Research Article

Explicit Wave Solutions and Qualitative Analysis of the (1 + 2)-Dimensional Nonlinear Schrödinger Equation with Dual-Power Law Nonlinearity

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The (1 + 2)-dimensional nonlinear Schrödinger equation with dual-power law nonlinearity is studied using the factorization technique, bifurcation theory of dynamical system, and phase portraits analysis. From a dynamic point of view, the existence of smooth solitary wave, and kink and antikink waves is proved and all possible explicit parametric representations of these waves are presented.

1. Introduction

The nonlinear Schrödinger equation (NLSE) is studied in various areas of applied mathematics, theoretical physics, and engineering. In particular, it appears in the study of nonlinear optics, plasma physics, fluid dynamics, biochemistry, and many other areas. This equation is completely integrable by the inverse scattering transform [1]. There have been various forms of this equation that arise in the study of different areas. They have different forms of nonlinearity. Some previous works concentrated on the Kerr law nonlinearity media [2–5]. However, as one increases the intensity of the incident light power to produce shorter (femtosecond) pulses, non-Kerr nonlinearity effects become prominent, and the dynamics of pulses should be described by the nonlinear Schrödinger family of equations with higher order nonlinear terms [6]. Hence, it is very important that all higher order effects are considered in the propagation of femtosecond pulses [7].

The (1 + 2)-dimensional nonlinear Schrödinger equation with dual-power law nonlinearity is given in [8–13]:

\[ \dot{q} + \frac{1}{2} \left( q_{xx} + q_{yy} \right) + \left( |q|^{2m} + k |q|^{4m} \right) q = 0, \tag{1} \]

where \( q(x, y, t) \) is a complex function that indicates the complex amplitude of the wave form, \( i^2 = -1 \), \( m \) is an arbitrary nonzero constant, and \( k \) is a constant which indicates the saturation of the nonlinear refractive index. Biswas [8] obtained the 1-soliton solution and calculated a few conserved quantities. With the aid of symbolic computation, Zhang and Si [9] obtained soliton solutions, combined soliton solutions, triangular periodic solutions, and rational function solutions. Some soliton solutions are obtained in [10] using the \( G/G' \) - expansion method. For \( m = 1 \), the dark 1-soliton solution is obtained by the ansatz method, the invariance, conservation laws, and double reductions in [11]. Bulut et al. [12] found some soliton solutions, rational and elliptic function solutions using the extended trial equation method. The bifurcation is considered in [13] and two implicit solutions are obtained (see (16) and (18) of [13]).

In this paper, using the factorization technique [14], bifurcation theory of dynamical system, and phase portraits analysis [15–17], we present some explicit expressions of the smooth solitary, kink, and antikink waves of (1). Throughout this paper, we always suppose that \( m \neq -1 \) and \( -1/2, k \neq 0 \).
2. Preliminaries

Using transformation

\[ q(x, y, t) = e^{i \eta} u(\xi), \]
\[ \eta = x + c_1 y + c_2 t, \]
\[ \xi = x + \omega_1 y + \omega_2 t, \]

where \( c_1, c_2, \omega_1, \) and \( \omega_2 \) are constants, (1) can be rewritten as

\[
\left[ \frac{1}{2} \left( 1 + \omega_1^2 \right) u'' + ku^{4m+1} + u^{2m+1} - \frac{1}{2} \left( 1 + c_1^2 + 2c_2 \right) u \right] \]
\[ + i \left( 1 + c_1 \omega_1 + c_2 \right) u' \right] e^{i \eta} = 0,
\]

where "r" is the derivative with respect to \( \xi \).

From (3), we have

\[
\left( 1 + c_1 \omega_1 + c_2 \right) u' = 0,
\]
\[
\frac{1}{2} \left( 1 + \omega_1^2 \right) u'' + ku^{4m+1} + u^{2m+1} - \frac{1}{2} \left( 1 + c_1^2 + 2c_2 \right) u = 0.
\]

Equation (5) is converted into the following equation:

\[
8km^2 \phi^4 + 8m^2 \phi^3 - 4 \left( 1 + c_1^2 + 2c_2 \right) m^2 \phi^2
\]
\[ + 2 \left( 1 + \omega_1^2 \right) m\phi'' + \left( 1 + \omega_1^2 \right) (1 - 2m) \left( \phi' \right)^2 = 0.
\]

Differentiating (7) once with respect to \( \xi \), we have

\[
2 \left( 1 + \omega_1^2 \right) m\phi''' + 2 \left( 1 + \omega_1^2 \right) (1 - m) \phi' \phi''
\]
\[ + \left( 32km^2 \phi^3 + 24m^2 \phi^2 - 8 \left( 1 + c_1^2 + 2c_2 \right) m^2 \phi \right) \phi' = 0.
\]

Using Proposition 2 of [14], we know that (8) has the factorization

\[
\left( f(\phi) \phi_1(\phi) - \phi_1(\phi) \phi' - \phi_2(\phi) \right) \cdot \left( \phi_3(\phi) - \phi_3(\phi) \phi_1(\phi) - \phi_4(\phi) \right) \phi = 0
\]

if and only if the following expressions are satisfied:

\[
g(\phi, \phi') = -f(\phi) \phi_3(\phi) - \phi_1(\phi) \phi' - \phi_2(\phi),
\]
\[
\phi_1(\phi) \phi_3(\phi) - f(\phi) \frac{d\phi_3(\phi)}{d\phi} = 0,
\]
\[
k(\phi) = \phi_2(\phi) \phi_4(\phi),
\]
\[
\frac{\phi_2(\phi) \phi_3(\phi) - f(\phi)}{d\phi_4(\phi)} - f(\phi) \phi_4(\phi)
\]
\[ + \phi_1(\phi) \phi_4(\phi) \phi_4(\phi) = 0.
\]

