x} D^k_{p,y} [ f(x_1, y_1) g(x_2, y_2) ] = \left( \frac{\partial}{\partial x_1} + \alpha \frac{\partial}{\partial x_2} \right)^m \left( \frac{\partial}{\partial y_1} + \alpha \frac{\partial}{\partial y_2} \right)^k \left|_{x_1=x_2=x, y_1=y_2=y} \right.,$$

where the powers of $\alpha$ are determined by

$$\alpha^i = (-1)^{r(i)},$$

where $i = r(i) \mod p$ with $0 \leq r(i) < p; i \geq 0.$

Obviously, the case of $p = 1$ gives the normal derivatives, and the cases of $p = 2k, k \in N$, reduce to Hirota bilinear operators.

In particular, when $m = 0$, we have

$$D^0_{p,x} (f \cdot g)(x) = (\partial_x + \alpha \partial_{x'})^n f(x) g(x'),$$

$$= \sum_{i=0}^{n} \alpha^i C^n_{i} \partial_{x'}^{n-i} f \partial_{x'}^i g.$$  

According to the definition of $D_p$-operator, when $p = 3$, we have

$$\alpha^0 = 1, \quad \alpha = -1, \quad \alpha^2 = \alpha^3 = 1, \quad \alpha^4 = -1, \quad \alpha^5 = \alpha^6 = 1, \quad \alpha^7 = -1, \quad \alpha^8 = \alpha^9 = 1, \ldots,$$

$$D^4_{3,x} f \cdot g = \sum_{i=0}^{4} \alpha^i C_4^i \partial_{x'}^{4-i} f \partial_{x'}^i g = f_{4x} g - 4 f_{3x} g_x + 6 f_{2x} g_{2x} + 4 f_x g_{3x} - f g_{4x};$$

when $p = 5$, we have

$$\alpha^0 = 1, \quad \alpha^1 = -1, \quad \alpha^2 = 1, \quad \alpha^3 = -1, \quad \alpha^4 = \alpha^5 = 1, \quad \alpha^6 = -1, \quad \alpha^7 = 1, \quad \alpha^8 = -1, \ldots,$$

$$D^5_{5,x} f \cdot g = f_{5x} g - f g_{5x}, \quad D_{3} D_{5,x} f \cdot g = D_{5} (f_{5x} g - f g_{5x}) = f_{5x} g - f_{x} g_{5x} - f_{5x} g_x + f g_{6x},$$

$$D^5_{5,x} f \cdot g = \sum_{i=0}^{5} \alpha^i C^5_{i} \partial_{x'}^{5-i} f \partial_{x'}^i g = f_{5x} g - 2 f_{4x} g_x + f g_{6x}.$$
\[ D_{5,\alpha}^4 f \cdot g = \sum_{i=0}^{4} \alpha_i C_i^4 \sum_{\alpha_i} f \partial_{x_i} g \]
\[ = f_{4,\alpha} - 4f_{3,\alpha}g_x + 6f_{2,\alpha}g_{xx} - 4f_{1,\alpha}g_{xxx} + f_{0,\alpha}g_{xxxx}, \]
\[ D_{5,\alpha}^5 f \cdot g = \sum_{i=0}^{5} \alpha_i C_i^5 \sum_{\alpha_i} f \partial_{x_i} g \]
\[ = f_{5,\alpha} - 5f_{4,\alpha}g_x + 10f_{3,\alpha}g_{xx} - 10f_{2,\alpha}g_{xxx} + 5f_{1,\alpha}g_{xxxx} + f_{0,\alpha}g_{xxxxx}. \]

Now, under \( u = 2(\ln f)_{xx} \), for KdV equation,
\[ u_t + 6uu_x + u_{xxx} = 0, \]
we have
\[ \frac{\partial}{\partial x} \left( \frac{f_{x}f - f_{x}f + f_{xx}f - 4f_{xx}f + 3f_{xx}f}{f^2} \right) = 0; \]
we can get its bilinear form with \( D_p \)-operators:
\[ \left( D_{5,\alpha} S_{\alpha} + D_{5,\beta} \right) f \cdot f = 0. \]

In fact, if we seek the bilinear form with \( D_p \)-operators of nonlinear integrable differential equations according to the definition of \( D_p \)-operators, this needs some special skills and complex computations. So we would like to explore the relations between \( D_p \)-operators and multivariate binary Bell polynomials. The bilinear forms with \( D_p \)-operators of nonlinear integrable differential equations are obtained quickly and easily by applying the relations.

