Research Article

Lie Symmetry Analysis of a First-Order Feedback Model of Option Pricing

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A first-order feedback model of option pricing consisting of a coupled system of two PDEs, a nonlinear generalised Black-Scholes equation and the classical Black-Scholes equation, is studied using Lie symmetry analysis. This model arises as an extension of the classical Black-Scholes model when liquidity is incorporated into the market. We compute the admitted Lie point symmetries of the system and construct an optimal system of the associated one-dimensional subalgebras. We also construct some invariant solutions of the model.

1. Introduction

Extensions of the Black-Scholes equation typically lead to nonlinear PDEs that are often tackled with numerical methods. In instances when analytic solutions exist, Lie symmetry methods [1–12] may be employed to construct the solutions. The model we consider in this paper arises as an extension of the Black-Scholes equation when liquidity is incorporated into the market. We start by introducing the Black-Scholes equation to provide a context. The Black-Scholes equation has played a very significant role in the development of modern derivative asset analysis. Based on a small set of assumptions on the price behavior of the underlying asset, the Black-Scholes equation allows investors to calculate the “fair” price of a derivative security. In its simplest form, the Black-Scholes equation requires the estimation of only one parameter that cannot be observed in the market, namely, the market volatility of the underlying asset price, which is assumed to be constant. Another assumption of the Black-Scholes model is that the market is perfectly liquid. However, increases in market volatility of asset prices have been observed in recent years and it seems they are caused by the extensive usage of the Black-Scholes model and the associated hedging strategies for pricing derivative securities [13]. This observed feedback effect of Black-Scholes pricing on the underlying’s price and consequently on the price of the derivatives has led

to extensions to the Black-Scholes model aimed at accounting for this phenomenon [13–16]. In this paper we consider one such extension, the first-order feedback model [17].

2. The First-Order Feedback Model

In the classical Black-Scholes model the price process S_t of the underlying asset is assumed to be governed by a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (1)$$

where S_t is the price of the underlying stock, μ and σ (assumed to be nonnegative constants) are the drift and volatility, respectively, and W_t is a standard Brownian motion. To incorporate liquidity into the Black-Scholes we modify the underlying asset price process. We introduce a forcing term, $f(S_t, t)$, dependent on the stock price and time so that in place of (1) we have

$$dS_t = \mu S_t dt + \sigma S_t dW_t + \lambda(S_t, t) df, \quad (2)$$

where $\lambda(S_t, t)$ is an arbitrary function that models the nature of price impact and liquidity [17].

Applying Itô’s formula on the function $f(S_t, t)$, we obtain

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} (dS_t)^2 + \dots, \quad (3)$$

which when substituted into (2) leads to

$$\left(1 - \lambda \frac{\partial f}{\partial S_t}\right) dS_t = \left(\mu S_t + \lambda \frac{\partial f}{\partial t}\right) dt + \frac{\lambda}{2} \frac{\partial^2 f}{\partial S_t^2} (dS_t)^2 + \sigma S_t dW_t. \quad (4)$$

By simply squaring both sides of (4) and applying the usual rules, $dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0 = 0$ and $dW_t \cdot dW_t = dt$ [18], we obtain

$$(dS_t)^2 = \frac{\sigma^2 S_t^2 dt}{(1 - \lambda (\partial f / \partial S_t))^2} + o(dt) \quad (\text{as } dt \rightarrow 0). \quad (5)$$

Substituting (5) into (4) and rearranging, we arrive at a stochastic process analogous to (1):

$$dS_t = \tilde{\mu}(S_t, t) dt + \tilde{\sigma}(S_t, t) dW_t, \quad (6)$$

where

$$\begin{aligned} \tilde{\mu}(S_t, t) &= \frac{1}{1 - \lambda (\partial f / \partial S_t)} \left[\mu S_t + \lambda \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial S_t^2} \right) \right], \\ \tilde{\sigma}(S_t, t) &= \frac{\sigma S_t}{1 - \lambda (\partial f / \partial S_t)}. \end{aligned} \quad (7)$$

Under this setting, where the stock price follows the modified stochastic process (6), it is deduced via standard arguments that V , the derivative price, satisfies the *generalised Black-Scholes* equation (see Duffie [19] for more details):

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\sigma^2 S_t^2}{(1 - \lambda (\partial f / \partial S_t))^2} \frac{\partial^2 V}{\partial S_t^2} + r S_t \frac{\partial V}{\partial S_t} - rV = 0, \quad (8)$$

where r is the risk-free interest rate, assumed to be a positive constant. When (8) is used as a model for the value of a *European call option*, for example, then at expiry time T we have that

$$V(S_T, T) = \max\{S_T - K, 0\}, \quad (9)$$

where K is the strike price.