Solving (10) and setting the integral constant being zero, we have

\[
\phi_1(\phi) = 2 \left( 1 + \omega_1^2 \right) (m - 1),
\]
\[
\phi_2(\phi) = \phi_3(\phi) = 0,
\]
\[
\phi_4(\phi) = -\frac{4m^2 \left( 4k (m + 1) \phi^2 + 3 (2m + 1) \phi - \left( 1 + c_1^2 + 2c_2 \right) (m + 1) (2m + 1) \right)}{(1 + \omega_1^2) (m + 1) (2m + 1)}.
\]

Thus, (8) has the following factorization:

\[
\left( 2 \left( 1 + \omega_1^2 \right) m\phi \phi_1(\phi) - 2 \left( 1 + \omega_1^2 \right) (m - 1) \phi' \right)
\]
\[ \cdot \left( \phi_2(\phi) \phi_3(\phi) - f(\phi) \phi_4(\phi) \right) \phi = 0,
\]

where \( A = 4k(m + 1), B = 3(2m + 1), \) and \( C = -(1 + c_1^2 + 2c_2)(m + 1)(2m + 1). \)

Some special solutions of (7) could be obtained by solving the following second-order differential equation:

\[
\phi'' = \rho \left( \phi^2 + E\phi + F \right) \phi,
\]

where \( \rho = \frac{32km^2 \phi^3 + 24m^2 \phi^2 - 8 \left( 1 + c_1^2 + 2c_2 \right) m^2 \phi}{A B C} \).
where \( \rho = -16 km^2/(1+\omega_1^2)(2m+1), \) \( E = 3(2m+1)/4k(m+1), \) and \( F = -(1+c_1^2+2c_2)(2m+1)/4k. \)

Letting \( y = d\phi/d\xi, \) we get the following planar dynamical system:

\[
\begin{align*}
\frac{d\phi}{d\xi} &= y, \\
\frac{dy}{d\xi} &= \rho \left( \phi^2 + E\phi + F \right) \phi
\end{align*}
\]  
(14)

with the first integral

\[
H(\phi, y) = y^2 - \frac{1}{6} \rho \left( 3\phi^2 + 4E\phi + 6F \right) \phi^2 = h. 
\]  
(15)

For a fixed \( h, \) the level curve \( H(\phi, y) = h \) defined by (15) determines a set of invariant curves of (14) which contain different branches. As \( h \) is varied, it defines different families of orbits of (14) with different dynamical behaviors.

The remainder of this paper is organized as follows. In Section 3, we consider bifurcation sets and phase portraits of (14). Some explicit smooth solitary, kink, and antikink wave solutions of (I) are presented in Section 4. A short conclusion will be provided in Section 5.

### 3. Bifurcation Sets and Phase Portraits of (14)

We will use the following notations:

\[
\begin{align*}
\Delta &= E^2 - 4F, \\
\phi_{1,2} &= \frac{1}{2} \left( -E \pm \sqrt{\Delta} \right).
\end{align*}
\]  
(16)

Obviously, (14) has only one equilibrium point at \((0,0)\) in \( \phi \)-axis when \( \Delta < 0. \) Equation (14) has two equilibrium points at \((-1/2)E, 0)\) and \((0, 0)\) in \( \phi \)-axis when \( \Delta = 0. \) Equation (14) has three equilibrium points at \((\phi_{1,2}, 0)\) and \((0, 0)\) in \( \phi \)-axis when \( \Delta > 0. \) From (15), the following conclusions hold:

\[
\begin{align*}
h_1 &= H(\phi_e, 0) \\
&= \frac{1}{48} \rho \left( E - \sqrt{\Delta} \right)^2 \left( E \left( E - \sqrt{\Delta} \right) - 6F \right), \\
h_2 &= H(\phi_e, 0) \\
&= \frac{1}{48} \rho \left( E + \sqrt{\Delta} \right)^2 \left( E \left( E + \sqrt{\Delta} \right) - 6F \right).
\end{align*}
\]  
(17)

Let \( M(\phi_e, 0) \) be the coefficient matrix of the linearized system of (14) at equilibrium point \((\phi_e, 0). \) We have

\[
J(\phi_e, 0) = \det(M(\phi_e, 0)) = -\rho \left( 3\phi_e^2 + 2E\phi_e + F \right). 
\]  
(18)

For an equilibrium point \((\phi_e, 0)\) of (14), we know that \((\phi_e, 0)\) is a saddle point if \( J(\phi_e, 0) < 0, \) a center point if \( J(\phi_e, 0) > 0, \) and a cusp if \( J(\phi_e, 0) = 0 \) and the Poincaré index of \((\phi_e, 0)\) is zero.