### 3. Relations with Bell Exponential Polynomials

#### 3.1. Relations with Bell Exponential Polynomials

As we all know, Bell proposed three kinds of exponential form polynomials. Later, Wang and Chen generalized the third type of Bell polynomials in \([23, 24]\) which is used mainly in this paper. The multidimensional binary Bell polynomials which we will use are defined as follows:

\[ Y_{n_1,\ldots,n_n}(y) = Y_{n_1,\ldots,n_n}(y_{r_1,\ldots,r_n}) \]
\[ = e^{-y} \partial_{x_1}^{n_1} \cdots \partial_{x_n}^{n_n} e^y \quad (n_1, \ldots, n_i \geq 0), \]

with \( y_{r_1,\ldots,r_n} = \partial_{x_1}^{r_1} \cdots \partial_{x_n}^{r_n}, r_1 = 0, \ldots, n_1, \ldots, r_i = 0, \ldots, n_i. \)

For example,
\[ Y_x = y_x, \]
\[ Y_{2x} = y_x^2 + y_{2x}, \]
\[ Y_{3x} = y_x^3 + 3y_x y_{2x} + y_{3x}, \]
\[ y_{4x} = y_x^4 + 4y_x y_{2x} + 6y_x y_{3x} + 3y_{2x}^2 + y_{4x}, \]
\[ Y_{5x} = y_x^5 + 5y_x y_{4x} + 15y_x y_{2x}^2 + 10y_x y_{3x} + 10y_{2x} y_{3x} + 10y_{2x} y_{4x} + y_{5x}, \]
\[ Y_{6x} = (y_x^6 + 5y_x y_{5x} + 15y_x y_{2x}^3 + 10y_x y_{3x} + 10y_{2x} y_{3x} + 10y_{2x} y_{4x} + y_{5x}) y_x, \]
\[ Y_{x,2x} = y_x^2 + y_{2x}, \]
\[ Y_{x,2x} = 2y_x y_{2x} + y_{2x,2x} + y_{4x} y_2 + y_{2x,2x}, \]
\[ Y_{x,3x} = 3y_x^2 y_2 + 3y_{2x} y_{2x} + 3y_{2x} y_{3x} + y_{3x,2x} + (y_x^3 + 3y_x y_{2x}^2 + y_{3x}) y_2, \]

For the sake of computational convenience, we assume that
\[ f = e^{\xi(x_1,\ldots,x_n)}, \]
\[ g = e^{\eta(x_1,\ldots,x_n)}. \]
we have
\[ (fg)^{-1} D_{p,x}^{n_1} \cdots D_{p,x}^{n_n} f \cdot g \]
\[ = \sum_{k=0}^{n_1} \cdots \sum_{k=0}^{n_n} \prod_{i=0}^{l} k_i (\xi) Y_{(1-k_1)x_1,\ldots,(1-k_n)x_n} (\xi) \]
\[ \cdot Y_{(1-k_1)x_1,\ldots,(1-k_n)x_n} (\eta) \]
\[ \cdot Y_{(1-k_1)x_1,\ldots,(1-k_n)x_n} (\xi + \eta), \]
\[ \cdot Y_{(1-k_1)x_1,\ldots,(1-k_n)x_n} (\xi - \eta). \]

where
\[ w = \xi + \eta, \]
\[ v = \xi - \eta. \]
We find that the link between $\Psi$-polynomials and the $D_p$-operator can be given in the following through the above deduction:

\[
(fg)^{-1}D_{p,x_1}^{n_1}, \ldots, D_{p,x_l}^{n_l} f \cdot g
= \Psi_{p,n_1,\ldots,n_l}(v = \ln f, \omega = \ln fg) \quad (17)
= Y_{n_1,\ldots,n_l}(y_{1,\ldots,n_l} = \xi_{1,\ldots,n_l} + \alpha^{r_1+\cdots+r_l}\eta_{1,\ldots,n_l}).
\]

