Research Article

Infinitely Many Standing Waves for the Nonlinear Chern-Simons-Schrödinger Equations

Jinmyoung Seok

Department of Mathematics, Kyonggi University, 154-42 Gwanggyosan-ro, Yeongtong-gu, Suwon 443-760, Republic of Korea

Correspondence should be addressed to Jinmyoung Seok; jmseok512@gmail.com

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We prove the existence of infinitely many solutions of the nonlinear Chern-Simons-Schrödinger equations under a wide class of nonlinearities. This class includes the standard power-type nonlinearity with exponent $p > 4$. This extends the previous result which covers the exponent $p > 6$.

1. Introduction

In [1, 2], Jackiw and Pi introduce a nonrelativistic model that the nonlinear Schrödinger dynamics is coupled with the Chern-Simons gauge terms as follows:

$$\begin{aligned} iD_0\phi + (D_1D_1 + D_2D_2)\phi &= -\lambda|\phi|^{p-2}\phi, \\ \partial_0A_1 - \partial_1A_0 &= -\operatorname{Im}(\bar{\phi}D_2\phi), \\ \partial_0A_2 - \partial_2A_0 &= \operatorname{Im}(\bar{\phi}D_1\phi), \\ \partial_1A_2 - \partial_2A_1 &= -\frac{1}{2|\phi|^2}. \end{aligned} \quad (1)$$

Here, i denotes the imaginary unit, $\partial_0 = \partial/\partial t$, $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}$ is a complex scalar field, $A_\mu : \mathbb{R}^{1+2} \rightarrow \mathbb{R}$ is a component of gauge potential and $\bar{D}_\mu = \partial_\mu + iA_\mu$ is a covariant derivative for μ running over 0, 1, 2, and $\lambda > 0$ is a parameter. The Chern-Simons gauge theory appears in the 1980s to explain electromagnetic phenomena of anyon physics such as the high temperature superconductivity or the fractional quantum Hall effect. In this paper, we are interested in standing wave solutions of (1). In [3], the authors introduce a standing wave ansatz of the following form:

$$\begin{aligned} \phi(t, x) &= u(|x|)e^{i\omega t}, \\ A_0(t, x) &= k(|x|), \end{aligned}$$

$$\begin{aligned} A_1(t, x) &= \frac{x_2}{|x|^2}h(|x|), \\ A_2(t, x) &= -\frac{x_1}{|x|^2}h(|x|), \end{aligned} \quad (2)$$

where $\omega > 0$ is a phase frequency and u, k, h are real valued functions on $[0, \infty)$ such that $h(0) = 0$. Inserting (2) into (1), one may check from direct computation that (1) is reduced to the following nonlinear nonlocal elliptic equation:

$$\begin{aligned} \Delta u - \omega u - \left(\int_{|x|}^{\infty} \frac{h_u(s)}{s} u^2(s) ds \right) u - \frac{h_u^2(|x|)}{|x|^2} u \\ + \lambda |u|^{p-2} u = 0 \quad \text{in } \mathbb{R}^2, \end{aligned} \quad (3)$$

where $h_u(s) = \int_0^s (1/2)u^2(\sigma)d\sigma$. See [3] for its derivation. It is shown in [3] that (3) is an Euler-Lagrange equation of a C^1 functional,

$$\begin{aligned} J(u) &:= \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \omega u^2 + \frac{h_u^2(|x|)}{|x|^2} u^2 dx \\ &\quad - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx \quad \text{on } H_r^1(\mathbb{R}^2), \end{aligned} \quad (4)$$

where $H_r^1(\mathbb{R}^2)$ denotes the set of radially symmetric functions in standard Sobolev space $H^1(\mathbb{R}^2)$. Investigating the structure

of J , the authors of [3] obtain several existence and nonexistence results for (3), depending on the range of $p > 2$ and $\lambda > 0$. Recently, Pomponio and Ruiz [4] improve the results in [3] for the case $p \in (2, 4)$. They find a threshold for the behavior of J , depending on $\omega > 0$. They also study (3) on bounded domain in [5].