Consistent with standard Black-Scholes arguments, the drift of the modified process $\tilde{\mu}(S_t, t)$ does not appear in the option pricing PDE. In the context of markets with finite elasticity, $f(S_t, t)$ can be interpreted as the number of extra shares that should be held due to some deterministic hedging/trading strategy and hence $df(S_t, t)$ specifies the number of shares needed to be bought or sold at time t and price S_t due to such a strategy. If we are interested in the price impact due to delta hedging then we can identify f as the delta of the option being replicated; that is,

$$f = \Delta^* = \frac{\partial V^*}{\partial S_t}. \quad (10)$$

This leads to the question of what strategy the hedgers are assumed to follow. Either they use the Black-Scholes

option delta to hedge, ignoring price impact, or they try to incorporate price impact into the hedging strategy by using the modified delta. The first case is called the first-order feedback, and in this case (8) becomes a linear PDE:

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\sigma^2 S_t^2}{(1 - \lambda (\partial^2 V^{\text{BS}} / \partial S_t^2))^2} \frac{\partial^2 V}{\partial S_t^2} + r S_t \frac{\partial V}{\partial S_t} - rV \\ = 0, \end{aligned} \quad (11)$$

where V^{BS} is the solution to the standard Black-Scholes equation:

$$\frac{\partial V^{\text{BS}}}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V^{\text{BS}}}{\partial S_t^2} + r S_t \frac{\partial V^{\text{BS}}}{\partial S_t} - rV^{\text{BS}} = 0. \quad (12)$$

Both (11) and (12) are subject to the same terminal condition (9), with V being replaced by V^{BS} in the case of (12).

The other scenario, in which the hedger is assumed to be aware of the feedback effect and so would change the hedging strategy accordingly, corresponds to the case when $V^* \equiv V$. In this case the trading strategy adopted has to be found as part of the problem. This is called the full feedback and leads to the nonlinear PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\sigma^2 S_t^2}{(1 - \lambda (\partial^2 V / \partial S_t^2))^2} \frac{\partial^2 V}{\partial S_t^2} + r S_t \frac{\partial V}{\partial S_t} - rV = 0. \quad (13)$$

We will apply Lie symmetry analysis to the first-order feedback model (11), which is coupled with the standard Black-Scholes equation (12). Investigation of solutions of differential equations via Lie symmetry analysis has been done to many problems in financial mathematics, for example, [20–24]. The primary objective of the present study is to determine general solutions (as invariant solutions) of the first-order feedback model (11).

3. The Admitted Lie Point Symmetries

For convenience, we let u , v , and x represent V^{BS} , V , and S_t , respectively. Further, we use the standard substitution

$$\tau = T - t, \quad (14)$$

representing time to expiry of the option (and rename τ as t), so that (11) and (12) are represented by

$$v_t - \frac{A^2 x^2 v_{xx}}{2(1 - \lambda u_{xx})^2} - Bxv_x + Cv = 0, \quad (15)$$

$$u_t - \frac{1}{2} A^2 x^2 u_{xx} - Bxu_x + Cu = 0, \quad (16)$$

respectively, with $A = \sigma$ and $B = C = r$. The constants A , B , and C are to be considered nonzero.

Remark 1. System (15)-(16) is a generalisation somewhat of (11) and (12). The symmetries reported in (19) and Tables 1 and 2 are based on this generalised system.

TABLE 1: Commutator table of the symmetry algebra admitted by (15)-(16).

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	0	0	0	CX_1	X_1	$-X_1$
X_2	0	0	0	0	CX_2	0	0
X_3	0	0	0	0	$(C-B)X_3$	X_3	$\left(\frac{B}{C} - \frac{3}{2}\right)X_3$
X_4	0	0	0	0	$(C-B)X_4$	0	$\left(\frac{B}{C} - \frac{1}{2}\right)X_4$
X_5	$-CX_1$	$-CX_2$	$(B-C)X_3$	$(B-C)X_4$	0	0	0
X_6	$-X_1$	0	$-X_3$	0	0	0	0
X_7	X_1	0	$\left(\frac{3}{2} - \frac{B}{C}\right)X_3$	$\left(\frac{1}{2} - \frac{B}{C}\right)X_4$	0	0	0

TABLE 2: Table of adjoint representations of the symmetries (19).