In particular, when $f = g$, (17) becomes

\[
f^{-2}D_{p,x_1}^{n_1}, \ldots, D_{p,x_l}^{n_l} f \cdot f = \Psi_{p,n_1,\ldots,n_l}(q) \quad (18)
= f^2Y_{p,n_1,\ldots,n_l}(v = 0, \omega = 2 \ln f = q).
\]

Equation (18) gives the relations between $D_p$-operators and multivariate binary Bell polynomials.

Then we have

\[
\begin{align*}
D_{p,x_1}^{n_1}, \ldots, D_{p,x_l}^{n_l} f \cdot f & = f^2\Psi_{p,n_1,\ldots,n_l}(q) \\
& = f^2Y_{p,n_1,\ldots,n_l}(v = 0, \omega = 2 \ln f = q) \quad (19).
\end{align*}
\]

From (13) and (18), we have

\[
\begin{align*}
\overline{P}_{5,2x} & = \overline{P}_{5,2x} = \overline{P}_{7,2x} = q_{xx}, \\
\overline{P}_{5,4x} & = q_{xx}, \\
\overline{P}_{5,3x} & = \overline{P}_{5,3x} = \overline{P}_{7,3x} = q_{3t}, \\
\overline{P}_{5,6x} & = \overline{P}_{7,4x} = q_{4x} + 3q_{2x}^2, \\
\overline{P}_{5,5x} & = \overline{P}_{7,5x} = q_{3x,y} + 3q_{xx}q_{xy}, \\
\overline{P}_{5,2x,4x} & = 0, \\
\overline{P}_{5,6x} & = 15q_{2x}^3 + 15q_{2x}q_{4x}, \\
\overline{P}_{7,6x} & = 15q_{3x}^3 + 15q_{2x}q_{4x} + q_{6x}.
\end{align*}
\]

3.2. Bilinear Form with $D_p$-Operators. In this section, we will construct the bilinear forms for KdV equation, (2+1)-dimensional KdV equation, and (2+1)-dimensional Sawada-Kotera equation with the $D_p$-operators quickly and easily by utilizing the relations between $D_p$-operators and multidimensional Bell polynomials.

Example 1 (KdV equation). Consider

\[
u_t + 6uu_x + u_{xxx} = 0. \quad (21)
\]

Setting $u = q_{2x}$, substituting it into (21), and integrating with respect to $x$ yield

\[
q_{xt} + 3q_{2x}^2 + q_{xx} - \lambda_1 = 0, \quad (22)
\]

where $\lambda_1$ is an arbitrary function of $t$.

Based on (20) and (22), (21) can be written as follows:

\[
\overline{P}_{5,2x}(q) + \overline{P}_{5,4x}(q) - \lambda_1 = 0. \quad (23)
\]

From (19) and (23), we get the bilinear form with $D_p$-operators of (21)

\[
(D_{5,x}D_{5,y} + D_{5,x}^3) f \cdot f - \lambda_1 f^2 = 0. \quad (24)
\]

Example 2 ((2+1)-dimensional KdV equation). Consider

\[
u_t + 3uu_y + u_{xy} + 3u_x \int u_j dx = 0. \quad (25)
\]

Setting $u = q_{2x}$, substituting it into (25), and integrating with respect to $x$ yield

\[
q_{xt} + 3q_{xy}q_{2x}^2 + q_{3x,y} - \lambda_2 = 0, \quad (26)
\]

where $\lambda_2$ is an arbitrary function of $y, t$. Based on (20) and (26), (25) can be written as follows:

\[
\overline{P}_{5,2x}(q) + \overline{P}_{5,3x,y}(q) - \lambda_2 = 0. \quad (27)
\]

From (19) and (27), we get the bilinear form with $D_p$-operators of (25):

\[
(D_{5,x}D_{5,y} + D_{5,x}^3D_{5,y}) f \cdot f - \lambda_2 f^2 = 0. \quad (28)
\]