In this paper, we are concerned with the existence of infinitely many solutions of (3). It is proved in [6] that if $p > 6$, J enjoys the symmetric mountain pass geometry and satisfies the (PS) condition so that the well-known symmetric mountain pass lemma (see [7]) applies to show there exist infinitely many critical points of J . For $p \in (4, 6]$, it turns out that J still enjoys the symmetric mountain pass geometry although checking the (PS) condition is not easy job. One aim of this paper is to show nevertheless J still admits infinitely many critical points for $p > 4$. Moreover, we will replace the power-type nonlinearity $\lambda|u|^{p-2}u$ of (3) with more general one as follows:

$$\Delta u - \omega u - \left(\int_{|x|}^{\infty} \frac{h_u(s)}{s} u^2(s) ds \right) u - \frac{h_u^2(|x|)}{|x|^2} u + V(u^2)u = 0. \quad (5)$$

The structure conditions for V are given by the following:

- (V1) Consider $V \in C^1(\mathbb{R}_+, \mathbb{R})$ such that $V(0) = V'(0) = 0$.
- (V2) Consider $\limsup_{s \rightarrow \infty} |V'(s)/s^{p-1}| < \infty$, $\limsup_{s \rightarrow \infty} |V(s)/s^p| < \infty$ for some $p > 1$.
- (V3) There exists some $\alpha > 1$ such that $\alpha V'(s)/s^{2-1/\alpha} - V(s)/s^{3-1/\alpha}$ is monotonically increasing to ∞ as $s \rightarrow \infty$.

Observe that assumptions (V1)–(V3) include the power-type nonlinearity $(\lambda/p)|u|^p$, $p > 4$.

Theorem 1. *Assume (V1)–(V3). Then, (5) admits infinitely many solutions.*

We refer to the work of Cunha et al. [8] that if we insert a sufficiently small parameter $q > 0$ into (5) as in

$$\Delta u - \omega u - q \left(\int_{|x|}^{\infty} \frac{h_u(s)}{s} u^2(s) ds \right) u - q \frac{h_u^2(|x|)}{|x|^2} u + V(u^2)u = 0, \quad (6)$$

then much more general assumptions for V , the so-called Berestycki-Lions conditions [9], are sufficient for guaranteeing the existence and multiplicity of solutions of (6). In our work, we assume further than the Berestycki-Lions conditions but we do not need small parameter $q > 0$. See also [10] in which Tan and Wan consider asymptotically linear nonlinearities.

To prove Theorem 1, we will apply the method employed in author's former paper [11] in which the Schrödinger-Poisson equation, another nonlocal field equation similar to (5), is dealt with. Instead of generating (PS) sequences, we will show the existence and compactness of the so-called approximate solution sequences of J which may be

considered as more refined version of (PS) sequences. In Section 2, we give a definition of the approximate solution sequences of J . Some auxiliary lemmas are also prepared in Section 2. In Section 3, we prove the compactness of approximate solution sequences. In Section 4, we construct infinitely many approximate solution sequences whose energy levels go to infinity and complete the proof of Theorem 1.

2. Mathematical Settings and Preliminaries

Let $H_r^1(\mathbb{R}^2)$ be the completion of $D := \{u \in C_c^\infty(\mathbb{R}^2) \mid u(x) = u(|x|)\}$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + \omega u^2 dx \right)^{1/2}. \quad (7)$$

The dual space of $H_r^1(\mathbb{R}^2)$ is denoted by $H_r^1(\mathbb{R}^2)^*$. Arguing similarly to [3], it is easy to show (5) is an Euler-Lagrange equation of the C^1 functional

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \omega u^2 + \frac{h_u^2(|x|)}{|x|^2} u^2 - V(u^2) dx \quad \text{on } H_r^1(\mathbb{R}^2). \quad (8)$$