$\text{Ad}(e^{\varepsilon X_i})$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	X_1	X_2	X_3	X_4	$X_5 - \varepsilon CX_1$	$X_6 - \varepsilon X_1$	$X_7 + \varepsilon X_1$
X_2	X_1	X_2	X_3	X_4	$X_5 - \varepsilon CX_2$	X_6	X_7
X_3	X_1	X_2	X_3	X_4	$X_5 + \varepsilon(B-C)X_3$	$X_6 - \varepsilon X_3$	$X_7 + \varepsilon\left(\frac{3}{2} - \frac{B}{C}\right)X_3$
X_4	X_1	X_2	X_3	X_4	$X_5 + \varepsilon(B-C)X_4$	X_6	$X_7 + \varepsilon\left(\frac{1}{2} - \frac{B}{C}\right)X_4$
X_5	$e^{\varepsilon C}X_1$	$e^{\varepsilon C}X_2$	$e^{\varepsilon(C-B)}X_3$	$e^{\varepsilon(C-B)}X_4$	X_5	X_6	X_7
X_6	$e^{\varepsilon}X_1$	X_2	$e^{\varepsilon}X_3$	X_4	X_5	X_6	X_7
X_7	$e^{-\varepsilon}X_1$	X_2	$e^{\varepsilon(B/C-3/2)}X_3$	$e^{\varepsilon(B/C-1/2)}X_4$	X_5	X_6	X_7

We will apply the standard machinery of symmetry analysis to find particular solutions of (15)-(16). Let us consider the one-parameter Lie group of transformations in the (x, t, u, v) -space given by

$$\begin{aligned}
 \tilde{x} &= x + \varepsilon \xi^1(x, t, u, v) + O(\varepsilon^2), \\
 \tilde{t} &= t + \varepsilon \xi^2(x, t, u, v) + O(\varepsilon^2), \\
 \tilde{u} &= u + \varepsilon \eta^1(x, t, u, v) + O(\varepsilon^2), \\
 \tilde{v} &= v + \varepsilon \eta^2(x, t, u, v) + O(\varepsilon^2),
 \end{aligned} \tag{17}$$

where ε is the group parameter, with infinitesimal generator

$$X = \xi^1 \partial_x + \xi^2 \partial_t + \eta^1 \partial_u + \eta^2 \partial_v. \tag{18}$$

For (17) to be admitted by (15)-(16), one requires that the transformation leaves invariant the set of solutions of system (15)-(16). Applying program Lie [25], we obtain that the Lie algebra L_7 of infinitesimal generators of system (15)-(16) is spanned by the seven operators:

$$\begin{aligned}
 X_1 &= e^{-Ct} \partial_v, \\
 X_2 &= e^{-Ct} \partial_u, \\
 X_3 &= e^{(B-C)t} x \partial_v, \\
 X_4 &= e^{(B-C)t} x \partial_u,
 \end{aligned}$$

$$X_5 = \partial_t,$$

$$X_6 = v \partial_v,$$

$$X_7 = \frac{x}{2} \partial_x - \frac{1}{C} \partial_t + u \partial_u.$$

(19)

Table 1 is the commutator table for the operators (19).

4. Determination of Invariant Solutions

An invariant solution is a solution that is mapped into itself by a group transformation. Such solutions are very important in that the majority of exact solutions that have important real world applications are reportedly invariant solutions [3, p. 29]. The algorithm for constructing invariant solutions of a given system of differential equations using admitted symmetries is well documented in many standard books on Lie symmetry analysis [1, 5, 7–9]. Each of the operators in (19) is of the form

$$X = \xi^1 \partial_x + \xi^2 \partial_t + \eta^1 \partial_u + \eta^2 \partial_v, \tag{20}$$

and generates a one-dimensional subalgebra $\langle X \rangle$ of the algebra spanned by (19). We consider invariant solutions associated with X . Variables in which (15)-(16) reduces to a coupled system of ordinary differential equations are determined by the invariants of X , which are obtained by solving the quasilinear PDE:

$$XJ(x, t, u, v) = 0, \tag{21}$$

where J is some function of the independent and dependent variables. The general solution of (21) is

$$L(I_1, I_2, I_3) = 0, \quad (22)$$

where L is an arbitrary function and

$$\begin{aligned} \psi_1(x, t, u, v) &= I_1, \\ \psi_2(x, t, u, v) &= I_2, \\ \psi_3(x, t, u, v) &= I_3 \end{aligned} \quad (23)$$

are functionally independent solutions of the characteristic system

$$\frac{dx}{\xi^1} = \frac{dt}{\xi^2} = \frac{du}{\eta^1} = \frac{dv}{\eta^2}. \quad (24)$$

The invariants I_1 , I_2 , and I_3 in (23) are now used to determine new variables in which (15)-(16) should be written so that it is reduced to a second-order system of ODEs, the solution of which leads to a family of invariant solutions of (15)-(16) associated with X .