Example 3 ((2+1)-dimensional Sawada-Kotera equation). Consider

\[
u_t - \left( u_{4x} + 5uu_{2x} + \frac{5}{3}u^3 + 5u_{xy} \right)_x + 5 \int u_j dx \\
- 5uu_y - 5u_x \int u_j dx = 0. \quad (29)
\]

Setting $u = 3q_{2x}$, substituting it into (29), and integrating with respect to $x$ yield

\[
q_{xt} + 3q_{xy}q_{2x}^2 - \left( q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3 \right) \\
- 5 \left( q_{3xy} + 3q_{2x}q_{xy} \right) - \lambda_3 = 0, \quad (30)
\]

where $\lambda_3$ is an arbitrary function of $y, t$. Based on (20) and (30), (29) can be written as follows:

\[
\overline{P}_{7,2x}(q) + 5\overline{P}_{7,2y}(q) - 5\overline{P}_{7,3x,y}(q) - \overline{P}_{7,6x}(q) - \lambda_3 = 0. \quad (31)
\]

From (19) and (31), we get the bilinear form with $D_p$-operators of (29):

\[
(D_{7,x}D_{7,y} + 5D_{7,y}^2 - 5D_{7,x}^3D_{7,y} - D_{7,x}^6) f \cdot f - \lambda_3 f^2 = 0. \quad (32)
\]

From the above computation process for seeking the bilinear forms of three nonlinear equations, we can find that the bilinear forms with $D_p$-operators of nonlinear integrable differential equations are obtained quickly and easily by applying the relations between $D_p$-operators and multidimensional Bell polynomials.
4. Periodic Wave Solution
of the (3+1)-Dimensional
Generalized Shallow Water Equation

In this section, firstly, we will give the bilinear form of a (3+1)-dimensional generalized shallow water equation with the help of \( P \)-polynomials and the \( D_p \)-operators. And then, we construct the exact periodic wave solution of the (3+1)-dimensional generalized shallow water equation with the aid of the Riemann theta function, \( D_p \)-operators, and the special property of the \( D_p \)-operators when acting on exponential functions.

The following is (3+1)-dimensional generalized shallow water equation:

\[
 u_{xxyz} + 3u_xuxy + 3u_xuy - u_{xzt} - u_{xxz} = 0. \tag{33}
\]

Setting \( u = q_x \), inserting it into (33), and integrating with respect to \( x \) yield

\[
 q_{xxyz} + 3q_xq_{xy} - q_{xzt} - q_{xxz} = \lambda = 0, \tag{34}
\]

where \( \lambda \) is an arbitrary function of \( y, z, t \). Based on (20) and (34), (33) can be expressed as

\[
 -P_{3x,y} - P_{5x,z} + P_{5x,y} - \lambda = 0. \tag{35}
\]

From the above, we can get the bilinear form of (33):

\[
 \left(-D_{5,y} D_{5,z} - D_{5,x} D_{5,z} + D_{3}^2 D_{5,y}\right) f \cdot f - \lambda \cdot f^2 = 0 \tag{36}
\]

with \( q = 2 \ln f \). When acting on exponential functions, we find that \( D_p \)-operators have a good property:

\[
 G \left(D_{p,x_1}, \ldots, D_{p,x_n}\right) e^{\xi_1} \cdot e^{\xi_2} = G \left(k_1 + a \alpha_2, \ldots, k_n + a \alpha_n\right) e^{\xi_1 + a \alpha_2}, \tag{37}
\]

where \( \alpha \) is an arbitrary vector.

\[
 \xi_i = k_i x + l_i y + h_i z + \omega_i t + \xi^{(0)}_i, \quad i = 1, 2, \ldots \tag{38}
\]

In order to construct periodic wave solutions of (33), we study the multidimensional Riemann theta function with genus \( N \) given by

\[
 f(\xi) = f(\xi, \tau) = \sum_{n \in z} e^{-\pi i (\langle (n, \tau) + 2m \rangle, \xi)} \tag{39}
\]