In this paper, we search for infinitely many critical points of J to prove Theorem 1. To do this, we insert parameter λ into J as follows:

$$J_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \omega u^2 + \frac{h_u^2(|x|)}{|x|^2} u^2 - \lambda V(u^2) dx. \quad (9)$$

Here λ ranges over $[1/2, 1]$. For a sequence $\{\lambda_j\} \in [1/2, 1]$ which converges to 1 as $j \rightarrow \infty$, we say $\{\lambda_j, u_j\}$ is an approximate solution sequence of J if $J'_{\lambda_j}(u_j) = 0$ for all j . In the following subsection, we state a variant of the famous Struwe's monotonicity trick [12], which plays a crucial role in constructing approximate solution sequences.

2.1. A Variant of Struwe's Monotonicity Trick. Let X be Banach space. We say a subset $A \subset X$ is symmetric if $-v \in A$ for every $v \in A$. Let A be a compact subset of X and B a closed subset of A . We denote by Γ the set of every continuous odd function $\gamma : A \rightarrow X$ such that $\gamma(v) = v$ on B . Let I be a closed interval in $\mathbb{R}$ and J_λ one parameter family of even C^1 functional on X . We define a minimax level by

$$c(\lambda) := \min_{\gamma \in \Gamma} \max_{v \in A} J_\lambda(\gamma(v)). \quad (10)$$

The following theorem is a variant of so-called Struwe's monotonicity trick [12]. A more general version of it is given in [11]. The property (H) below is first proposed by Jeanjean and Toland in [13].

Theorem 2 (see [11]). *Suppose that, for all $\lambda \in I$,*

$$c(\lambda) > \max_{v \in B} J_\lambda(v). \quad (11)$$

Then, for almost every $\lambda \in I$, there exists a norm-bounded (PS) sequence of J_λ at level $c(\lambda)$, provided the following property (H) for J_λ holds:

(H) For given $\lambda_0 \in I$, let $\{\lambda_j\} \in I$ be a sequence strictly increasing to λ_0 and $\{v_j\}$ a sequence in X such that

$$\begin{aligned} & -J_{\lambda_0}(v_j), \\ & J_{\lambda_j}(v_j), \\ & \frac{J_{\lambda_j}(v_j) - J_{\lambda_0}(v_j)}{\lambda_0 - \lambda_j} \end{aligned} \quad (12)$$

are all uniformly bounded above for j . Then the following holds:

- (i) $\{v_j\}$ is norm-bounded in X .
- (ii) For given $\varepsilon > 0$, there exists $N > 0$ such that

$$J_{\lambda_0}(v_j) \leq J_{\lambda_j}(v_j) + \varepsilon \quad \forall j \geq N. \quad (13)$$

2.2. Some Auxiliary Lemmas. Here, we prepare some lemmas which will be necessarily used for proving the main result. Define

$$\widehat{J}_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 - \lambda V(u^2) dx. \quad (14)$$

Lemma 3 (Lemma 3.2 in [3]). Let $\{u_j\} \in H_r^1(\mathbb{R}^2)$ be a sequence weakly converging to some u_0 in $H_r^1(\mathbb{R}^2)$ as $j \rightarrow \infty$. Then, for each $\varphi \in H_r^1(\mathbb{R}^2)$, it holds that $\widehat{J}_\lambda(u_n) \rightarrow \widehat{J}_\lambda(u_0)$, $\widehat{J}'_\lambda(u_n)\varphi \rightarrow \widehat{J}'_\lambda(u_0)\varphi$, and $\widehat{J}'_\lambda(u_n)u_n \rightarrow \widehat{J}'_\lambda(u_0)u_0$ as $j \rightarrow \infty$, up to a subsequence.