Using the properties of equilibrium points and the bifurcation method, we can obtain four bifurcation curves of (14) as follows:

\[
\begin{align*}
C_1 &: E = 0, \\
C_2 &: F = 0, \\
C_3 &: F = \frac{2}{9} E^2, \\
C_4 &: F = \frac{1}{4} E^2.
\end{align*}
\]  
(19)

The bifurcation curves \( C_1, C_2, C_3, \) and \( C_4 \) divide the \((E, F)\)-parameter plane into 14 subregions. The bifurcation sets and phase portraits of (14) are drawn in Figures 1-2.

### 4. Explicit Smooth Solitary, Kink and Anti-Kink Wave Solutions of (1)

In this section we present all possible explicit smooth solitary, kink, and antikink wave solutions of (1) in the following propositions.

**Proposition 1.** When \( \rho < 0, E \neq 0, \) and \( F < 0, \) (1) has two smooth solitary wave solutions as follows:

\[
q(x, y, t) = \frac{\left( 2\phi_m\phi_M \right)^{1/2m} e^{i(x+c_1y+c_2t)}}{\left[ (\phi_m + \phi_M) + (\phi_M - \phi_m) \cosh(\omega(x + \omega_1 y - (1 + c_1\omega_1 t))) \right]^{1/2m}},
\]  
(20)

\[
q(x, y, t) = \frac{\left( 2\phi_m\phi_M \right)^{1/2m} e^{i(x+c_1y+c_2t)}}{\left[ (\phi_m + \phi_M) - (\phi_M - \phi_m) \cosh(\omega(x + \omega_1 y - (1 + c_1\omega_1 t))) \right]^{1/2m}},
\]  
(21)
Figure 1: Continued.
\begin{align*}
  \phi \, (g) & \quad E < 0, \quad F = \frac{2}{9} E^2 \\
  \phi \, (h) & \quad E > 0, \quad F = \frac{2}{9} E^2 \\
  \phi \, (i) & \quad E < 0, \quad \frac{2}{9} E^2 < F < \frac{1}{4} E^2 \\
  \phi \, (j) & \quad E > 0, \quad \frac{2}{9} E^2 < F < \frac{1}{4} E^2 \\
  \phi \, (k) & \quad E < 0, \quad F = \frac{1}{4} E^2 \\
  \phi \, (l) & \quad E > 0, \quad F = \frac{1}{4} E^2
\end{align*}

\textbf{Figure 1: Continued.}
where \( \phi_{M,m} = \frac{1}{(1/2)\rho} \sqrt{(1/2)\rho} \phi_m \phi_M \).

\[
q(x, y, t) = \left[ (\phi_1 (\phi_M - \phi_m) \cosh(\omega (x + \omega_1 y - (1 + \epsilon_1 \omega_1) t)) - (\phi_1 (\phi_m + \phi_M) - 2\phi_m \phi_M) \right]^{1/2m} \cdot e^{i(x+c_1 y+c_2 t)},
\]

\[
q(x, y, t) = \left[ (\phi_2 (\phi_M - \phi_m) \cosh(\omega (x + \omega_1 y - (1 + \epsilon_1 \omega_1) t)) + (\phi_2 (\phi_m + \phi_M) - 2\phi_m \phi_M) \right]^{1/2m} \cdot e^{i(x+c_1 y+c_2 t)},
\]

where \( \phi_{M,m} = \frac{1}{(1/6)(3\sqrt{\Delta} - E \pm \sqrt{E^2 - 9F})}, \omega = \sqrt{(1/2)\rho} \sqrt{(1/2)\rho} (\phi_m - \phi_M) \phi_m - \phi_m \).

When \( \rho < 0, E < 0, 0 < F < (1/4)E^2 \), and \( F \neq (2/9)E^2 \), (1) has two smooth solitary wave solutions as follows:

\[
q(x, y, t) = \left[ \frac{1}{3} E (-1 \pm \sqrt{2}) \right]
\]

where \( \omega = -(E/3) \sqrt{-\rho} \).

When \( \rho < 0, E = 0, 0 < F < (1/4)E^2 \), and \( F \neq (2/9)E^2 \), (1) has two smooth solitary wave solutions as follows:

\[
q(x, y, t) = \left[ \frac{1}{3} E (-1 \pm \sqrt{2}) \right]
\]

where \( \omega = -(E/3) \sqrt{-\rho} \).

When \( \rho < 0, E < 0, 0 < F < (1/4)E^2 \), and \( F \neq (2/9)E^2 \), (1) has two smooth solitary wave solutions as follows:

\[
q(x, y, t) = \left[ \frac{1}{3} E (-1 \pm \sqrt{2}) \right]
\]

where \( \omega = -(E/3) \sqrt{-\rho} \).
Figure 2: Continued.
Figure 2: Continued.
Proof of Proposition 1. From Figures 1(a) and 1(b), when $\rho < 0$, $E \neq 0$, and $F < 0$, there are two homoclinic orbits connecting with the saddle point $(0, 0)$ and passing through the points $(\phi_m, 0)$ and $(\phi_M, 0)$, respectively, where $\phi_{M,m} = (1/3)(-2E \pm \sqrt{2(2E^2 - 9F)})$. Their expressions are, respectively,

$$y = \pm \sqrt{-\frac{1}{2}\rho\phi\sqrt{(\phi_M - \phi)(\phi - \phi_m)}}, \quad \phi_m \leq \phi < 0,$$

$$y = \pm \sqrt{-\frac{1}{2}\rho\phi\sqrt{(\phi_M - \phi)(\phi - \phi_m)}}, \quad 0 < \phi \leq \phi_M. \quad (27)$$