In which \( n = (n_1, n_2, \ldots, n_N) \in z^N \) denotes the integer value vector and \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \) is complex phase variable. In addition, for the given two vectors \( h = (h_1, h_2, \ldots, h_N) \) and \( g = (g_1, g_2, \ldots, g_N) \) their inner product can be written by

\[
 \langle h, g \rangle = h_1 g_1 + h_2 g_2 + \cdots + h_N g_N. \tag{40}
\]

\[-i \tau = (-i) \tau_{ij} \] in which \( \tau \) is a positive definite and real-valued symmetric \( N \times N \) matrix, which can be called the period matrix of the theta function. The entries \( \tau_{ij} \) of \( \tau \) are free parameters of the theta function (39); we consider that Riemann's (39) converges to a real-valued function with an arbitrary vector \( \xi \in C^N \).

In what follows we construct the one-periodic wave solutions of (33). For \( N = 1 \), Riemann theta function (39) reduces Fourier series in \( n \) as follows:

\[
 f = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 + 2\pi \eta n}, \tag{41}
\]

where \( n \in Z, \tau \in C, \mathrm{Im} \tau > 0 \), and \( \eta = k x + l y + h z + \omega t \), with \( k, l, h, \) and \( \omega \) being constants to be determined.

Riemann theta function (41) satisfying the bilinear equation (36) yields the sufficient conditions for obtaining periodic waves. Substituting the theta function (41) into the left of (36) and using the property (37), we have

\[
 G \left(D_{p,x}, D_{p,y}, D_{p,z}, D_{p,t}\right) f \cdot f = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G \left(D_{p,x}, D_{p,y}, D_{p,z}, D_{p,t}\right) e^{2\pi i n^2 + 2\pi \eta n} e^{2\pi i m^2 + 2\pi \eta m} \tau \tag{42}
\]

\[
 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G \left(2\pi (n + m) k, 2\pi (n + m) l, 2\pi (n + m) h, 2\pi (n + m) \omega\right) e^{2\pi i (n + m) \eta} e^{2\pi i (n + m) \eta} \tau
\]

\[
 = \sum_{\delta = -\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} G \left(2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) k, 2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) l, 2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) h, 2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) \omega
\]

\[
 \cdot e^{2\pi i (\eta + \alpha \delta) \eta} \right\} e^{2\pi i (\eta - \alpha \delta) \eta} = \sum_{\delta = -\infty}^{\infty} G(\delta) e^{2\pi i (\eta - \alpha \delta) \eta},
\]

where \( \delta = -(1/\alpha)(m + n) \). To the bilinear form of (33), \( G(\delta) \) satisfies the period characters when \( p = 5 \). The powers of \( \alpha \) obey rule (5), noting that

\[
 \bar{G}(\delta) = \sum_{n=-\infty}^{\infty} G \left(2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) k, 2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) \omega
\]

\[
 \cdot k, 2\pi i \left(1 - \alpha\right) n - \alpha^2 \delta\right) \omega \]
\[ l \cdot (2 \pi i ((1 - \alpha) n - \alpha^2 \delta)) \]
\[ h \cdot (2 \pi i ((1 - \alpha) n - \alpha^2 \delta)) \omega \cdot e^{n(n^2 + (n + \alpha)^2) \tau} \]
\[ = \sum_{n=-\infty}^{\infty} G(2 \pi i (2 n - \delta) k, 2 \pi i (2 n - \delta) h, 2 \pi i (2 n - \delta) \omega) \cdot e^{n(n^2 + (\delta - n)^2) \tau} \]
\[ a_{12} = \sum_{n=-\infty}^{\infty} \rho_1 (n), \]
\[ b_1 = -\sum_{n=-\infty}^{\infty} \left\{ 16 \pi^2 n^2 k h + 256 \pi^4 n^4 k^3 l \right\} \rho_1 (n), \]
\[ \rho_2 (n) = e^{2 \pi i (2 n - 2 + \delta) \tau}, \]
\[ a_{21} = \sum_{n=-\infty}^{\infty} n^2 \pi^2 (2 n - 1)^2 l p_2 (n), \]
\[ a_{22} = \sum_{n=-\infty}^{\infty} \rho_2 (n), \]
\[ b_2 = -\sum_{n=-\infty}^{\infty} \left( 4 \pi^2 (2 n - 1)^2 n^2 k h + 16 \pi^4 (2 n - 1)^4 k^3 l \right) \rho_2 (n). \]

