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Research Article

Mannheim Curves in Nonflat 3-Dimensional Space Forms

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We consider the Mannheim curves in nonflat 3-dimensional space forms (Riemannian or Lorentzian) and we give the concept of Mannheim curves. In addition, we investigate the properties of nonnull Mannheim curves and their partner curves. We come to the conclusion that a necessary and sufficient condition is that a linear relationship with constant coefficients will exist between the curvature and the torsion of the given original curves. In the case of null curve, we reveal that there are no null Mannheim curves in the 3-dimensional de Sitter space.

1. Introduction

Semi-Riemannian geometry is important both in differential geometry and in physics, where it plays a central role in the theory of relativity. Curve which is the basic object of study has attracted much more attention by many mathematicians and physicists. In particular, there has been an increase in research on null curves in geometry and physics [1–4]. From physical significance point of view, there is a particle model entirely based on geometry of null curves [2, 3]. The other important physical reason is the application of null curves theory to general relativity. Many basic properties about curves could be seen in [5]. The renewed interest in the theory of curves has developed from the need to observe the properties of special curves such as Mannheim curves and Bertrand curves in different space. Mannheim curves are well studied classical curves and may be defined by their property that space curves whose principle normals are binormals of another curve at corresponding points. The notion of Mannheim curve was put forward by Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows: a space curve is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the relation $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a nonzero constant. On this basis, many mathematicians systematically investigated Mannheim curves in different space and they obtained a number of important results [6]. In addition, the properties of Mannheim curve in Minkowski space have been studied extensively by, among others, Liu and Wang who studied Mannheim partner curves in Euclidean 3-space and Minkowski 3-space in detail; in particular, they provided the necessary and sufficient condition for the judgment of the Mannheim partner curves [7].

The other kind of curve which has been focusing a lot of researchers' attention since the beginning is Bertrand curve [8–11]. In particular, Lucas and Ortega-Yagües have devoted their work to the research of properties of Bertrand curves which include nonnull Bertrand curves and null Bertrand curves and they obtained many perfect characterizations of Bertrand curves in nonflat 3-dimensional space forms [9].

Inspired by their work, we use some fundamental results of differential geometry as basic tools in our research on the Mannheim curve in nonflat 3-dimensional space forms. It is well known that, in nonflat semi-Euclidean space, there are two types of pseudospheres: pseudosphere with a positive radius squared, $\langle x, x\rangle = r^2$, which we call de Sitter space. The other type is pseudosphere with a negative radius squared, $\langle x, x\rangle = -r^2$, which we call anti-de Sitter space. When we choose one of these spheres as the ambient space of curves, the curves will show some special properties. The first goal of this paper is to define the Mannheim curve in 3-dimensional nonflat space forms which include two types of sphere as above. In addition, the definition of angle in Euclidean is well known by us. However, the concept of general angle in semi-Euclidean space has been drawing our attention [9]. On this basis, we show and proof some properties of

the angle between tangent vectors or binormal vectors of Mannheim curves and their partner curves at corresponding points. Furthermore, we investigate the characterization of Mannheim curves and Mannheim partner curves in nonflat 3-dimensional space forms. We give a necessary and sufficient condition for a curve to be Mannheim curve or Mannheim partner curve and obtain some explicit equations. Meanwhile, as is known, the Mannheim curves in nonflat space forms have two cases: one case is the nonnull Mannheim curves; the other case is the null Mannheim curves which are mainly considered in 3-dimensional de Sitter space $\mathbb{S}_1^3$. Grbović et al. discussed the null Mannheim curves in 3-dimensional Minkowski space [12]. We know that the case of null curve that is immersed in a three-dimensional de Sitter space is more sophisticated and interesting than nonnull curve in de Sitter space. To the authors' knowledge, there is no article dedicated to studying the existence of null Mannheim curves in de Sitter space. For this reason, we consider the null Mannheim curve and discover that there is no null Mannheim curve in 3-dimensional de Sitter space.

The brief description of the organization of this paper is as follows. In Section 2, we review some basic notions about the space $\mathbb{R}_\nu^{n+1}$ and curves which include nonnull curves and null curves. In addition, we give the definition of Mannheim curve in nonflat 3-dimensional space forms in Section 3. One of the main results in this paper is stated in Section 3, Theorem 5. Section 4 is devoted to claiming and showing that a null Mannheim curve that lies in 3-dimensional de Sitter space is nonexistent.

Throughout the paper, all the symbols here could be found in [13].

2. Preliminaries

Now we introduce some basic notions in semi-Euclidean space and curves.

Let $\mathbb{R}_\nu^{n+1}$ denote the $(n+1)$-dimensional *pseudo-Euclidean space* of index $\nu \geq 0$; let $E = \{e_1, \ldots, e_{n+1}\}$ be an canonical basis of $\mathbb{R}_\nu^{n+1}$. We choose two vectors $x, y \in \mathbb{R}_\nu^{n+1}$, and the standard metric of $\mathbb{R}_\nu^{n+1}$ is given by

$$\langle x, y\rangle = -\sum_{i=1}^{\nu} x_i y_i + \sum_{j=\nu+1}^{n+1} x_j y_j, \tag{1}$$

where x_i, y_i stand for the coordinate components of x and y with respect to E in $\mathbb{R}_\nu^{n+1}$, respectively.

Letting $x \in \mathbb{R}_\nu^{n+1}$, the vector x is said to be

$$\begin{aligned} &\text{spacelike}, \quad \text{if } \langle x, x\rangle > 0 \text{ or } x = 0, \\ &\text{timelike}, \quad \text{if } \langle x, x\rangle < 0, \\ &\text{lightlike (null)}, \quad \text{if } \langle x, x\rangle = 0,\ x \neq 0. \end{aligned} \tag{2}$$

Define the *norm* of a nonnull vector u by $\|u\| = |\langle u, u\rangle|^{1/2}$, where $u \in \mathbb{R}_\nu^{n+1}$. We call u the *unit vector* if $\|u\| = 1$.