Substituting (27) and (28) into $d\phi/d\xi = y$, respectively, and integrating them along the homoclinic orbits, we have

$$\int_{\phi_m}^{\phi} \frac{ds}{s\sqrt{(\phi_M - s)(s - \phi_m)}} = -\sqrt{-\frac{1}{2}\rho |\xi|}, \quad (29)$$

$$\int_{\phi_M}^{\phi} \frac{ds}{s\sqrt{(\phi_M - s)(s - \phi_m)}} = \sqrt{-\frac{1}{2}\rho |\xi|}. \quad (30)$$

Completing the integrals in (29) and (30), we obtain two smooth solitary wave solutions of (7) as follows:

$$\phi(\xi) = \frac{2\phi_m\phi_M}{(\phi_m + \phi_M) \pm (\phi_M - \phi_m) \cosh(\omega \xi)}, \quad (31)$$

where $\omega = \sqrt{(1/2)\rho\phi_M\phi_m}$. The profiles of (31) are shown in Figures 3(a) and 3(b). From (2), (6), (31), and $\omega = -(1 + c_1\omega_1)$, we obtain two smooth solitary wave solutions of (1) with $\rho < 0$, $E \neq 0$, and $F < 0$ given in (20) and (21).

From Figures 1(e) and 1(i), when $\rho < 0$, $E < 0$, $0 < F < (1/4)E^2$, and $F \neq (2/9)E^2$, there are two homoclinic orbits connecting with the saddle point $(\phi_2, 0)$ and passing through the points $(\phi_m, 0)$ and $(\phi_M, 0)$, respectively, where $\phi_{M,m} = (1/6)(3\sqrt{3} - E \pm 2\sqrt{E - 3\sqrt{3} \Delta})$. Their expressions are, respectively,

$$y = \pm \sqrt{-\frac{1}{2}\rho(\phi_2 - \phi)\sqrt{(\phi_M - \phi)(\phi - \phi_m)}}, \quad \phi_m \leq \phi < \phi_2,$$

$$y = \pm \sqrt{-\frac{1}{2}\rho(\phi - \phi_2)\sqrt{(\phi_M - \phi)(\phi - \phi_m)}}, \quad \phi_2 < \phi \leq \phi_M. \quad (32)$$

Substituting (32) and (33) into $d\phi/d\xi = y$, respectively, and integrating them along the homoclinic orbits, we have

$$\int_{\phi_m}^{\phi} \frac{ds}{(\phi_2 - s)\sqrt{(\phi_M - s)(s - \phi_m)}} = \sqrt{-\frac{1}{2}\rho |\xi|}, \quad (34)$$

$$\int_{\phi}^{\phi_M} \frac{ds}{(s - \phi_2)\sqrt{(\phi_M - s)(s - \phi_m)}} = \sqrt{-\frac{1}{2}\rho |\xi|}. \quad (35)$$

Completing the integrals in (34) and (35), we obtain two smooth solitary wave solutions of (7) as follows:

$$\phi(\xi) = \frac{\phi_2(\phi_M - \phi_m)\cosh(\omega \xi) - (\phi_2 \phi_m + \phi_M) - 2\phi_m\phi_M}{(\phi_M - \phi_m) \cosh(\omega \xi) + (\phi_m + \phi_M - 2\phi_2)}, \quad (36)$$

$$\phi(\xi) = \frac{\phi_2(\phi_M - \phi_m)\cosh(\omega \xi) + (\phi_2 \phi_m + \phi_M) - 2\phi_m\phi_M}{(\phi_M - \phi_m) \cosh(\omega \xi) - (\phi_m + \phi_M - 2\phi_2)}, \quad (37)$$

where $\omega = \sqrt{-1/2)(\phi_M - \phi_2)(\phi_2 - \phi_m)}$. The profiles of (36) and (37) are shown in Figures 3(c) and 3(d), respectively.
From (2), (6), (36), (37), and \( \omega_2 = -(1 + c_1 \omega_1) \), we obtain two smooth solitary wave solutions of (1) with \( \rho < 0, E < 0, 0 < F < (1/4)E^2, \) and \( F \neq (2/9)E^2 \) given in (22) and (23).

From Figures 1(g) and 1(h), when \( \rho < 0, E > 0, 0 < F < (1/4)E^2, \) and \( F \neq (2/9)E^2 \), there are two homoclinic orbits connecting with the saddle point \( (\phi_m, 0) \) and passing through the points \( (\phi_m, 0) \) and \( (\phi_m, 0) \), respectively, where \( \phi_m = -(1/6)(3 \sqrt{\Delta} + E \mp 2 \sqrt{E(E + 3 \sqrt{\Delta})}) \). Their expressions are, respectively,

\[
y = \pm \sqrt{1/2 \rho} (\phi_1 - \phi) \sqrt{\phi_M - \phi} (\phi - \phi_m),
\]

\( \phi_m \leq \rho < \phi_1, \) \hspace{1cm} (38)

\[
y = \pm \sqrt{1/2 \rho} (\phi - \phi_1) \sqrt{\phi_M - \phi} (\phi - \phi_m),
\]

\( \phi_1 < \rho \leq \phi_M. \) \hspace{1cm} (39)