where \( s = n + \alpha \). From (43) we can infer that

\[ G(\delta) = \begin{cases} G(0) e^{n \pi i \delta \tau}, & \delta = 2n; \\ G(1) e^{n(2 \pi i + \pi i (\delta + 1) \tau)}, & \delta = 2n + 1, \end{cases} \]

\[ G(0) = \sum_{n=-\infty}^{\infty} \left\{ -2 \pi i (1 - \alpha) n^2 l \omega \right\} e^{\pi i n^2 \tau} \]
\[ + 2 \pi i (1 - \alpha) n^2 k h \]
\[ + 2 \pi i (1 - \alpha) n \omega \]
\[ = \sum_{n=-\infty}^{\infty} \left( 16 \pi^2 n^2 l \omega + 16 n^2 \pi^2 k h + 256 \pi^4 n^4 k^3 l - \lambda \right) e^{\pi i n^2 \tau} = 0, \]

\[ G(1) = \sum_{n=-\infty}^{\infty} \left\{ -2 \pi i (1 - \alpha) n^2 l \right\} e^{\pi i n^2 \tau} = \sum_{n=-\infty}^{\infty} \left[ 4 \pi^2 (2 n - 1)^2 l \omega \right. \\
\[ + 4 \pi^2 (2 n - 1)^2 k h + 16 \pi^4 (2 n - 1)^4 k^3 l - \lambda \] \\
\[ \cdot e^{\pi i (2 n^2 - 2 n + 1) \tau} = 8 \pi^2 l (\gamma + 9 \gamma^5 + \cdots). \]

Also, the powers of \( \alpha \) obey rule (5). For the sake of computational convenience, we denote that

\[ \rho_1 (n) = e^{\pi i n^2 \tau}, \]
\[ a_{11} = \sum_{n=-\infty}^{\infty} 16 \pi^2 n^2 l \rho_1 (n), \]
\[ b_1 = -\sum_{n=-\infty}^{\infty} \left\{ 16 \pi^2 n^2 k h + 256 \pi^4 n^4 k^3 l \right\} \rho_1 (n), \]
\[ \rho_2 (n) = e^{2 \pi i (2 n - 2 + \delta) \tau}, \]
\[ a_{21} = \sum_{n=-\infty}^{\infty} n^2 \pi^2 (2 n - 1)^2 l p_2 (n), \]
\[ a_{22} = \sum_{n=-\infty}^{\infty} \rho_2 (n), \]
\[ b_2 = -\sum_{n=-\infty}^{\infty} \left( 4 \pi^2 (2 n - 1)^2 n^2 k h + 16 \pi^4 (2 n - 1)^4 k^3 l \right) \rho_2 (n). \]

Then (45) and (46) can be written as

\[ a_{11} \omega + a_{12} \lambda + b_1 = 0, \]
\[ a_{21} \omega + a_{22} \lambda + b_2 = 0. \]

Solving this system, we get

\[ \omega = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}}, \]
\[ \lambda = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}. \]

Finally, we get one-periodic wave solution:

\[ u = 2 (\ln f) \chi, \]