Lemma 4 (Pohozaev identity). Let $u \in H_r^1(\mathbb{R}^2)$ be a critical point of J_λ . Then one has

$$\begin{aligned} & \int_{\mathbb{R}^2} \omega u^2 dx + 2 \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 dx - \int_{\mathbb{R}^2} \lambda V(u^2) dx \\ & = 0. \end{aligned} \quad (15)$$

For each $u \in H_r^1(\mathbb{R}^2)$ and $t > 0$, we define one parameter family of functions $u_t \in H_r^1(\mathbb{R}^2)$ by

$$u_t(x) := t^\alpha u(tx). \quad (16)$$

For fixed $t > 0$, we define a map $L_t : H^1(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2)$ by $L_t(u) = u_t$. It is easy to see that L_t is a continuous and linear map with the inverse $L_{1/t}$. Thus L_t is a linear isomorphism.

For each $u \in H^1(\mathbb{R}^2)$ and $\lambda \in [1/2, 1]$, let $c_{u,\lambda} : [0, \infty) \rightarrow \mathbb{R}$ be a function defined by

$$c_{u,\lambda}(t) := J_\lambda(u_t). \quad (17)$$

Lemma 5. For any $u \in H_r^1(\mathbb{R}^2)$ and $\lambda \in [1/2, 1]$, $c_{u,\lambda}$ admits a unique critical point $t_0 > 0$ on $(0, \infty)$; that is, $c'_{u,\lambda}(t_0) = 0$, such that $c_{u,\lambda}(t)$ is increasing on $(0, t_0)$, attains its maximum at t_0 , and is decreasing to $-\infty$ on (t_0, ∞) .

Proof. By the change of variable, one can compute

$$\begin{aligned} c_{u,\lambda}(t) &= J_\lambda(t^\alpha u(t \cdot)) = \frac{1}{2} \left(t^{2\alpha} \int_{\mathbb{R}^2} |\nabla u|^2 dx \right. \\ &+ t^{2\alpha-2} \int_{\mathbb{R}^2} \omega u^2 dx + t^{6\alpha-4} \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 dx \\ &\left. - t^{-2} \int_{\mathbb{R}^2} \lambda V(t^{2\alpha} u^2) dx \right). \end{aligned} \quad (18)$$

We differentiate it with respect to t to get

$$\begin{aligned} c'_{u,\lambda}(t) &= \alpha t^{2\alpha-1} \int_{\mathbb{R}^2} |\nabla u|^2 dx + (\alpha-1) t^{2\alpha-3} \int_{\mathbb{R}^2} \omega u^2 dx + (3\alpha-2) t^{6\alpha-5} \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 dx + t^{-3} \int_{\mathbb{R}^2} \lambda V(t^{2\alpha} u^2) dx \\ &- t^{-2} \int_{\mathbb{R}^2} \lambda V'(t^{2\alpha} u^2) \alpha t^{2\alpha-1} u^2 dx = t^{6\alpha-5} \left(\underbrace{\frac{\alpha}{t^{4\alpha-4}} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{\alpha-1}{t^{4\alpha-2}} \int_{\mathbb{R}^2} \omega u^2 dx + (3\alpha-2) \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 dx}_{A(t)} \right. \\ &\left. - \underbrace{\int_{\mathbb{R}^2} \lambda \left[\frac{\alpha V'((t^\alpha u)^2)}{((t^\alpha u)^2)^{2-1/\alpha}} - \frac{V((t^\alpha u)^2)}{((t^\alpha u)^2)^{3-1/\alpha}} \right] (u^2)^{3-1/\alpha} dx}_{B(t)} \right). \end{aligned} \quad (19)$$

Observe from assumption (V3) that $A(t)$ is strictly monotonically decreasing from infinity to the positive number $(3\alpha - 2) \int_{\mathbb{R}^2} (h_u^2(|x|)/|x|^2) u^2 dx$ on $(0, \infty)$ and $B(t)$ is monotonically increasing to infinity on $(0, \infty)$. Therefore there is $t_0 > 0$ such that $A(t) > B(t)$ on $(0, t_0)$, $A(t_0) = B(t_0)$, and $A(t) < B(t)$ on $(0, t_0)$. Also from assumption (V3), we deduce $B(t) \rightarrow \infty$ as $t \rightarrow \infty$. This proves the proposition. $\square$

We define a function $T : H^1(\mathbb{R}^2) \rightarrow \mathbb{R}$ by assigning a positive number t satisfying $\|t^{-\alpha} u(t^{-1} \cdot)\| = 1$ for any nonzero $u \in H_r^1(\mathbb{R}^2)$. The value $T(0)$ is defined by 0.