4.1. Invariant Solutions of an Arbitrary Element of L_7 . In principle every element from L_7 can be used to construct an invariant solution of system (15)-(16). Take, for example, X taken as

$$X = X_1 + X_2 + X_3 + X_4 + \left(\frac{1}{C}\right) X_5 + X_6 + X_7, \quad (25)$$

$(B = C = r),$

in which case the infinitesimal coefficients are

$$\begin{aligned} \xi^1 &= \frac{x}{2}, \\ \xi^2 &= 0, \\ \eta^1 &= e^{-rt} + u + x, \\ \eta^2 &= e^{-rt} + v + x. \end{aligned} \quad (26)$$

The invariants of (25) are obtained as solutions of the characteristic equation (24), with the infinitesimals given in (26). We obtain

$$\begin{aligned} I_1 &= t, \\ I_2 &= \frac{e^{-rt} + 2x + u}{x^2}, \\ I_3 &= \frac{e^{-rt} + 2x + v}{x^2}. \end{aligned} \quad (27)$$

The functional form of the invariant solution is now constructed from these invariants. Taking the solution in the form

$$\begin{aligned} I_2 &= f(I_1), \\ I_3 &= g(I_1), \end{aligned} \quad (28)$$

we obtain that invariant solutions of (15)-(16) arising from (25) have the form

$$\begin{aligned} u(x, t) &= x^2 f(t) - 2x - e^{-rt}, \\ v(x, t) &= x^2 g(t) - 2x - e^{-rt}, \end{aligned} \quad (29)$$

where f and g are arbitrary functions. When v and u as prescribed in (29) are substituted in (15)-(16) we obtain the following pair of ODEs:

$$\begin{aligned} f'(t) - \omega f(t) &= 0, \\ (1 - 2\lambda)^2 g'(t) - [4r\lambda(\lambda f^2(t) - f(t)) + \omega] g(t) &= 0, \quad \omega = \sigma^2 + r. \end{aligned} \quad (30)$$

These equations are easily solved as linear first-order ODEs, and we obtain

$$\begin{aligned} f(t) &= K_1 e^{\omega t}, \\ g(t) &= K_2 \exp \left\{ \omega t + \frac{\sigma^2}{\omega(1 - \lambda K_1 e^{\omega t})} \right\} \\ &\quad \cdot (2\lambda K_1 e^{\omega t} - 1)^{(-\sigma^2/\omega)}, \end{aligned} \quad (31)$$

where K_1 and K_2 are arbitrary constants. Therefore, invariant solutions of (15)-(16) arising from (25) are of the form

$$\begin{aligned} u(x, t) &= e^{-rt} (K_1 x^2 e^{(r+\omega)t} - 1 - 2xe^{rt}), \\ v(x, t) &= e^{-rt} (\Psi(x, t) - 1 - 2xe^{rt}), \end{aligned} \quad (32)$$

where

$$\begin{aligned} \Psi(x, t) &= \exp \left[(r + \omega)t + \frac{\sigma^2}{\omega - e^{\omega t} K_1 \lambda \omega} \right] \\ &\quad \cdot K_2 x^2 (2e^{\omega t} \lambda K_1 - 1)^{(-\sigma^2/\omega)}. \end{aligned} \quad (33)$$

4.2. Optimal System of One-Dimensional Subalgebras. How every one-dimensional subalgebra of the algebra L_7 may be used to find a family of invariant solutions of (15)-(16) has just been illustrated. The exercise of finding all such families from all possible elements of L_7 is reduced to that of finding families of invariant solutions associated with only a small number of inequivalent symmetries, called an optimal system of one-dimensional subalgebras [6, 8]. Two elements of L_7 are equivalent if the family of invariant solutions associated with one element can be transformed into a family of invariant solutions associated with the other by one of the Lie point symmetries of L_7 . We set out to construct an optimal system of one-dimensional subalgebras of L_7 .

Following Olver's approach [8] we start by constructing the adjoint presentation of the Lie group generated by (19) on its Lie algebra. Each X_i of the basis symmetries (19) generates

an adjoint representation $\text{Ad}(\exp(\varepsilon X_i))$ defined by the Lie series

$$\begin{aligned} \text{Ad}(\exp(\varepsilon X_i)) X_j &= X_j - \varepsilon [X_i, X_j] \\ &+ \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] - \dots \end{aligned} \quad (34)$$

The meaning of (34) is that the generator $\text{Ad}(\exp(\varepsilon X_i)) X_j$ is equivalent to X_j under the Lie group generated by X_i . All the adjoint actions for the operators (19) are given in Table 2, with the (i, j) th entry indicating $\text{Ad}(\exp(\varepsilon X_i)) X_j$.