where \( f \) is given by (41) and \( \omega, \lambda \) are satisfied with (49). And if we assume that \( k = 0.01, l = 0.01, h = 0.01, \) and \( \tau = i \) to (50), the solution (50) of (33) can be shown in Figure 1.
Figure 1: A one-periodic wave (50) of the (3+1)-dimensional shallow water wave equation (33) with parameters $k = 0.01$, $l = 0.01$, $h = 0.01$, and $\tau = i$. This figure shows that every one-periodic wave is one-dimensional, and it can be viewed as a superposition of overlapping solitary waves, placed one period apart. (a) Perspective view of the periodic wave $\text{Abs}(u)$ on $xot$-plane. (b) Perspective view of the periodic wave $\text{Abs}(u)$ on $yot$-plane. (c) Perspective view of the periodic wave $\text{Abs}(u)$ on $zot$-plane. (d) Wave propagation pattern of the wave along the $x$-axis. (e) Wave propagation pattern of the wave along the $y$-axis. (f) Wave propagation pattern of the wave along the $z$-axis.
\[ b_1 = - \sum_{n=-\infty}^{\infty} \{16\pi^2 n^2 k h + 256\pi^4 n^4 k \} e^{2\pi i n \tau} \]
\[ = -32\pi^2 k \left( (h + 16\pi^2 k^2) \gamma^2 + 4 (h + 16\pi^2 k^2) \gamma^8 + \cdots \right), \]
\[ b_2 = - \sum_{n=-\infty}^{\infty} \left( 4\pi^2 (2n-1)^2 n^2 k h + 16\pi^2 (2n-1)^4 k^3 l \right) e^{2\pi i n \tau} \]
\[ = 8\pi^2 k \left( (h + 4\pi^2 k^2) \gamma^2 + 9 (h + 4\pi^2 k^2) \gamma^5 + \cdots \right), \]
(51)

which lead to
\[ a_{11} a_{22} - a_{12} a_{21} = 8\pi^2 l \gamma + o(\gamma), \]
\[ a_{21} b_1 - a_{12} b_2 = 8\pi^2 k \left( (h + 4\pi^2 k^2 l) \gamma + o(\gamma) \right). \]
(52)

So, we have \( \omega \rightarrow (hk + 4\pi^2 k^2 l)/l \) as \( \text{Im} \tau \rightarrow +\infty (\gamma \rightarrow 0) \). And we can write \( f \) as
\[ f = \sum_{n=-\infty}^{\infty} e^{i n \tau + 2\pi i n \eta} = \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} e^{2\pi i n \eta} \]
\[ = 1 + \sum_{n=1}^{\infty} e^{i n \tau} \left( e^{2\pi i n \eta} + e^{-2\pi i n \eta} \right) \]
\[ + e^{\gamma i n \tau} \left( e^{2\pi i n \eta} + e^{-2\pi i n \eta} \right) + \cdots \]
\[ = 1 + e^{\frac{i}{\gamma} \pi \eta} + (e^{\frac{i}{\gamma} \pi \eta} - e^{-\frac{i}{\gamma} \pi \eta}) + e^{\frac{i}{\gamma} \pi \eta} + e^{-\frac{i}{\gamma} \pi \eta} + \cdots \]
(53)

It is interesting that if we set \( \eta' = 2\pi i \eta + \pi i \tau \), (53) can be rewritten as
\[ f = 1 + e^{\eta'} + \gamma^2 \left( e^{-\eta'} + e^{2\eta'} \right) + \gamma^6 \left( e^{-2\eta'} + e^{3\eta'} \right) + \cdots \]
(54)

From (54), we have \( f \rightarrow 1 + e^{\eta} \) as
\[ \text{Im} \tau \rightarrow +\infty (\gamma \rightarrow 0), \]
(55)

Then the periodic wave solution (50) of (33) turns to the soliton
\[ u = 2 (\ln f)_{x\tau}, \]
\[ f = 1 + e^{\eta'} = 1 + e^{\pi i (2k t + 2 l y + 2 n z + 2 \omega t + \tau)} \]
(56)

5. Conclusions and Remarks

In this paper, we investigate a (3+1)-dimensional generalized shallow water wave equation (33). Its bilinear form is given by applying the relations \( D_p \)-operators and binary Bell polynomials, which has proved to be a quick and direct method. Then, we successfully get the exact periodic wave solution with the help of \( D_p \)-operators and Riemann theta function in terms of Hirota direct method. Furthermore, we obtain the corresponding soliton solutions via asymptotic analysis for their periodic wave solutions.

There are many other interesting questions on bilinear differential equations; for example, can the approach be generalized to solve trilinear equations with trilinear differential operators? How to apply the \( D_p \)-operators into the discrete equations? Besides, we will try to explore how to construct more nonlinear evolution equations with other operators simply and directly. We will continue to explore these problems in the near future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References


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