We define the *signature* of a vector x as follows:

$$\operatorname{sign}(x) = \begin{cases} 1, & x \text{ is spacelike}, \\ 0, & x \text{ is null}, \\ -1, & x \text{ is timelike}. \end{cases} \tag{3}$$

Let $\mathbb{M}_q^3(c) \subset \mathbb{R}_\nu^4$ denote the *nonflat 3-dimensional space forms* of $q = 0, 1$, and constant curvature $c \neq 0$. Meanwhile, $\nu = q$, if $c = 1$, and $\nu = q + 1$, if $c = -1$. Moreover, we will denote $\mathbb{M}_q^3(c)$ by the pseudo-Euclidean hypersphere $\mathbb{S}_q^3(1)$ or the pseudo-Euclidean hyperbolic space $\mathbb{H}_q^3(-1)$ according to $c = 1$ or $c = -1$, respectively, where $\mathbb{S}_q^3(1)$ is denoted by

$$\mathbb{S}_q^3(1) = \left\{x = (x_1, \ldots, x_4) \in \mathbb{R}_q^4 \mid \langle x, x\rangle = 1\right\}, \tag{4}$$

and the pseudo-Euclidean hyperbolic space of index $q \geq 0$ and curvature $c = -1$ is given by $\mathbb{H}_q^3(-1)$:

$$\mathbb{H}_q^3(-1) = \left\{x = (x_1, \ldots, x_4) \in \mathbb{R}_{q+1}^4 \mid \langle x, x\rangle = -1\right\}. \tag{5}$$

Many features of inner product spaces have analogues in the pseudo-Euclidean case. In $\mathbb{R}^n$, the Schwarz inequality permits the definition of the Euclidean angle θ between vectors X and Y as the unique number $0 \leq \theta \leq \pi$, such that $\langle X, Y\rangle = |X||Y|\cos\theta$. For two nonnull vectors X, Y which are not spacelike vectors in the Lorentzian space $\mathbb{R}_1^n$, the definition of the angle θ between X and Y is of great interest and importance. Then definition of general angle which is similar to Euclidean angle is as Definition 1. We consider two nonnull vectors $X, Y \in \mathbb{R}_1^n$ such that they span a plane $\mathbb{R}_1^2$. In this plane, we can choose an orthogonal basis $\{e_1, e_2\}$, with $\langle e_1, e_1\rangle = -1$ and $\langle e_2, e_2\rangle = 1$. Then we can write vectors X, Y in this basis as $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$.

Definition 1 (see [9]). Let one consider two nonnull vectors $X, Y \in \mathbb{R}_1^n$.

(a) Let one assume that X and Y are spacelike vectors; then

  (i) if they span a spacelike plane, there is a unique number $0 \leq \theta \leq \pi$ such that $\langle X, Y\rangle = |X||Y|\cos\theta$;

  (ii) if they span a timelike plane, there is a unique number $\theta \geq 0$ such that $\langle X, Y\rangle = \varepsilon|X||Y|\cosh\theta$, where $\varepsilon = 1$ or $\varepsilon = -1$ according to $\operatorname{sgn}(X_2) = \operatorname{sgn}(Y_2)$ or $\operatorname{sgn}(X_2) \neq \operatorname{sgn}(Y_2)$, respectively.

(b) Let one assume that X and Y are timelike vectors; then there is a unique number $\theta \geq 0$ such that $\langle X, Y\rangle = \varepsilon|X||Y|\cosh\theta$, where $\varepsilon = 1$ or $\varepsilon = -1$ according to X and Y have different time orientation or the same time orientation, respectively.

(c) Let one assume that X is spacelike vector and Y is timelike vector; then there is a unique number $\theta \geq 0$ such that $\langle X, Y\rangle = \varepsilon|X||Y|\sinh\theta$, where $\varepsilon = 1$ or $\varepsilon = -1$ according to $\operatorname{sgn}(X_2) = \operatorname{sgn}(Y_1)$ or $\operatorname{sgn}(X_2) \neq \operatorname{sgn}(Y_1)$, respectively.

Given two nonnull vectors $X, Y \in \mathbb{R}^n_1$, the corresponding number θ given above will be called simply the angle between X and Y.

Definition 2. Let $\alpha : I \to \mathbb{R}^{n+1}_\nu$ be a curve in $\mathbb{R}^{n+1}_\nu$ and let α' be the velocity vector of α, where I is an open interval of $\mathbb{R}$. For any $s \in I$, the curve α is called timelike curve, spacelike curve, or lightlike (null) curve if, for each s, $\langle \alpha'(s), \alpha'(s)\rangle < 0$, $\langle \alpha'(s), \alpha'(s)\rangle > 0$, or $\langle \alpha'(s), \alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, respectively. We call α a nonnull curve if α is a timelike curve or a spacelike curve.

The Frenet frame of a nonnull curve in $\mathbb{M}^3_q(c)$ is as follows.

Let $\alpha = \alpha(s) : I \to \mathbb{M}^3_q(c)$, $q = 0, 1$ be a nonnull curve immersed in the 3-dimensional space $\mathbb{M}^3_q(c)$, where I is an open interval. If $\|\alpha'(s)\| = 1$ for some $s \in I$, the curve α is called a *unit speed curve*. Then in this paper we assume without loss of generality that α is parameterized by the arc length parameter s. Letting ∇ be the Levi-Civita connection of $\mathbb{R}^4_\nu$, there exists the Frenet frame $\{T, N, B\}$ along α and smooth functions κ, τ in $\mathbb{M}^3_q(c)$ such that

$$\begin{aligned} \nabla_T T &= -\varepsilon_1 c\alpha + \varepsilon_2 \kappa N, \\ \nabla_T N &= -\varepsilon_1 \kappa T + \varepsilon_3 \tau B, \\ \nabla_T B &= -\varepsilon_2 \tau N, \end{aligned} \tag{6}$$

where κ and τ are called the *curvature* and *torsion* of α, respectively. Considering $\langle T, T\rangle = \varepsilon_1$, $\langle N, N\rangle = \varepsilon_2$, and $\langle B, B\rangle = \varepsilon_3$, we denote by $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ the *casual characters* of $\{T, N, B\}$. When $\{T, N, B\}$ are spacelike, then $\varepsilon_i = 1$, and otherwise, $\varepsilon_i = -1$, where $i \in \{1, 2, 3\}$. It is well known that curvature and torsion are invariant under the isometries of $\mathbb{M}^3_q(c)$. Three vector fields T, N, B consisting of the Frenet frame of α are called the tangent vector field, principal normal vector field, and binormal vector field, respectively.

A vector field X on $\mathbb{M}^3_q(c)$ along α is said to be *parallel along* α if $\nabla_s X = 0$, where ∇_s denotes the covariant derivative along α. A vector $X_{\alpha(s)}$ at $\alpha(s)$ is called parallel displacement of vector $X_{\alpha(t)}$ at $\alpha(t)$ along α. If its tangent vector field α' of curve α is parallel along α, then the curve is called geodesic.

We can denote the exponential map at $x \in \mathbb{M}^3_q(c)$ by $\exp_x$ and review the *exponential map* $\exp_x : T_x\mathbb{M}^3_q(c) \to \mathbb{M}^3_q(c)$ at $x \in \mathbb{M}^3_q(c)$ which is defined by $\exp_x(v) = \gamma_v(1)$, where $\gamma_v : [0, \infty) \to \mathbb{M}^3_q(c)$ is the constant speed geodesic starting from x with the initial velocity $\gamma'_v(0) = v$. For any point $\alpha(s)$ in the curve α, the *principal normal geodesic* in $\mathbb{M}^3_q(c)$ starting at α is defined as the geodesic curve $\gamma^\alpha_s(t) = \exp_{\alpha(s)}(tN(s)) = f(t)\alpha(s) + g(t)N(s), t \in \mathbb{R}$, where the functions f and g are given by

$$\begin{aligned} f(t) &= \cos t, \quad g(t) = \sin t, \quad \text{if } \varepsilon_2 c = 1, \\ f(t) &= \cosh t, \quad g(t) = \sinh t, \quad \text{if } \varepsilon_2 c = -1. \end{aligned} \tag{7}$$

In the following, we will recall the Frenet frame of a null curve in $\mathbb{S}^3_1$.