Lemma 6. *The function T is well-defined and continuous even map on $[0, \infty)$.*

Proof. To show the well-definedness of T , we have to show that there exists unique $t > 0$ satisfying $\|t^{-\alpha} u(t^{-1} \cdot)\| = 1$ for given nonzero $u \in H_r^1(\mathbb{R}^2)$. We note that this is equivalent to prove there is a unique solution $t > 0$ of the equation

$$g(t) := t^{2\alpha} - \left(\int_{\mathbb{R}^2} \omega u^2 dx \right) t^2 - \int_{\mathbb{R}^2} |\nabla u|^2 dx = 0. \quad (20)$$

Arguing similarly to the proof of Lemma 5, we are able to see that $g(t)$ is monotonically decreasing on $(0, t_0)$ for some $t_0 > 0$, attains its unique local minimum at t_0 , and is monotonically increasing to infinity on (t_0, ∞) . Therefore there is a unique positive zero of $g(t)$ since $g(0) = 0$. Also, the implicit function theorem says that $T(u)$ is continuous on $[0, \infty)$ because $g'(T(u)) \neq 0$. The evenness of T follows from the fact that each coefficient of (20) is even. This completes the proof. $\square$

3. Compactness of Approximate Solution Sequences

In this section, we prove the compactness of an approximate solution sequence $\{u_j\}$ of J_{λ_j} when its energy $\{J_{\lambda_j}(u_j)\}$ is bounded above.

Proposition 7. *Let $\lambda_j \in [1/2, 1]$ be such that $\lambda_j \rightarrow 1$ as $j \rightarrow \infty$. Let $\{u_j\} \in H_r^1(\mathbb{R}^2)$ be a sequence of critical points of J_{λ_j} ; that is, $J'_{\lambda_j}(u_j) = 0$. Suppose that $J_{\lambda_j}(u_j) < C$ for some $C > 0$, independent of j . Then $u_j \rightarrow u_0$ in $H^1(\mathbb{R}^2)$ for some critical point $u_0 \in H_r^1(\mathbb{R}^2)$ of $J(= J_1)$ up to a subsequence.*

Proof. We divide the proof into two steps.

Step 1 (boundedness of u_j). We first prove that $\{u_j\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Arguing indirectly, suppose that $\{u_j\}$ is unbounded. Let $T_j := T(u_j)$, where the function T is defined in Section 2. Equation (20) says T_j is unbounded. Let $v_j(x) := T_j^{-\alpha} u_j(T_j^{-1} x)$ so that $\|v_j\| = 1$. Then, up to a subsequence, $\{v_j\}$ converges weakly in $H_r^1(\mathbb{R}^2)$ and strongly in $L^q(\mathbb{R}^2)$ for all $q > 2$ to some $v_0 \in H_r^1(\mathbb{R}^2)$. Since u_j is a critical point of J_{λ_j} , we see that

$$\int_{\mathbb{R}^2} |\nabla u_j|^2 dx + \int_{\mathbb{R}^2} \omega u_j^2 dx + 3 \int_{\mathbb{R}^2} \frac{h_{u_j}^2(|x|)}{|x|^2} u_j^2 dx \quad (21)$$

$$- \int_{\mathbb{R}^2} \lambda_j V'(u_j^2) u_j^2 dx = J'_{\lambda_j}(u_j) u_j = 0.$$