Substituting (38) and (39) into \( d\phi/d\xi = y \), respectively, and integrating along the homoclinic orbits, we have

\[
\int_{\phi_m}^{\phi} \frac{ds}{(\phi_M - \phi) (\phi - \phi_m)} = \frac{\sqrt{1/2 \rho} \left| \xi \right|}{\left| \phi_1 - \phi \right|}, \hspace{1cm} (40)
\]

\[
\int_{\phi}^{\phi_M} \frac{ds}{(\phi_M - \phi) (\phi - \phi_m)} = \frac{\sqrt{1/2 \rho} \left| \xi \right|}{\left| \phi_1 - \phi \right|}. \hspace{1cm} (41)
\]

Completing the integrals in (40) and (41), we obtain two smooth solitary wave solutions of (7) as follows:

\[
\phi_1(\xi) = \frac{\phi_1(\phi_M - \phi_m) \cosh(\omega_1 \xi) - (\phi_1 \phi_m + \phi_M - 2 \phi_m \phi_M)}{(\phi_M - \phi_m) \cosh(\omega_1 \xi) + (\phi_m + \phi_M - 2 \phi_1)}, \hspace{1cm} (42)
\]

\[
\phi_1(\xi) = \frac{\phi_1(\phi_M - \phi_m) \cosh(\omega_1 \xi) + (\phi_1 \phi_m + \phi_M - 2 \phi_m \phi_M)}{(\phi_M - \phi_m) \cosh(\omega_1 \xi) - (\phi_m + \phi_M - 2 \phi_1)}, \hspace{1cm} (43)
\]

where \( \omega = \sqrt{(1/2 \rho)(\phi_M - \phi_m)(\phi_1 - \phi_m)} \). The profiles of (42) and (43) are shown in Figures 3(e) and 3(f), respectively. From (2), (6), (42), (43), and \( \omega_2 = -(1 + c_1 \omega_1) \), we obtain two smooth solitary wave solutions of (1) with \( \rho < 0, E > 0, 0 < F < (1/4)E^2, \) and \( F \neq (2/9)E^2 \) given in (24) and (25).

From Figures 1(g) and 1(h), when \( \rho < 0, E \neq 0, \) and \( F = (2/9)E^2, \) there are two homoclinic orbits connecting with the saddle point \(-1/3)E,0\) and passing through the points \( (\phi_m, 0) \) and \( (\phi_m, 0) \), respectively, where \( \phi_{m,m} = -(1/3)(1 \pm \sqrt{2})E \) when \( \rho < 0, E < 0, F = (2/9)E^2, \) and 

\[
\phi_{m,m} = -(1/3)(1 \pm \sqrt{2})E \text{ when } \rho < 0, E > 0, \text{ and } F = (2/9)E^2.
\]

Their expressions are, respectively,

\[
y = \pm \sqrt{1/2 \rho} \left( -1/3 - E \right) \sqrt{(\phi_m - \phi) (\phi - \phi_m)}, \hspace{1cm} (44)
\]

\( \phi_m \leq \rho < -1/3 E, \)

\[
y = \pm \sqrt{1/2 \rho} \left( 1/3 + E \right) \sqrt{(\phi_m - \phi) (\phi - \phi_m)}, \hspace{1cm} (45)
\]

\( -1/3 E < \rho \leq \phi_M. \)

Substituting (44) and (45) into \( d\phi/d\xi = y \), respectively, and integrating along the homoclinic orbits, we have

\[
\int_{\phi_m}^{\phi} \frac{ds}{(\phi - (1/3)E - s) (\phi_m - s - \phi_m)} = \frac{\sqrt{1/2 \rho} \left| \xi \right|}{\left| \phi_1 - \phi \right|}, \hspace{1cm} (46)
\]

\[
\int_{\phi}^{\phi_M} \frac{ds}{(s + (1/3)E) (\phi_m - s - \phi_m)} = \frac{\sqrt{1/2 \rho} \left| \xi \right|}{\left| \phi_1 - \phi \right|}. \hspace{1cm} (47)
\]

Completing the integrals in (46) and (47), we obtain two smooth solitary wave solutions of (7) as follows:

\[
\phi_1(\xi) = \frac{1}{3} E \left( -1 \pm \sqrt{2} \right) \text{sech}(\omega_1 \xi), \hspace{1cm} (48)
\]

where \( \omega = -(E/3) \sqrt{-\rho} \). The profiles of (48) are shown in Figures 3(g) and 3(h). From (2), (6), (48), and \( \omega_2 = -(1 + c_1 \omega_1) \), we obtain two smooth solitary wave solutions of (1) with \( \rho < 0, E \neq 0, \) and \( F = (2/9)E^2 \) given in (26).

The proof of the Proposition 1 is completed. \( \square \)

**Proposition 2.** When \( \rho < 0, E \neq 0, \) and \( F = 0, (1) \) has one smooth solitary wave solution as follows:

\[
q(x, y, t) = \frac{[(-4E)^{1/2m}] e^{i(x+c_1 y + c_2 t)}}{\left[3 \left( 1 + (\omega (x + \omega_1 y - (1 + c_1 \omega_1) t)^2 \right) \right]^{1/2m}}, \hspace{1cm} (49)
\]

where \( \omega = (E/3) \sqrt{-(1/2)\rho} \).