Let $\alpha : I \to \mathbb{S}^3_1$ be a null curve in $\mathbb{S}^3_1$, where I is an open interval of $\mathbb{R}$. Then there exists the Frenet frame $\{T = \alpha', N, B\}$ along α and smooth functions κ_1, κ_2 in $\mathbb{S}^3_1$ such that

$$\begin{aligned} \nabla_T T &= \kappa_1 N, \\ \nabla_T N &= -\kappa_2 T + \kappa_1 B, \\ \nabla_T B &= -\kappa_2 N + \alpha, \end{aligned} \tag{8}$$

where the first curvature $\kappa_1 = 0$ if α is geodesic; otherwise $\kappa_1 = 1$. In addition, the following conditions are satisfied:

$$\begin{aligned} \langle T, T\rangle = \langle B, B\rangle = \langle T, N\rangle = \langle B, N\rangle = 0, \\ \langle N, N\rangle = 1, \qquad \langle T, B\rangle = 1. \end{aligned} \tag{9}$$

A null curve α which is not a null geodesic is parameterized by pseudoarc length parameter if $\langle \nabla_T T, \nabla_T T\rangle = 1$. In the circumstances, $\kappa_1 = 1$. For any point $\alpha(s)$ in the null curve α, we define the principal normal geodesic in $\mathbb{S}^3_1$ starting at α as the geodesic curve:

$$\gamma^\alpha_s(t) = \exp_{\alpha(s)}(tN(s)) = f(t)\alpha(s) + g(t)N(s), \tag{10}$$

where $t \in \mathbb{R}$, and the functions $f(t) = \cos t$, $g(t) = \sin t$.

3. Mannheim Curves in $\mathbb{M}^3_q(c)$

In this section, we will discuss Mannheim curves in nonflat 3-dimensional space forms $\mathbb{M}^3_q(c)$, and then we give the definition of Mannheim curves as follows.

Definition 3. A curve α with nonzero curvature is said to be a Mannheim curve if there exists another immersed curve $\beta = \beta(\sigma) : I \to \mathbb{M}^3_q(c)$ and a one-to-one corresponding between α and β, $\beta \neq \pm\alpha$, such that the principal normal geodesics of curve α coincide with the binormal geodesics of curve β at corresponding points. One says that β is a Mannheim curve mate (or Mannheim partner curve) of curve α. The curves α and β are called a pair of Mannheim curves.

Let $\alpha = \alpha(s)$ be a unit speed curve in $\mathbb{M}^3_q(c)$ and let $\beta = \beta(\sigma)$ be the Mannheim partner curve of α, where σ is the arc-length parameter of β. From the definition, we know that there exists a differentiable function $a(s)$ such that

$$\beta(\sigma(s)) = f(a(s))\alpha(s) + g(a(s))N_\alpha(s), \tag{11}$$

where $\{T_\alpha, N_\alpha, B_\alpha\}$ denotes the Frenet frame along α and $\beta(\sigma(s))$ is the point in β corresponding to $\alpha(s)$. We introduce the distance function $d(s)$ as the distance in 3-dimensional nonflat space between $\alpha(s)$ and its corresponding point $\beta(\sigma(s))$.

Proposition 4. *Let $\alpha(s)$ and $\beta(\sigma)$ be a pair of nonnull Mannheim curves in $\mathbb{M}^3_q(c)$; then the following properties hold:*

(1) *the function $a(s)$ is constant;*

(2) *the angle between binormal vectors of Mannheim curves α and β at corresponding points is constant;*

(3) *the angle between tangent vectors of Mannheim curves α and β at corresponding points is constant if and only if the curvature of α is nonzero constant:*

$$\kappa_\alpha = \frac{-\varepsilon_1\varepsilon_2 cg(a(s))}{f(a(s))}. \tag{12}$$

Proof. (1) Suppose that $\alpha(s)$ and $\beta(\sigma)$ are a pair of nonnull Mannheim curves in $\mathbb{M}^3_q(c)$. Since the principal normal geodesics of curve α coincide with the binormal geodesics of curve β at corresponding points, we obtain

$$\left.\frac{d}{dt}\right|_{t=a(s)} \gamma_s^\alpha(t) = \varepsilon B_\beta(\sigma(s)), \quad \varepsilon = \pm 1, \tag{13}$$

and because of $f' = -\varepsilon_2 cg$ and $g' = f$, then

$$B_\beta(\sigma(s)) = -\varepsilon\varepsilon_2 cg(a(s))\alpha(s) + \varepsilon f(a(s)) N_\alpha(s), \tag{14}$$

where $\{T_\beta, N_\beta, B_\beta\}$ with causal characters given by $\{\delta_1, \delta_2, \delta_3\}$ is the Frenet frame along curve β. On the other hand, by the definition we get

$$\beta(\sigma(s)) = f(a(s))\alpha(s) + g(a(s)) N_\alpha(s). \tag{15}$$

Then the tangent vector of β is given by

$$\begin{aligned}
&\sigma'(s) T_\beta(\sigma(s)) \\
&\quad = \frac{d}{ds}\beta(\sigma(s)) \\
&\quad = -\varepsilon_2 cg(a(s)) a'(s)\alpha(s) \\
&\qquad + \left(-\varepsilon_1\kappa_\alpha g(a(s)) + f(a(s))\right) T_\alpha(s) \\
&\qquad + a'(s) f(a(s)) N_\alpha(s) \\
&\qquad + \varepsilon_3 g(a(s)) \tau_\alpha B_\alpha(s).
\end{aligned} \tag{16}$$

By taking the scalar product of (14) with (16), hence

$$\begin{aligned}
0 &= \left\langle B_\beta(\sigma(s)), \frac{d}{ds}\beta(\sigma(s))\right\rangle \\
&= \varepsilon a'(s)\left(cg^2(a(s)) + \varepsilon_2 f^2(a(s))\right) \\
&= \varepsilon\varepsilon_2 a'(s).
\end{aligned} \tag{17}$$

Thus (17) is reduced to $a'(s) = 0$, which implies $a(s)$ is a constant.