We construct the optimal system via the naive approach of taking a general element X and subjecting it to various adjoint transformations to simplify it as much as possible [8]. Given a nonzero vector

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + \dots + a_7 X_7 \quad (35)$$

we endeavour to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to X . We deal with particular cases depending on which of the coefficients a_i are nonzero.

4.3. $a_7 \neq 0$. Scaling away a_7 from (35), since $a_7 \neq 0$, we can write (35) equivalently as

$$\begin{aligned} X &= a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 + a_6 X_6 \\ &+ X_7. \end{aligned} \quad (36)$$

If we now act on (36) by $\text{Ad}(e^{\varepsilon_1 X_1})$ we obtain

$$\begin{aligned} (a_1 - a_6 - r a_5 + 1) X_1 &+ a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 \\ &+ a_6 X_6 + X_7. \end{aligned} \quad (37)$$

We can make the coefficient of X_1 vanish by setting

$$\varepsilon_1 = \frac{a_1}{a_6 + r a_5 - 1}, \quad a_6 \neq 1 - r a_5. \quad (38)$$

With this choice of ε_1 , we obtain

$$\begin{aligned} X' &= \text{Ad}(e^{\varepsilon_1 X_1}) X \\ &= a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 + a_6 X_6 + X_7. \end{aligned} \quad (39)$$

Similarly X' in (39) is simplified further by using $\text{Ad}(e^{\varepsilon_2 X_2})$ with

$$\varepsilon_2 = \frac{a_2}{r a_5}, \quad a_5 \neq 0. \quad (40)$$

We obtain

$$X'' = a_3 X_3 + a_4 X_4 + a_5 X_5 + a_6 X_6 + X_7. \quad (41)$$

Further simplification of X'' is achieved with $\text{Ad}(e^{\varepsilon_3 X_3})$ followed by $\text{Ad}(e^{\varepsilon_4 X_4})$, with $\varepsilon_3 = 2a_3/(2a_6 - 1)$ and $\varepsilon_4 = 2a_4$. We obtain the following simplifications:

$$\begin{aligned} X''' &= \text{Ad}(e^{\varepsilon_3 X_3}) X'' = a_4 X_4 + a_5 X_5 + a_6 X_6 + X_7, \\ a_6 &\neq \frac{1}{2}, \end{aligned} \quad (42)$$

$$X'''' = \text{Ad}(e^{\varepsilon_4 X_4}) X''' = a_5 X_5 + a_6 X_6 + X_7.$$

No further simplification is possible through the action of adjoint maps. This means that every one-dimensional subalgebra generated by a general element X with $a_7 \neq 0$ is equivalent to the subalgebra spanned by

$$X''' = a_5 X_5 + a_6 X_6 + X_7. \quad (43)$$

The remaining subalgebras are spanned by vectors of the form (35) with $a_7 = 0$.

4.4. $a_6 \neq 0$. In this case the general element can be written in the form

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 + X_6. \quad (44)$$

The simplification of (44) is achieved as follows:

$$\begin{aligned} X' &= \text{Ad}(e^{\varepsilon_1 X_1}) X \\ &= a_2 X_2 + a_3 X_3 + a_4 X_4 + a_5 X_5 + X_6, \end{aligned} \quad (45)$$

$$X'' = \text{Ad}(e^{\varepsilon_2 X_2}) X' = a_3 X_3 + a_4 X_4 + a_5 X_5 + X_6, \quad (46)$$

$$X''' = \text{Ad}(e^{\varepsilon_3 X_3}) X'' = a_4 X_4 + a_5 X_5 + X_6, \quad (47)$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{a_1}{1 + a_5 r}, \\ \varepsilon_2 &= \frac{a_2}{a_5 r}, \\ \varepsilon_3 &= a_3, \end{aligned} \quad (48)$$

$$a_5 \neq 0, \quad a_5 \neq -\frac{1}{r}.$$

Note that none of the X_i 's in (47) can be eliminated by adjoint transformations, but we can scale away the coefficient of X_4 to obtain

$$\begin{aligned} X'''' &= \text{Ad}(e^{\varepsilon_4 X_4}) X''' = X_4 + a_5 X_5 + X_6, \\ \varepsilon_4 &= 2 \ln \left(\frac{1}{a_4} \right), \quad a_4 > 0. \end{aligned} \quad (49)$$