When \( \rho < 0, E \neq 0, \) and \( F = (1/4)E^2, (1) \) has one smooth solitary wave solution as follows:

\[
q(x, y, t) = \frac{[E(1 - 3(\omega (x + \omega_1 y - (1 + c_1 \omega_1) t)^2)]^{1/2m} e^{i(x+c_1 y + c_2 t)}}{\left[6 \left( 1 + (\omega (x + \omega_1 y - (1 + c_1 \omega_1) t)^2 \right) \right]^{1/2m}}, \hspace{1cm} (50)
\]

where \( \omega = (E/3) \sqrt{-(1/2)\rho} \).

When \( \rho > 0, E < 0, \) and \( F < 0 \) (or \( \rho > 0, E > 0, \) and \( (2/9)E^2 < F < (1/4)E^2 \)), (1) has one smooth solitary wave solution the same as in (22).

When \( \rho > 0, E > 0, \) and \( F < 0 \) (or \( \rho > 0, E < 0, \) and \( (2/9)E^2 < F < (1/4)E^2 \)), (1) has one smooth solitary wave solution the same as in (25).
Figure 3: Continued.
Figure 3: Smooth solitary waves of (7).
When $\rho > 0$, $E < 0$, and $0 < F < (2/9)E^2$, (1) has one smooth solitary wave solution the same as in (20).

When $\rho > 0$, $E > 0$, and $0 < F < (2/9)E^2$, (1) has one smooth solitary wave solution the same as in (21).

Proof of Proposition 2. From Figures 1(c) and 1(d), when $\rho < 0$, $E \neq 0$, and $F = 0$, there is a homoclinic orbit connecting with the cusp $(0, 0)$ and passing through the point $(-4/3)E, 0)$. When $\rho < 0$, $E < 0$, and $F = 0$, its expression is

$$y = \pm \sqrt{-\frac{1}{2}\rho \phi \sqrt{\phi (\frac{4}{3}E - \phi)}}, \quad 0 < \phi \leq -\frac{4}{3}E.$$  \tag{51}

When $\rho < 0$, $E > 0$, and $F = 0$, its expression is

$$y = \pm \sqrt{-\frac{1}{2}\rho \phi \sqrt{\phi (\frac{4}{3}E + \phi)}}, \quad -\frac{4}{3}E \leq \phi < 0.$$  \tag{52}

Substituting (51) and (52) into $d\phi / d\xi = y$, respectively, and integrating them along the homoclinic orbits, we have

$$\int_{\phi}^{\phi_{M}} \frac{ds}{s \sqrt{s (-\frac{4}{3}E - s)}} = \sqrt{-\frac{1}{2}\rho |\xi|},$$  \tag{53}

$$\int_{-\frac{4}{3}E}^{\phi} \frac{ds}{s \sqrt{s (s + \frac{4}{3}E)}} = -\sqrt{-\frac{1}{2}\rho |\xi|}.$$  \tag{54}

Completing the integral in (53), we obtain a smooth solitary wave solution of (7) as follows:

$$\phi (\xi) = -\frac{4E}{3 (1 + (\omega \xi)^2)},$$  \tag{55}

where $\omega = (2E/3) \sqrt{-1/2}\rho$. Completing the integral in (54), we obtain a smooth solitary wave solution of (7) the same as in (55). The profiles of (55) are shown in Figures 3(i) and 3(j).

From (2), (6), (55), and $\omega_2 = -(1 + c_1\omega_1)$, we obtain a smooth solitary wave solution of (1) with $\rho < 0, E \neq 0$, and $F = 0$ given in (49).

From Figures 1(k) and 1(l), when $\rho < 0, E \neq 0$, and $F = (1/4)E^2$, there is a homoclinic orbit connecting with the cusp $(-1/2)E, 0$ and passing through the point $(1/6)E, 0)$. When $\rho < 0, E < 0$, and $F = (1/4)E^2$, its expression is

$$y = \pm \sqrt{-\frac{1}{2}\rho \left(\frac{1}{2}E - \phi\right) \sqrt{\left(-\frac{1}{2}E - \phi\right) \left(\phi - \frac{1}{6}E\right)}}, \quad \frac{1}{6}E \leq \phi < -\frac{1}{2}E.$$  \tag{56}

When $\rho < 0, E > 0$, and $F = (1/4)E^2$, its expression is

$$y = \pm \sqrt{-\frac{1}{2}\rho \left(\phi + \frac{1}{2}E\right) \sqrt{\left(\phi + \frac{1}{2}E\right) \left(\phi - \frac{1}{6}E\right)}}, \quad -\frac{1}{2}E < \phi \leq -\frac{1}{6}E.$$  \tag{57}

Substituting (56) and (57) into $d\phi / d\xi = y$, respectively, and integrating them along the homoclinic orbits, we have

$$\int_{\phi}^{\phi_{M}} \frac{ds}{s \sqrt{(s - (1/2)E)(s - (1/6)E)}} \sqrt{(-\frac{1}{2}\rho |s|)},$$  \tag{58}

$$\int_{\phi}^{\phi_{M}} \frac{ds}{s \sqrt{(s + (1/2)E)((1/6)E - s)}} = \sqrt{-\frac{1}{2}\rho |\xi|}.$$  \tag{59}

Completing the integral in (58), we obtain a smooth solitary wave solution of (7) as follows:

$$\phi (\xi) = \frac{E \left(1 - 3 (\omega \xi)^2\right)}{6 \left(1 + (\omega \xi)^2\right)},$$  \tag{60}

where $\omega = (E/3) \sqrt{-1/2}\rho$. Completing the integral in (59), we obtain a smooth solitary wave solution of (7) the same as in (60). The profiles of (60) are shown in Figures 3(k) and 3(l).