(2) A straightforward computation shows that

$$\begin{aligned}
&\frac{d}{ds}\left\langle B_\alpha(s), B_\beta(\sigma(s))\right\rangle \\
&\quad = \left\langle -\varepsilon_2\tau_\alpha N_\alpha(s), B_\beta(\sigma(s))\right\rangle \\
&\qquad + \sigma'(s)\left\langle B_\alpha(s), -\delta_2\tau_\beta N_\beta(\sigma(s))\right\rangle.
\end{aligned} \tag{18}$$

According to (14),

$$B_\beta(\sigma(s)) = -\varepsilon\varepsilon_2 cg(a(s))\alpha(s) + \varepsilon f(a(s)) N_\alpha(s), \tag{19}$$

and, by taking the derivative of this equation,

$$\begin{aligned}
&\nabla_{T_\beta} B_\beta(\sigma(s)) \\
&\quad = -\sigma'(s)\delta_2\tau_\beta(\sigma(s)) N_\beta(\sigma(s)) \\
&\quad = -\varepsilon\left(\varepsilon_2 cg(a(s)) + \varepsilon_1 f(a(s))\kappa_\alpha\right) T_\alpha(s) \\
&\qquad + \varepsilon\varepsilon_3 f(a(s))\tau_\alpha B_\alpha(s).
\end{aligned} \tag{20}$$

Then apply (20) to (18):

$$\begin{aligned}
&\frac{d}{ds}\left\langle B_\alpha(s), B_\beta(\sigma(s))\right\rangle \\
&\quad = -\varepsilon\tau_\alpha(s) f(a(s)) + \varepsilon f(a(s))\tau_\alpha(s) = 0
\end{aligned} \tag{21}$$

and finish the proof.

(3) By a direct computation,

$$\begin{aligned}
&\frac{d}{ds}\left\langle T_\alpha(s), T_\beta(\sigma(s))\right\rangle \\
&\quad = \left\langle -\varepsilon_1 c\alpha(s) + \varepsilon_2\kappa_\alpha N_\alpha(s), T_\beta(\sigma(s))\right\rangle \\
&\qquad + \sigma'(s)\left\langle T_\alpha, -\delta_1 c\beta(\sigma(s)) + \delta_2\kappa_\beta N_\beta(\sigma(s))\right\rangle.
\end{aligned} \tag{22}$$

On the other hand, by taking the derivative of $B_\beta(\sigma(s))$ with respect to s,

$$\begin{aligned}
&\nabla_{T_\beta} B_\beta(\sigma(s)) \\
&\quad = -\delta_2\sigma'(s)\tau_\beta N_\beta(\sigma(s)) \\
&\quad = -\varepsilon\varepsilon_2 cg(a(s)) T_\alpha(s) \\
&\qquad + \varepsilon f(a(s))\left(-\varepsilon_1\kappa_\alpha T_\alpha(s) + \varepsilon_3\tau_\alpha B_\alpha(s)\right) \\
&\quad = -\varepsilon\left(\varepsilon_2 cg(a(s)) + \varepsilon_1 f(a(s))\kappa_\alpha\right) T_\alpha(s) \\
&\qquad + \varepsilon\varepsilon_3 f(a(s))\tau_\alpha B_\alpha(s).
\end{aligned} \tag{23}$$

We apply (16), (23), and Proposition 4(1) to (22), obtaining that if

$$\kappa_\alpha = \frac{-\varepsilon_1\varepsilon_2 cg(a(s))}{f(a(s))}, \tag{24}$$

then

$$\frac{d}{ds}\left\langle T_\alpha(s), T_\beta(\sigma(s))\right\rangle = 0. \tag{25}$$

□

In the following, we consider the characterizations in terms of the curvature and the torsion of Mannheim curves in nonflat 3-dimensional space forms.

Theorem 5. *A curve α in $\mathbb{M}^3_q(c)$ with curvature κ and torsion τ is a Mannheim curve if and only if it satisfies*

$$\varepsilon_3\tau_\alpha^2 + \varepsilon_1\kappa_\alpha^2 = \frac{f^2(\theta_0) - c\varepsilon_2 g^2(\theta_0)}{f(\theta_0) g(\theta_0)}\kappa_\alpha + c\varepsilon_1\varepsilon_2, \tag{26}$$

where $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ are the casual characters of $\{T_\alpha, N_\alpha, B_\alpha\}$, and $c = \pm 1$.

Proof. Let $\alpha = \alpha(s)$ be a unit speed curve in $\mathbb{M}_q^3(c)$, and let $\beta = \beta(\sigma)$ be a Mannheim partner curve of α, where σ is the arc length parameter of β. We know that $a(s)$ is a nonzero constant from Proposition 4(1); we denote $a(s) = \theta_0$.

Let $\beta(\sigma(s)) = f(\theta_0)\alpha(s) + g(\theta_0)N_\alpha(s)$; by differentiating $\beta(\sigma(s))$ with respect to parameter σ,

$$\begin{aligned}&\sigma'(s)\,T_\beta(\sigma(s))\\&\quad=\left(f(\theta_0)-\varepsilon_1 g(\theta_0)\kappa_\alpha\right)T_\alpha(s)+g(\theta_0)\,\varepsilon_3\tau_\alpha B_\alpha(s),\end{aligned}\tag{27}$$

where $\sigma'(s) = \sqrt{(f(\theta_0)-\varepsilon_1 g(\theta_0)\kappa_\alpha)^2 + g^2(\theta_0)\tau_\alpha^2}$.

Let $a_1(s) = (1/\sigma'(s))(f(\theta_0) - \varepsilon_1 g(\theta_0)\kappa_\alpha)$, $a_2(s) = (1/\sigma'(s))\varepsilon_3\tau_\alpha g(\theta_0)$, so we write

$$T_\beta(\sigma(s)) = a_1(s)\,T_\alpha(s) + a_2(s)\,B_\alpha(s).\tag{28}$$

By differentiating (28) with respect to s,

$$\begin{aligned}&\sigma'(s)\left[-\delta_1 c\beta(\sigma(s))+\delta_2\kappa_\beta N_\beta(\sigma(s))\right]\\&\quad=a_1'(s)\,T_\alpha(s)-\varepsilon_1 c a_1(s)\,\alpha(s)\\&\qquad+\left(\varepsilon_2 a_1(s)\,\kappa_\alpha-\varepsilon_2 a_2(s)\,\tau_\alpha\right)N_\alpha+a_2'(s)\,B_\alpha(s).\end{aligned}\tag{29}$$

Since B_β is orthogonal to N_β,

$$\begin{aligned}\beta(\sigma(s))&=f(\theta_0)\,\alpha(s)+g(\theta_0)\,N_\alpha(s),\\B_\beta(\sigma(s))&=-\varepsilon\varepsilon_2 c g(\theta_0)\,\alpha(s)+\varepsilon f(\theta_0)\,N_\alpha(s).\end{aligned}\tag{30}$$

Therefore, we apply these equations to a computation:

$$\begin{aligned}&\delta_2\sigma'(s)\,\kappa_\beta N_\beta(\sigma(s))\\&\quad=a_1'(s)\,T_\alpha(s)+\left(\delta_1 c\sigma'(s)\,f(\theta_0)-\varepsilon_1 c a_1(s)\right)\alpha(s)\\&\qquad+\Big(\varepsilon_2 a_1(s)\,\kappa_\alpha-\varepsilon_2 a_2(s)\,\tau_\alpha\\&\qquad\quad+\delta_1 c\sigma'(s)\,g(\theta_0)\Big)N_\alpha(s)+a_2'(s)\,B_\alpha(s),\end{aligned}\tag{31}$$

$$\begin{aligned}&\delta_1 c\sigma'(s)\,f(\theta_0)-\varepsilon_1 c a_1(s)=0,\\&\varepsilon_2 a_1(s)\,\kappa_\alpha-\varepsilon_2 a_2(s)\,\tau_\alpha+\delta_1 c\sigma'(s)\,g(\theta_0)=0.\end{aligned}\tag{32}$$