Combining this with the Pohozaev identity (15), we obtain

$$\begin{aligned} & \alpha \int_{\mathbb{R}^2} |\nabla u_j|^2 dx + (\alpha - 1) \int_{\mathbb{R}^2} \omega u_j^2 dx \\ & + (3\alpha - 2) \int_{\mathbb{R}^2} \frac{h_{u_j}^2(|x|)}{|x|^2} u_j^2 dx + \int_{\mathbb{R}^2} \lambda_j V(u_j^2) dx \\ & - \alpha \int_{\mathbb{R}^2} \lambda_j V'(u_j^2) u_j^2 dx = 0. \end{aligned} \quad (22)$$

Then, from the change of variable and dividing by $T_j^{6\alpha-4}$, (22) transforms to

$$\begin{aligned} & \underbrace{\frac{\alpha}{T_j^{4\alpha-4}} \int_{\mathbb{R}^2} |\nabla v_j|^2 dx + \frac{\alpha-1}{T_j^{4\alpha-2}} \int_{\mathbb{R}^2} \omega v_j^2 dx + (3\alpha-2) \int_{\mathbb{R}^2} \frac{h_{v_j}^2(|x|)}{|x|^2} v_j^2 dx}_{A_j} \\ & = \underbrace{\lambda_j \int_{\mathbb{R}^2} \left[\frac{\alpha V'((T_j^\alpha v_j)^2)}{((T_j^\alpha v_j)^2)^{2-1/\alpha}} - \frac{V((T_j^\alpha v_j)^2)}{((T_j^\alpha v_j)^2)^{3-1/\alpha}} \right] (v_j^2)^{3-1/\alpha} dx}_{B_j}. \end{aligned} \quad (23)$$

Since $\|v_j\| = 1$ and T_j is unbounded, A_j is bounded for j but the structure assumption (V3) implies that B_j tends to infinity as $j \rightarrow \infty$ provided v_0 is not identically zero. We claim that v_0 is nonzero. Suppose v_0 is identically zero. From (19) and (22), we see that

$$\begin{aligned} c'_{u_j, \lambda_j}(1) &= \alpha \int_{\mathbb{R}^2} |\nabla u_j|^2 dx + (\alpha - 1) \int_{\mathbb{R}^2} \omega u_j^2 dx \\ &+ (3\alpha - 2) \int_{\mathbb{R}^2} \frac{h_{u_j}^2(|x|)}{|x|^2} u_j^2 dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^2} \lambda_j V(u_j^2) dx \\
& - \alpha \int_{\mathbb{R}^2} \lambda_j V'(u_j^2) u_j^2 dx = 0.
\end{aligned} \tag{24}$$

Then, Lemma 5 implies that $c_{u_j, \lambda_j}(1)$ is the global maximum of $c_{u_j, \lambda_j}(t)$ on $(0, \infty)$. Thus we see that, for each $R > 1$,

$$\begin{aligned}
C & > J_{\lambda_j}(u_j) = c_{u_j, \lambda_j}(1) \geq c_{u_j, \lambda_j}(RT_j^{-1}) \\
& = \frac{1}{2} \left(R^{2\alpha} \int_{\mathbb{R}^2} |\nabla v_j|^2 dx + R^{2\alpha-2} \int_{\mathbb{R}^2} \omega v_j dx \right. \\
& \quad + R^{6\alpha-4} \int_{\mathbb{R}^2} \frac{h_{v_j}^2(|x|)}{|x|^2} v_j^2 dx \\
& \quad \left. - \int_{\mathbb{R}^2} \lambda_j V((R^\alpha v_j(Rx))^2) dx \right) \geq \frac{1}{2} R^{2\alpha-2} \|v_j\|^2 \\
& \quad - \frac{1}{2} \int_{\mathbb{R}^2} \lambda_j V((R^\alpha v_j(Rx))^2) dx = \frac{1}{2} R^{2\alpha-2} + o(1).
\end{aligned} \tag{25}$$

The last equality follows from $\|v_j\| = 1$, the convergence of $\{v_j\}$ to 0 in $L^q(\mathbb{R}^2)$ for all $q > 2$, and the structure conditions (V1)-(V2). However, taking large $R > 1$, this makes a contradiction and shows v_0 is not identically zero. This proves the boundedness of $\{u_j\}$ in $H_r^1(\mathbb{R}^2)$.