From Figures 2(a) and 2(j), when $\rho > 0, E < 0$, and $F < 0$ (or $\rho > 0, E > 0$, and $(2/9)E^2 < F < (1/4)E^2$), there is a homoclinic orbit connecting with the saddle point $(\phi_2, 0)$ and passing through the point $(\phi_m, 0)$, and its expression is

$$y = \pm \sqrt{-\frac{1}{2}\rho \phi_2 \sqrt{((\phi_m) - \phi) (\phi_m - \phi)}}, \quad \phi_2 < \phi \leq \phi_m.$$  \tag{61}

where $\phi_{m,m} = (1/6)(3\sqrt{\Delta} + E \pm 2 \sqrt{E(3\sqrt{\Delta})})$. Substituting (61) into $d\phi / d\xi = y$ and integrating it along the homoclinic orbit, we have

$$\int_{\phi}^{\phi_{m}} \frac{ds}{s \sqrt{(s - (\phi_m) - s)}} = \sqrt{-\frac{1}{2}\rho |\xi|}.$$  \tag{62}

Completing the integral in (62), and using (2), (6), and $\omega_2 = -(1 + c_1\omega_1)$, we obtain a smooth solitary wave solution of (1) with $\rho > 0, E < 0$, and $F < 0$ (or $\rho > 0, E > 0$, and $(2/9)E^2 < F < (1/4)E^2$) given in (22).

From Figures 2(b) and 2(i), when $\rho > 0, E > 0$, and $F < 0$ (or $\rho > 0, E < 0$, and $(2/9)E^2 < F < (1/4)E^2$), there is a homoclinic orbit connecting with the saddle point $(\phi_1, 0)$ and passing through the point $(\phi_M, 0)$, and its expression is

$$y = \pm \sqrt{-\frac{1}{2}\rho \phi_1 \sqrt{((\phi_M - \phi) (\phi - \phi_m)}}, \quad \phi_M \leq \phi < \phi_1.$$  \tag{63}
where \( \phi_{M,m} = -(1/6)(3\sqrt{\Delta} + E + 2\sqrt{(E + 3\sqrt{\Delta})}) \). Substituting (63) into \( d\phi/d\xi = y \) and integrating it along the homoclinic orbit, we have
\[
\int_{\phi_{M}}^{\phi} \frac{ds}{s\sqrt{(s - \phi_{M})(s - \phi_{m})}} = \frac{\sqrt{2\rho}}{2} |\xi|.
\] (64)

Completing the integral in (64), and using (2), (6), and \( \omega_{3} = -(1 + c_{1} \omega_{1}) \), we obtain a smooth solitary wave solution of (1) with \( \rho > 0, E > 0, \) and \( F < 0 \) (or \( \rho > 0, E < 0, \) and \( (2/9)E^{2} < F < (1/4)E^{2} \)) given in (25).

From Figure 2(e), when \( \rho > 0, E < 0, \) and \( 0 < F < (2/9)E^{2} \), there is a homoclinic orbit connecting with the saddle point \((0,0)\) and passing through the point \((\phi_{m},0)\), and its expression is
\[
y = \pm \sqrt{\frac{1}{2}\rho \phi \left((\phi_{M} - \phi)(\phi_{m} - \phi)\right)}, \quad 0 < \phi < \phi_{m}, \quad (65)
\]
where \( \phi_{M,m} = (1/3)(-2E \pm \sqrt{2(2E^{2} - 9F)}) \). Substituting (65) into \( d\phi/d\xi = y \) and integrating it along the homoclinic orbit, we have
\[
\int_{\phi_{M}}^{\phi} \frac{ds}{s\sqrt{(s - \phi_{M})(s - \phi_{m})}} = \frac{\sqrt{2\rho}}{2} |\xi|.
\] (66)

Completing the integral in (66), and using (2), (6), and \( \omega_{3} = -(1 + c_{1} \omega_{1}) \), we obtain a smooth solitary wave solution of (1) with \( \rho > 0, E < 0, \) and \( 0 < F < (2/9)E^{2} \) given in (20).

From Figure 2(f), when \( \rho > 0, E > 0, \) and \( 0 < F < (2/9)E^{2} \), there is a homoclinic orbit connecting with the saddle point \((0,0)\) and passing through the point \((\phi_{M},0)\), and its expression is
\[
y = \pm \sqrt{\frac{1}{2}\rho \phi \left((\phi_{M} - \phi)(\phi_{M} - \phi)\right)}, \quad \phi_{M} \leq \phi < 0, \quad (67)
\]
where \( \phi_{M,m} = (1/3)(-2E \pm \sqrt{2(2E^{2} - 9F)}) \). Substituting (67) into \( d\phi/d\xi = y \) and integrating it along the homoclinic orbit, we have
\[
\int_{\phi_{M}}^{\phi} \frac{ds}{s\sqrt{(s - \phi_{M})(s - \phi_{m})}} = -\frac{\sqrt{2\rho}}{2} |\xi|.
\] (68)

Completing the integral in (68), and using (2), (6), and \( \omega_{3} = -(1 + c_{1} \omega_{1}) \), we obtain a smooth solitary wave solution of (1) with \( \rho > 0, E > 0, \) and \( 0 < F < (2/9)E^{2} \) given in (21).