Then

$$\begin{aligned}f(\theta_0)&=\frac{\varepsilon_1\delta_1 a_1(s)}{\sigma'(s)},\\a_2(s)\,\tau_\alpha&=\sigma'(s)\left[\varepsilon_1\delta_1 f(\theta_0)\,\kappa_\alpha(s)+\delta_1\varepsilon_2 c g(\theta_0)\right].\end{aligned}\tag{33}$$

We change it as follows:

$$\begin{aligned}&\frac{f(\theta_0)}{\varepsilon_1\delta_1 f(\theta_0)\,\kappa_\alpha(s)+\delta_1\varepsilon_2 c g(\theta_0)}\\&\quad=\frac{\varepsilon_1\delta_1}{\tau_\alpha}\cdot\frac{f(\theta_0)-\varepsilon_1 g(\theta_0)\,\kappa_\alpha}{\varepsilon_3\tau_\alpha g(\theta_0)}.\end{aligned}\tag{34}$$

Moreover, we have the following conclusion:

$$\varepsilon_3\tau_\alpha^2+\varepsilon_1\kappa_\alpha^2=\frac{f^2(\theta_0)-c\varepsilon_2 g^2(\theta_0)}{f(\theta_0)\,g(\theta_0)}\kappa_\alpha+c\varepsilon_1\varepsilon_2.\tag{35}$$

Conversely, for some curve α in $\mathbb{M}_q^3(c)$, its curvature and torsion satisfy (35); that is, there exists a constant λ that satisfies $\varepsilon_3\tau_\alpha^2 + \varepsilon_1\kappa_\alpha^2 = \lambda\kappa_\alpha + c\varepsilon_1\varepsilon_2$. Then we can choose some nonzero constant $\theta_0 \in \mathbb{R}$, such that

$$\lambda=\frac{f^2(\theta_0)-c\varepsilon_2 g^2(\theta_0)}{f(\theta_0)\,g(\theta_0)}.\tag{36}$$

Thus

$$\varepsilon_1\delta_1\left(f(\theta_0)-\varepsilon_1 g(\theta_0)\,\kappa_\alpha\right)=\frac{\varepsilon_3 f(\theta_0)\,g(\theta_0)\,\tau_\alpha^2}{\delta_1\varepsilon_1 f(\theta_0)\,\kappa_\alpha+\delta_1\varepsilon_2 c g(\theta_0)}.\tag{37}$$

We define a curve β and denote $a(s) = \theta_0$.

Let $\beta(\sigma(s)) = f(\theta_0)\alpha(s) + g(\theta_0)N_\alpha(s)$; from (37), it is easy to know that (32) and (33) are satisfied. Moreover,

$$\delta_2\sigma'(s)\,\kappa_\beta N_\beta(\sigma(s))=a_1'(s)\,T_\alpha(s)+a_2'(s)\,B_\alpha(s).\tag{38}$$

By taking the derivative of (38) and applying (34), (28),

$$B_\beta(\sigma(s))=-\varepsilon\varepsilon_2 c g(a(s))\,\alpha(s)+\varepsilon f(a(s))\,N_\alpha(s).\tag{39}$$

The binormal geodesic of curve β at $\beta(\sigma(s_0))$ is denoted by

$$\gamma_{\sigma_0}^\beta(t)=f(t)\,\beta(\sigma(s_0))+g(t)\,B_\beta(\sigma(s_0)).\tag{40}$$

Then we put (39) in (40):

$$\gamma_{\sigma_0}^\beta(t_0)=f(t+a)\,\alpha(s_0)+g(t+a)\,N_\alpha(s_0)=\gamma_{\sigma_0}^\alpha(s_0).\tag{41}$$

Therefore, β is a Mannheim partner curve of α; it means that the curve α is a Mannheim curve. □

After that, we investigate the relationship with respect to the curvature and the torsion of Mannheim partner curves. In similar way to Theorem 5, we obtain the following theorem.

Theorem 6. *Let β be a curve with the arc length parameter σ in $\mathbb{M}_q^3(c)$; then β is the Mannheim partner curve of the Mannheim curve α in $\mathbb{M}_q^3(c)$ if and only if the curvature and torsion of curve β satisfy the following equation:*

$$\tau_\beta'=\frac{\delta_1\kappa_\beta\left(f^2(\theta_1)+g^2(\theta_1)\,\tau_\beta^2\right)}{\delta_2 f(\theta_1)\,g(\theta_1)},\tag{42}$$

where $\{\delta_1,\delta_2,\delta_3\}$ are the causal characters of $\{T_\beta, N_\beta, B_\beta\}$ and κ_β and τ_β are the curvature and torsion of curve β, respectively.

As is known, a helix in a 3-dimensional manifold M is defined by a curve whose curvature and torsion are constants. In [7], Liu and Wang claimed that the Mannheim partner curve of a helix in $\mathbb{R}^3$ is a straight line. Inspired by this result, we give the following example.

Example 7. Let $\mathbb{M}_q^3(c)$ be a 3-dimensional nonflat space form, which is equivalent to $\mathbb{M}_q^3(c) = \mathbb{S}_1^3(1)$ or $\mathbb{M}_q^3(c) = \mathbb{H}_0^3(-1)$, respectively. $\alpha = \alpha(s)$ is a helix in $\mathbb{M}_q^3(c)$ parameterized by the arc length with curvature κ_α and the torsion τ_α.

According to (35), we have

$$\lambda = \frac{\varepsilon_3\tau_\alpha^2 + \varepsilon_1\kappa_\alpha^2 - c\varepsilon_1\varepsilon_2}{\kappa_\alpha}. \tag{43}$$

It means that α is a Mannheim curve in $\mathbb{M}_q^3(c)$. Thus, the Mannheim partner curve β of α in $\mathbb{M}_q^3(c)$ is given by

$$\beta(\sigma) = f(\theta)\,\alpha(s) + g(\theta)\,N_\alpha(s), \tag{44}$$

where N_α is the principal normal vector field of α and σ is the arc length parameter of β.