4.5. $a_5 \neq 0$. The general element has the form $X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + X_5$ and is simplified as follows:

$$\begin{aligned} X' &= \text{Ad}(e^{\varepsilon_1 X_1}) X = a_2 X_2 + a_3 X_3 + a_4 X_4 + X_5, \\ \varepsilon_1 &= \frac{a_1}{r}, \end{aligned}$$

$$X'' = \text{Ad}(e^{\varepsilon_2 X_2}) X' = a_3 X_3 + a_4 X_4 + X_5, \quad \varepsilon_2 = \frac{a_2}{r}, \quad (50)$$

$$X''' = \text{Ad}(e^{\varepsilon_3 X_3}) X'' = X_3 + a_4 X_4 + X_5,$$

$$\varepsilon_3 = \ln \left(\frac{1}{a_3} \right), \quad a_3 > 0.$$

4.6. $a_4 \neq 0$. Simplification of the general element

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + X_4 \quad (51)$$

through the action of adjoint maps is not possible in this case!

4.7. $a_3 \neq 0$. In this case, the general element $X = a_1 X_1 + a_2 X_2 + X_3$ is simplified through the adjoint map $\text{Ad}(e^{\varepsilon_1 X_6})X$ with $\varepsilon_1 = \ln(a_2)$, $a_2 > 0$. The resulting operator is then divided by e^{ε_1} . We obtain

$$X' = \frac{\text{Ad}(e^{\varepsilon_1 X_6})X}{e^{\varepsilon_1}} = a_1 X_1 + X_2 + X_3. \quad (52)$$

4.8. $a_2 \neq 0$. For $X = a_1 X_1 + X_2$, we can only scale away the coefficient a_1 :

$$X' = \text{Ad}(e^{\varepsilon_1 X_6})X = X_1 + X_2, \quad (53)$$

$$\varepsilon_1 = \ln\left(\frac{1}{a_1}\right), \quad a_1 > 0.$$

We have therefore constructed an optimal system, (54), of one-dimensional subalgebras for the first-order feedback model (15)-(16):

Γ_i	$\sum X_i$	$\xi^1 \partial_x + \xi^2 \partial_t + \eta^1 \partial_v + \eta^2 \partial_u$
Γ_1	X_1	$e^{-rt} \partial_v$
Γ_2	$X_1 + X_2$	$e^{-rt} (\partial_u + \partial_v)$
Γ_3	$a_1 X_1 + X_2 + X_3$	$e^{-rt} \partial_u + (a_1 e^{-rt} + x) \partial_v$
Γ_4	$a_1 X_1 + a_2 X_2 + a_3 X_3 + X_4$	$(a_2 e^{-rt} + x) \partial_u + (a_1 e^{-rt} + a_3 x) \partial_v$
Γ_5	$X_3 + a_4 X_4 + X_5$	$\partial_t + a_4 x \partial_u + x \partial_v$
Γ_6	$X_4 + a_5 X_5 + X_6$	$a_5 \partial_t + x \partial_u + v \partial_v$
Γ_7	$a_5 X_5 + a_6 X_6 + X_7$	$\frac{x}{2} \partial_x + \left(a_5 - \frac{1}{r}\right) \partial_t + u \partial_u + a_6 v \partial_v$

5. Invariant Solutions of Elements of the Optimal System

In this section we construct invariant solutions associated with elements of the constructed optimal system. We follow the procedure outlined and illustrated in Section 4.

5.1. $X = \alpha X_1 + \beta X_2 + \gamma X_3$, $\beta \neq 0$. Solving the characteristic equation, (24), corresponding to $X = \alpha X_1 + \beta X_2 + \gamma X_3$ we obtain the invariants

$$I_1 = x,$$

$$I_2 = t,$$

$$I_3 = v - \frac{u(\alpha + \gamma x e^{rt})}{\beta}, \quad (55)$$

from which we deduce the form of the invariant solution,

$$v(x, t) = \frac{(\alpha + \gamma e^{rt} x) u(x, t)}{\beta} + f(t), \quad (56)$$

where $u(x, t)$ is any solution to the *Black-Scholes* equation, (16), and $f(t)$ is an arbitrary function. Taking, for example, the simple solution

$$u(x, t) = e^{-rt} \quad (57)$$

of (16), (15) reduces to the ordinary differential equation:

$$rf(t) + f'(t) = 0, \quad (58)$$

the solution of which is

$$f(t) = K_1 e^{-rt}, \quad (59)$$

where K_1 is an arbitrary constant. Hence one family of invariant solutions of system (15)-(16) arising from $X = \alpha X_1 + \beta X_2 + \gamma X_3$ have the form

$$u(x, t) = e^{-rt},$$

$$v(x, t) = \frac{e^{-rt} (\alpha + \beta K_1 + \gamma e^{rt} x)}{\beta}. \quad (60)$$