The proof of the Proposition 2 is completed. \( \square \)

**Proposition 3.** When \( \rho > 0, E \neq 0, \) and \( F = (2/9)E^{2}, \) (1) has the kink and antikink wave solutions as follows:
\[
q(x, y, \tau) = \frac{\left[-(2E)^{1/2m}\right]e^{i(x+c_{1}y+ct)}}{\left[3(1 + e^{i(x+c_{1}y+ct)})\right]^{1/2m}},
\] (69)
where \( \omega = (2E/3)\sqrt{(1/2)\rho} \).

**Proof of Proposition 3.** From Figures 2(g) and 2(h), when \( \rho > 0, E \neq 0, \) and \( F = (2/9)E^{2} \), there are two heteroclinic orbits connecting with the saddle points \((0,0)\) and \((-(2/3)E,0)\). When \( \rho > 0, E < 0, \) and \( F = (2/9)E^{2} \), its expression is
\[
y = \pm \sqrt{\frac{1}{2}\rho \phi \left((\phi + 2E/3)\right)}, \quad 2E/3 < \phi < 0. \quad (70)
\]

When \( \rho > 0, E > 0, \) and \( F = (2/9)E^{2} \), its expression is
\[
y = \pm \sqrt{\frac{1}{2}\rho \phi \left((\phi - 2E/3)\right)}, \quad -2E/3 < \phi < 0. \quad (71)
\]

Substituting (70) and (71) into \( d\phi/d\xi = y \) respectively, and integrating them along the heteroclinic orbits, we have
\[
\int_{\phi}^{-E/3} \frac{ds}{s(-2/3E - s)} = \pm \frac{\sqrt{2\rho}}{2} |\xi|,
\] (72)
\[
\int_{-E/3}^{\phi} \frac{ds}{s(s + 2/3E)} = \pm \frac{\sqrt{2\rho}}{2} |\xi|.
\] (73)

Completing the integral in (72), we obtain the kink and antikink wave solutions of (7) as follows:
\[
\phi(\xi) = -\frac{2E}{3(1 + e^{i\omega_{2}t})}.
\] (74)

where \( \omega = (2E/3)\sqrt{(1/2)\rho} \). Completing the integral in (73), we obtain the kink and antikink wave solutions of (7) the same as in (74). The profiles of (74) are shown in Figures 4(a) and 4(b). From (2), (6), (74), and \( \omega_{2} = -(1 + c_{1} \omega_{1}) \), we obtain the kink and antikink wave solutions of (1) with \( \rho > 0, E \neq 0, \) and \( F = (2/9)E^{2} \) given in (69).

The proof of Proposition 3 is completed. \( \square \)

**Remark 4.** The implicit solutions in [13] can be deduced if setting that concrete values of a set of the parameters. For instance, if one set \( a_{1} \rightarrow 1, a_{2} \rightarrow c_{1}, a_{3} \rightarrow c_{2}, a_{4} \rightarrow 0, A_{1} \rightarrow 1, A_{2} \rightarrow \omega_{1}, A_{3} \rightarrow \omega_{2} = -(1+c_{1} \omega_{1}), A_{4} \rightarrow 0, m \rightarrow 2m, \) and \( v \rightarrow \phi \) in [13], (16) and (18) in [13], respectively, can be deduced as follows:

\[
\frac{4m \left(4m^{2} \left(1 + 2c_{2} + c_{4}^{2}\right) + 2m \left(1 + 2c_{2} + c_{4}^{2}\right) - 2m \phi^{2} + (2m + 1) \Phi \left(1 + 2c_{2} + c_{4}^{2}\right) \left(1 + \omega_{1}^{2}\right)\right)}{(2m + 1) \left(1 + \omega_{1}^{2}\right) \phi} = e^{\pm 4m \sqrt{(1 + 2c_{2} + c_{4}^{2})/1 + \omega_{1}^{2}}},
\] (75)
\[ \xi = \pm \left( \frac{4m^2\phi^2\left((1+2c_2+c_1^2)(8m^2+6m+1) - 2(1+4m)\phi^2 - 2k(2m+1)\phi^4\right) + 2h\left(8m^2+6m+1\right)\phi^{2-1/m}}{(8m^2+6m+1)(1+\omega_1^2)} \right)^{-1/2} d\phi, \] (76)

where

\[ \xi = x + \omega_1 y - (1 + c_1 \omega_1) t, \]

\[ \Phi = 2 \sqrt{\frac{m^2\left((1+2c_2+c_1^2)(2m+1)(4m+1) - 2k(2m+1)\phi^4 - 2(4m+1)\phi^2\right)}{(2m+1)(4m+1)(1+\omega_1^2)}} \] (77)

and \( h \) is given in (10) of [13]. Equations (75) and (76) maybe can be reduced to some of results in this paper under concrete values of some sets of the parameters. We omit the discussions here.

5. Conclusion

In this paper, we present some explicit smooth solitary wave solutions for (1) expressed in (20)–(26), (49), and (50), and also some explicit kink and antikink wave solutions shown in (69). We will continue to discuss the properties of (1), and more results will be obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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