In the following, we consider the curve α in $\mathbb{S}_1^3$. By taking the derivative of $\beta(\sigma)$, we get

$$\begin{aligned} T_\beta(s)\frac{d\sigma}{ds} &= \cos\theta T_\alpha(s) + \sin\theta\left(-\varepsilon_1\kappa_\alpha T_\alpha(s) + \varepsilon_3\tau_\alpha B_\alpha(s)\right) \\ &= \left(\cos\theta - \varepsilon_1\kappa_\alpha\sin\theta\right)T_\alpha(s) + \varepsilon_3\tau_\alpha\sin\theta B_\alpha(s). \end{aligned} \tag{45}$$

Then

$$\sigma = \sqrt{\left(\cos\theta - \varepsilon_1\kappa_\alpha\sin\theta\right)^2 + \tau_\alpha^2\sin^2\theta}s. \tag{46}$$

According to

$$\lambda = \frac{f^2(\theta) - c\varepsilon_2 g^2(\theta)}{f(\theta)\,g(\theta)}, \tag{47}$$

we obtain that

$$\theta = \frac{1}{2}\arctan\left(\frac{2\kappa_\alpha}{\kappa_\alpha^2 + \tau_\alpha^2 - 1}\right). \tag{48}$$

On the other hand, by using relation (27), (28), and (32), we have $\nabla_\sigma T_\beta = 0$. Moreover, β is a geodesic in $\mathbb{S}_1^3$.

In the following, we take a concrete example.

Let α be a curve in $\mathbb{S}_1^3$ with equation

$$\alpha(s) = \left(\frac{\sqrt{2}}{2}\sinh(2s), \frac{\sqrt{2}}{2}\cosh(2s), \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\cos s\right). \tag{49}$$

By simple calculation, we get

$$\begin{aligned} &\alpha'(s) \\ &\quad = \left(\sqrt{2}\cosh(2s), \sqrt{2}\sinh(2s), \frac{\sqrt{2}}{2}\cos s, -\frac{\sqrt{2}}{2}\sin s\right), \\ &\alpha''(s) \\ &\quad = \left(2\sqrt{2}\sinh(2s), 2\sqrt{2}\cosh(2s), -\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s\right), \\ &\alpha'''(s) \\ &\quad = \left(4\sqrt{2}\cosh(2s), 4\sqrt{2}\sinh(2s), -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\sin s\right). \end{aligned} \tag{50}$$

By using the Schmidt orthogonalization method, we have

$$\begin{aligned} T_\alpha(s) &= \left(\frac{2\sqrt{3}}{3}\cosh(2s), \frac{2\sqrt{3}}{3}\sinh(2s), \frac{\sqrt{3}}{3}\cos s, -\frac{2\sqrt{3}}{3}\sin s\right), \\ N_\alpha(s) &= \left(\frac{4}{\sqrt{17}}\sinh(2s), \frac{4}{\sqrt{17}}\cosh(2s), -\frac{1}{\sqrt{17}}\sin s, -\frac{1}{\sqrt{17}}\cos s\right), \\ B_\alpha(s) &= \left(\frac{8}{3\sqrt{7}}\cosh(2s), \frac{8}{3\sqrt{7}}\sinh(2s), -\frac{1}{3\sqrt{7}}\cos s, \frac{1}{3\sqrt{7}}\sin s\right). \end{aligned} \tag{51}$$

We can easily see that $\langle T, T\rangle = -1$, $\langle N, N\rangle = 1$, and $\langle B, B\rangle = -1$; that is, T is a timelike vector, N is a spacelike vector, and B is a timelike vector. By using the Frenet frame, we obtain

$$\begin{aligned} \nabla_T B &= \left(\frac{16}{3\sqrt{7}}\sinh(2s), \frac{16}{3\sqrt{7}}\cosh(2s), \frac{1}{3\sqrt{7}}\sin s, \frac{1}{3\sqrt{7}}\cos s\right), \\ \nabla_T N &= \left(\frac{8}{\sqrt{17}}\cosh(2s), \frac{8}{\sqrt{17}}\sinh(2s), -\frac{1}{\sqrt{17}}\cos s, \frac{1}{\sqrt{17}}\sin s\right). \end{aligned} \tag{52}$$

Moreover, by the formulae of the curvature and the torsion for a general parameter, we can calculate that

$$\begin{aligned} \langle\nabla_T B, N\rangle &= -\tau = \frac{3\sqrt{7}}{\sqrt{17}}, \\ \langle\nabla_T N, T\rangle &= -\kappa = -\frac{\sqrt{17}}{\sqrt{3}}. \end{aligned} \tag{53}$$