Remark 2. We remark here that in place of (57) one could use any solution of (16). One could, for example, use an invariant solution of (16) such as

$$u(x, t) = \exp \left\{ -\frac{(\sigma^2 + 2r)^2 t^2 + 4(\ln x)^2}{8\sigma^2 t} \right\} \cdot x^{((\sigma^2 - 2r)/2\sigma^2)} t^{-1/2} \quad (61)$$

which arises from the following symmetry of (15):

$$X = \sigma^2 t x \partial_x + u \left[\left(\frac{\sigma^2}{2} - r \right) t - \ln x \right] \partial_u. \quad (62)$$

5.2. $X = X_4 + \gamma X_5 + X_6$, $\gamma \neq 0$. Invariant solutions of system (15)-(16) arising from X in this case are found to be of the form

$$u(x, t) = \frac{tx + \gamma f(x)}{\gamma}, \quad (63)$$

$$v(x, t) = e^{t/\gamma} g(x).$$

Substituting (63) in system (15)-(16) leads to the following set of ordinary differential equations:

$$x + r\gamma [f(x) - xf'(x)] - \frac{\gamma \sigma^2 x^2}{2} f''(x) = 0, \quad (64)$$

$$\gamma x [2rg'(x) (\lambda f''(x) - 1)^2 + \sigma^2 x g''(x)] - 2(1 + \gamma r) (\lambda f''(x) - 1)^2 g(x) = 0. \quad (65)$$

Solving (64) for f , we obtain

$$f(x) = K_1 x^{-2r/\sigma^2} + K_2 x + \frac{2x [(\sigma^2 + 2r) \ln x - \sigma^2]}{\gamma (\sigma^2 + 2r)^2}, \quad (66)$$

where K_1 and K_2 are arbitrary constants. The solution of (65) taking into account the (66) completes the process. It is however not easy to solve (65) analytically.

5.3. $X = X_3 + \gamma X_4 + X_5$. In this case X dictates that u and v that solve (15)-(16) take the form

$$\begin{aligned} u(x, t) &= \gamma tx + f(x), \\ v(x, t) &= tx + g(x). \end{aligned} \quad (67)$$

Substituting (67) into system (15)-(16), we obtain the following pair of coupled ODEs:

$$2rf(x) - x[\sigma^2 x f''(x) - 2\gamma + 2rf'(x)] = 0, \quad (68)$$

$$\begin{aligned} [\lambda f''(x) - 1]^2 [2rg(x) + x(2 - 2rg'(x))] \\ - \sigma^2 x^2 g''(x) = 0. \end{aligned} \quad (69)$$

Equation (68) is a linear ODE and is solved to give

$$\begin{aligned} f(x) &= C_1 x^{-2r/\sigma^2} + C_2 x \\ &+ 2\gamma x \left(\frac{\ln x}{\sigma^2 + 2r} - \frac{\sigma^2}{(\sigma^2 + 2r)^2} \right), \end{aligned} \quad (70)$$

where C_1 and C_2 are arbitrary constants. Equation (69) is to be solved for g .

5.4. $X = \alpha X_5 + \beta X_6 + X_7$, $\alpha \neq 1/r$. Invariant solutions arising from this element of the optimal system are such that v and u have the forms

$$\begin{aligned} u(x, t) &= \exp\left(-\frac{rt}{1-\alpha r}\right) f(\zeta), \\ v(x, t) &= \exp\left(-\frac{\beta rt}{1-\alpha r}\right) g(\zeta), \end{aligned} \quad (71)$$

where f and g are arbitrary functions ζ , and

$$\zeta = x \exp\left(\frac{rt}{2(1-\alpha r)}\right). \quad (72)$$

Substituting (71) into system (15)-(16), we obtain the following coupled system of ODEs:

$$\begin{aligned} 2\alpha r^2 f(\zeta) + (r - 2\alpha r^2) \zeta f'(\zeta) + \sigma^2 (1 - \alpha r) \zeta^2 f''(\zeta) \\ = 0, \end{aligned} \quad (73)$$

$$a(\zeta) g(\zeta) + b(\zeta) g'(\zeta) + c(\zeta) g''(\zeta) = 0, \quad (74)$$

where

$$\begin{aligned} a(\zeta) &= [(2\beta - 1)r - \gamma_1] [\lambda f''(\zeta) - 1]^2 = 0, \\ b(\zeta) &= \gamma_1 \zeta (\lambda f''(\zeta) - 1)^2, \\ c(\zeta) &= \sigma^2 \zeta^2 - (\sigma^2 - \gamma_2) \zeta^2, \\ \gamma_0 &= 2\alpha r^2, \\ \gamma_1 &= r - 2\alpha r^2, \\ \gamma_2 &= \sigma^2 (1 - \alpha r). \end{aligned} \quad (75)$$