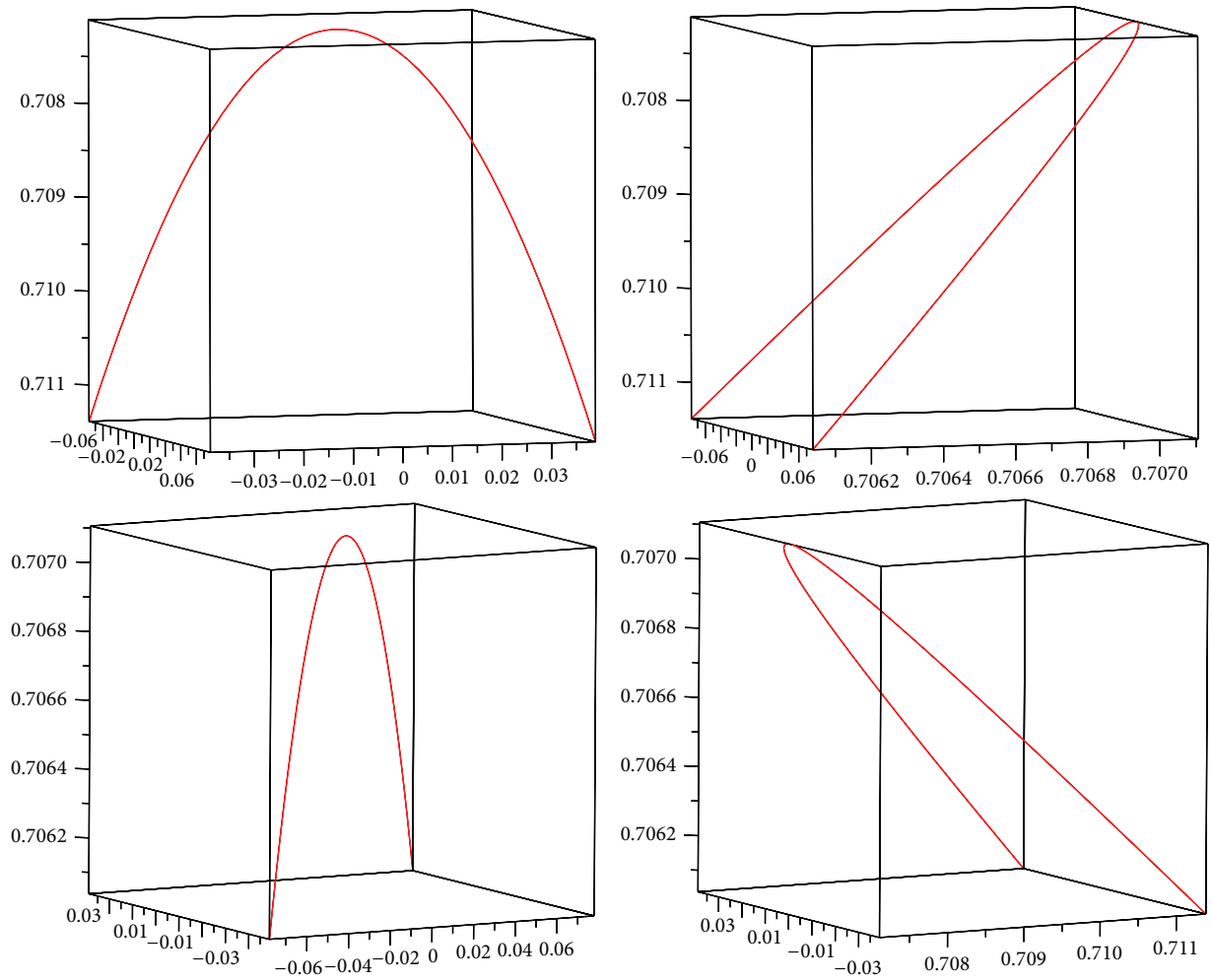

Figure 1: The projected image of Mannheim curve α in $\mathbb{S}_1^3$.

Then $\tau = -3\sqrt{7}/\sqrt{17}$ and $\kappa = \sqrt{17}/\sqrt{3}$. Therefore, the curve α is a helix in $\mathbb{S}_1^3$. Moreover, we know the helix in $\mathbb{S}_1^3$ is Mannheim curve and satisfies Theorem 5.

The projected image of Mannheim curve α in $\mathbb{S}_1^3$ is obtained as Figure 1.

In general, we can take curve α with

$$\alpha(s) = a\sinh(ps), a\cosh(ps), b\sin(qs), b\cos(qs), \tag{54}$$

where a, b, p, and q are constants such that $a^2 + b^2 = 1$ and $-a^2p^2 + b^2q^2 = \pm 1$; it is easy to know that curve α is a unit speed curve in $\mathbb{S}_1^3$. In similar way to Example 7, we get that curve α is Mannheim curve in $\mathbb{S}_1^3$.

4. Null Mannheim Curves in $\mathbb{S}_1^3$, $\mathbb{H}_1^3$

In this section, we will apply ourself to discussing the null Mannheim curves in de Sitter space $\mathbb{S}_1^3$ and anti-de Sitter space $\mathbb{H}_1^3$. Due to the fact that the conclusion about the null Mannheim curve in $\mathbb{H}_1^3$ is similar to the case in $\mathbb{S}_1^3$, then we just give the proof of null Mannheim curve in $\mathbb{S}_1^3$.

Theorem 8. *There is no null Mannheim curve in de Sitter space $\mathbb{S}_1^3$.*

Proof. Let $\alpha = \alpha(s)$ be a null curve in $\mathbb{S}_1^3$, and $\beta = \beta(\sigma)$ a Mannheim partner curve of α in $\mathbb{S}_1^3$, where σ is the arc length parameter of β. Suppose that $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ with casual characters $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ are the frames of curves α and β, respectively. Then there exists a differential function $a(s)$ such that $\beta(\sigma(s)) = f(a(s))\alpha(s) + g(a(s))N_\alpha(s)$, where $f(t) = \cos t$ and $g(t) = \sin t$. Since the principal normal geodesics of curve α coincide with the binormal geodesics of curve β at corresponding points, we obtain

$$\left.\frac{d}{dt}\right|_{t=a(s)} \gamma_s^\alpha(t) = \varepsilon B_\beta(\sigma(s)), \quad \varepsilon = \pm 1, \tag{55}$$

and because of $f'(t) = -g(t)$ and $g'(t) = f(t)$, then

$$\begin{aligned} B_\beta(\sigma(s)) &= \varepsilon\left(f'(a(s))\alpha(s) + g'(a(s))N_\alpha(s)\right) \\ &= -\varepsilon g(a(s))\alpha(s) + \varepsilon f(a(s))N_\alpha(s), \end{aligned} \tag{56}$$

so that $\langle B_\beta(\sigma(s)), B_\beta(\sigma(s))\rangle = \varepsilon^2(f^2 + g^2) = 1$. It means that B_β is a spacelike vector. Thus β is a nonnull curve whose

Frenet equations are similar to Frenet equations (6) in $\mathbb{S}_1^3$. The tangent vector of β is given by

$$\begin{aligned}\sigma'(s)\,T_\beta(\sigma(s)) &= \frac{d}{ds}\beta(\sigma(s))\\ &= -g(a(s))\,a'(s)\,\alpha(s)\\ &\quad + \left(-\kappa_2 g(a(s)) + f(a(s))\right)T_\alpha(s)\\ &\quad + a'(s)\,f(a(s))\,N_\alpha(s) + g(a(s))\,\kappa_1 B_\alpha(s).\end{aligned} \tag{57}$$

By taking the scalar product of (56) with (57), we get $a'(s) = 0$, and it means that $a(s)$ is constant. For the sake of simplicity, we denote $a(s) = \theta_0$; let $\alpha(s) = f(\theta_0)\beta(\sigma(s)) + g(\theta_0)B_\beta(\sigma(s))$; by differentiating $\alpha(s)$ with respect to parameter σ,

$$\frac{ds}{d\sigma}T_\alpha(s) = f(\theta_0)\,T_\beta(\sigma(s)) - \varepsilon_2 g(\theta_0)\,\tau_\beta N_\beta(\sigma(s)). \tag{58}$$

Next, we may distinguish two cases: (A) $\tau_\beta(\sigma(s)) = 0$ and (B) $\tau_\beta(\sigma(s)) \neq 0$.

(A) If $\tau_\beta(\sigma(s)) = 0$ holds,

$$\frac{ds}{d\sigma}T_\alpha(s) = f(\theta_0)\,T_\beta(\sigma(s)). \tag{59}$$

This implies that null vector T_α is collinear with nonnull vector T_β, which is impossible.

(B) If $\tau_\beta(\sigma(s)) \neq 0$ holds,

$$\frac{ds}{d\sigma}T_\alpha(s) = f(\theta_0)\,T_\beta(\sigma(s)) - \varepsilon_2 g(\theta_0)\,\tau_\beta N_\beta(\sigma(s)). \tag{60}$$

By using Frenet equations of curve β, it follows that

$$0 = \left\langle \frac{ds}{d\sigma}T_\alpha(s), \frac{ds}{d\sigma}T_\alpha(s)\right\rangle = \varepsilon_1 f^2(\theta_0) + \varepsilon_2 g^2(\theta_0)\,\tau_\beta^2. \tag{61}$$

Then

$$\tau_\beta^2 = \frac{-\varepsilon_1 f^2(\theta_0)}{\varepsilon_2 g^2(\theta_0)}. \tag{62}$$

Relation (62) implies $\varepsilon_1\varepsilon_2 = -1$, and

$$\tau_\beta = \pm\frac{f(\theta_0)}{g(\theta_0)}. \tag{63}$$

In addition, we assume that $g(\theta_0) \neq 0$, and otherwise $\beta = \pm\alpha$. We distinguish two subcases according to whether α is geodesic or not: (B.1) $\kappa_1 = 0$ and (B.2) $\kappa_1 = 1$.

(B.1) If $\kappa_1 = 0$, then $T_\alpha(s)$ is constant. By taking the derivative of (60) with respect to σ,

$$\begin{aligned}&\frac{d^2s}{d\sigma^2}T_\alpha(s)\\ &\quad = -\varepsilon_1 f(\theta_0)\,\beta(\sigma(s)) - g(\theta_0)\,\kappa_\beta\tau_\beta T_\beta(\sigma(s))\\ &\qquad + \varepsilon_2 f(\theta_0)\,\kappa_\beta N_\beta(\sigma(s)) - \varepsilon_2 g(\theta_0)\,\tau_\beta^2 B_\beta(\sigma(s)).\end{aligned} \tag{64}$$

By taking the scalar product of (56) and (64),

$$0 = \left\langle \frac{d^2s}{d\sigma^2}T_\alpha(s), B_\beta(\sigma(s))\right\rangle = -\varepsilon_2 g(\theta_0)\,\tau_\beta^2. \tag{65}$$

Moreover, $\tau_\beta = 0$ which is a contradiction to the fact that τ_β is nonzero constant.

(B.2) If $\kappa_1 = 1$, by applying (63) to (60), it follows that

$$\frac{ds}{d\sigma}T_\alpha(s) = f(\theta_0)\,T_\beta(\sigma(s)) \mp \varepsilon_2 f(\theta_0)\,N_\beta(\sigma(s)). \tag{66}$$

By differentiating (66) with respect to σ,

$$\begin{aligned}&\frac{d^2s}{d\sigma^2}T_\alpha(s) + \left(\frac{ds}{d\sigma}\right)^2 N_\alpha(s)\\ &\quad = -\varepsilon_1 f(\theta_0)\,\beta(\sigma(s)) \mp f(\theta_0)\,\kappa_\beta T_\beta(\sigma(s))\\ &\qquad + \varepsilon_2 f(\theta_0)\,\kappa_\beta N_\beta(\sigma(s)) \mp \varepsilon_2 f(\theta_0)\,\tau_\beta B_\beta(\sigma(s)).\end{aligned} \tag{67}$$

Then

$$\begin{aligned}\left(\frac{ds}{d\sigma}\right)^4 = \Bigg\langle &\frac{d^2s}{d\sigma^2}T_\alpha(s) + \left(\frac{ds}{d\sigma}\right)^2 N_\alpha(s),\\ &\frac{d^2s}{d\sigma^2}T_\alpha(s) + \left(\frac{ds}{d\sigma}\right)^2 N_\alpha(s)\Bigg\rangle = \tau_\beta^2.\end{aligned} \tag{68}$$

By taking the derivative of $B_\beta(\sigma(s))$ with respect to s,

$$\begin{aligned}&\frac{d}{ds}B_\beta(\sigma(s))\\ &\quad = -\sigma'(s)\,\tau_\beta N_\beta(\sigma(s))\\ &\quad = -\varepsilon\left(g(\theta_0) + f(\theta_0)\,\kappa_2\right)T_\alpha(s) + \varepsilon f(\theta_0)\,B_\alpha(s).\end{aligned} \tag{69}$$

By taking the scalar product of (67) with $N_\beta(\sigma(s))$ and using (68), we obtain $\kappa_\beta = 0$. Since $a(s)$ is constant, (57) is reduced to

$$T_\beta(\sigma(s)) = \frac{ds}{d\sigma}\left[\left(-\kappa_2 g(\theta_0) + f(\theta_0)\right)T_\alpha(s) + g(\theta_0)\,B_\alpha(s)\right]. \tag{70}$$

By applying $\kappa_\beta = 0$ to the Frenet frame of β, we get

$$\nabla_{T_\beta}T_\beta = -\varepsilon_1\beta. \tag{71}$$

On the other hand,

$$\nabla_{T_\beta}T_\beta = \left(\frac{ds}{d\sigma}\right)^2\left[\left(f(\theta_0) - 2\kappa_2 g(\theta_0)\right)N_\alpha(s) + g(\theta_0)\,\alpha(s)\right]. \tag{72}$$

Then

$$\begin{aligned}
&\beta(\sigma(s)) \\
&\quad = -\varepsilon_1 \left(\frac{ds}{d\sigma}\right)^2 \left[\left(f(\theta_0) - 2\kappa_2 g(\theta_0)\right) N_\alpha(s) + g(\theta_0)\alpha(s)\right] \\
&\quad = f(\theta_0)\alpha(s) + g(\theta_0) N_\alpha(s), \\
&\langle \beta(\sigma(s)), \beta(\sigma(s)) \rangle \\
&\quad = 1 \\
&\quad = -\varepsilon_1 \left(\frac{ds}{d\sigma}\right)^2 \left[2f(\theta_0) g(\theta_0) - 2\kappa_2 g^2(\theta_0)\right].
\end{aligned} \tag{73}$$

Therefore,

$$\begin{aligned}
\kappa_2 &= \frac{(d\sigma/ds)^2 + 2\varepsilon_1 f(\theta_0) g(\theta_0)}{2\varepsilon_1 g^2(\theta_0)} \\
&= \frac{f(\theta_0)}{g(\theta_0)} + \frac{\varepsilon_1}{2g^2(\theta_0)} \left(\frac{d\sigma}{ds}\right)^2.
\end{aligned} \tag{74}$$

By taking the scalar product of (69) with itself,

$$\begin{aligned}
&\left\langle -\sigma'(s)\tau_\beta N_\beta(\sigma(s)), -\sigma'(s)\tau_\beta N_\beta(\sigma(s)) \right\rangle \\
&\quad = -f(\theta_0)\left(g(\theta_0) + f(\theta_0)\kappa_2\right) \\
&\quad = \varepsilon_2 \left(\frac{d\sigma}{ds}\right)^2 \tau_\beta^2.
\end{aligned} \tag{75}$$

Then we apply (68) to the last relation

$$\tau_\beta^2 \left[\tau_\beta^2 - f^2(\theta_0)\left(g(\theta_0) + f(\theta_0)\kappa_2\right)^2\right] = 0. \tag{76}$$

Then $\tau_\beta = 0$ or $\tau_\beta^2 - f^2(\theta_0)(g(\theta_0) + f(\theta_0)\kappa_2)^2 = 0$.

If $\tau_\beta^2 - f^2(\theta_0)(g(\theta_0) + f(\theta_0)\kappa_2)^2 = 0$ holds, thus

$$\kappa_2 = \frac{f(\theta_0)}{g(\theta_0)}, \tag{77}$$

which is a contradiction to (74). It is impossible that $\tau_\beta = 0$.

Therefore, there is no null Mannheim curve in de Sitter space. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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