Equation (73) is a linear second-order ODE with constant coefficients. The solution is

$$\begin{aligned} f(\zeta) &= \zeta^{-(\gamma_1 - \gamma_2 + \sqrt{\gamma_1^2 - 4\gamma_0\gamma_2 - 2\gamma_1\gamma_2 + \gamma_2^2})/2\gamma_2} \left[C_1 \right. \\ &\left. + C_2 \zeta^{\sqrt{\gamma_1^2 - 4\gamma_0\gamma_2 - 2\gamma_1\gamma_2 + \gamma_2^2}/\gamma_2} \right], \end{aligned} \quad (76)$$

where C_1 and C_2 are arbitrary constants. This leaves (74) to be solved for g .

Remark 3. We remark that (65), (69), and (74) are variable coefficient and linear second-order ODEs, which are of the form

$$p(x) \frac{d^2 y}{dx^2} + q(x) \frac{dy}{dx} + r(x) y = f(x), \quad (77)$$

with $f(x) = 0$.

The analysis of such equations can be found in many standard books on ordinary differential equations such as [26, 27]. Also numerical methods are available for solution of differential equations of the form (77).

5.5. *Interpretation of the Invariant Solutions.* Analytic solutions of the first-order feedback problem obtained as group invariant solutions in this study have various uses. They may be used to benchmark numerical schemes developed for solution of the first-order feedback problem. They may also be used to differentiate between types of auxiliary conditions which lead to qualitatively different forms of option prices. Perhaps more importantly, the solutions can be used to provide insight into valuation dynamics of the first-order feedback model. A phenomenon of particular interest which can be investigated using the invariant solutions is the relationship between the value of the option and the liquidity parameter λ . However, not all the invariant solutions generated have direct relevance to the context of the first-order feedback problem. Among the solutions constructed, we identify two solutions (corresponding to $X = X_4 + \gamma X_5 + X_6$ and $X = X_3 + \gamma X_4 + X_5$) that depict an important feature of the first-order feedback model. In Figure 1 we illustrate the option value in these cases for the liquidity parameters $\lambda = 1/4$, $\lambda = 1/2$, $\lambda = 3/4$, and $\lambda = 1$, with time to maturity $T = 1$, risk-free rate $r = 0.04$, and volatility $\sigma = 0.2$. There are other free parameters in the solution that ought to be assigned suitable values. The graphs in the two cases considered were generated with the help of Mathematica's `NDSolve` command [28]. In both cases the option value exhibits the important feature whereby as λ is increased, the option value is eroded monotonically.

6. Concluding Remarks

In this paper we have used Lie symmetry analysis to study a nonlinear equation that arises as a generalisation of the Black-Scholes equation when liquidity is incorporated into

FIGURE 1: Option value with first-order feedback as a function of the liquidity parameter λ when $T = 1$, $r = 0.04$, and $\sigma = 0.2$, and $K_1 = 12$, $K_2 = 5$, and $\gamma = 1$ for the solution arising from (a); $K_1 = K_2 = 1$ and $\gamma = 1/10$ for the solution arising from (b).

the market. The generalised Black-Scholes equation is called a first-order feedback model and is coupled with the standard Black-Scholes equation as a result of the assumption that a hedger holds the number of stocks dictated by the standard Black-Scholes equation delta rather than the delta from the modified option price in the market with liquidity. We have exploited the Lie point symmetries admitted by the coupled system to construct particular solutions that represent option prices in the modified Black-Scholes market.

As explained in Section 2, an improvement to the first-order feedback model is the full feedback (13), a fully nonlinear PDE. Existence and uniqueness of the solution to (13) are established by Frey [14], who has shown that options in such a market can be perfectly replicated, an attestation to the completeness of the market. Unfortunately (13) is poorly endowed with Lie point symmetries and, therefore, does not lend itself to any interesting analysis via Lie point symmetries. Perhaps analysis of this equation would benefit from the use of more general symmetries.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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