Step 2 (compactness of u_j). Compactness of $\{u_j\}$ follows from a standard procedure. Since $\{u_j\}$ is bounded, there exists $u_0 \in H_r^1(\mathbb{R}^2)$ such that $\{u_j\}$ converges, up to a subsequence, to u_0 weakly in $H_r^1(\mathbb{R}^2)$ and strongly in $L^q(\mathbb{R}^2)$ for all $q > 2$. Then it follows from Lemma 3 that u_0 is a critical point of J . Also, it is easy to see from the boundedness of $\{u_j\}$ that $J'(u_j) \rightarrow 0$ in $H_r^1(\mathbb{R}^2)^*$. Recall that

$$J(u) = \frac{1}{2} \|u\|^2 + \widehat{J}(u). \tag{26}$$

Using Lemma 3 once again, one can observe that

$$\begin{aligned}
o(1) \|u_j\| & = J'(u_j) u_j = \|u_j\|^2 + \widehat{J}'(u_j) u_j \\
& = \|u_j\|^2 + \widehat{J}'(u_0) u_0 + o(1) \\
& = \|u_j\|^2 - \|u_0\|^2 + o(1),
\end{aligned} \tag{27}$$

which shows $\|u_j\| \rightarrow \|u_0\|$ as $j \rightarrow \infty$. Therefore we have $u_j \rightarrow u_0$ in $H_r^1(\mathbb{R}^2)$ as $j \rightarrow \infty$. $\square$

4. Construction of Approximate Solution Sequences

In this section, we construct infinitely many approximate solution sequences. Choose an orthonormal basis $\{u_i\}_{i=1}^\infty$ of

$H_r^1(\mathbb{R}^2)$. For given $k \in \mathbb{N}$, let Y_k and Z_k be linear subspaces of $H_r^1(\mathbb{R}^2)$ spanned by $\{u_1, u_2, \dots, u_k\}$ and $\{u_k, u_{k+1}, \dots\}$, respectively. We will show that J_λ enjoys a variant of symmetric mountain pass geometry (see [14]).

Lemma 8. *There exist a sequence $\{k_n\} \in \mathbb{N}$ such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$ and sequences $\{\rho_n\}, \{r_n\} \in \mathbb{R}$ satisfying $\rho_n > r_n > 0$ for each n and*

- (i) $\inf\{J_\lambda(u) \mid u \in Z_{k_n}, T(u) = r_n\} \geq n$ for all $\lambda \in [1/2, 1]$ and all n ;
- (ii) $\max\{J_\lambda(u) \mid u \in Y_{k_n}, T(u) = \rho_n\} \leq 0$ for all $\lambda \in [1/2, 1]$ and all n .

Proof. We first show (i). The structure assumptions (V1)-(V2) imply that

$$|V(s)| \leq \frac{\omega}{2} |s| + C |s|^p \tag{28}$$

for some $C > 0$ and all $s \geq 0$.

Then,

$$\begin{aligned}
J_\lambda(u) & \geq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \left(\frac{1}{2} - \frac{\lambda}{4}\right) \int_{\mathbb{R}^2} \omega u^2 dx \\
& \quad + \frac{1}{2} \int_{\mathbb{R}^2} \frac{h_u^2(|x|)}{|x|^2} u^2 dx - \frac{C\lambda}{2} \int_{\mathbb{R}^2} |u|^{2p} dx \\
& \geq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} \omega u^2 dx \\
& \quad - \frac{C}{2} \int_{\mathbb{R}^2} |u|^{2p} dx.
\end{aligned} \tag{29}$$

We recall the function T and the linear isomorphism L_t in Section 2. Let $v(x) := T(u)^{-\alpha} u(T(u)^{-1}x)$ so that $\|v\| = 1$. By a change of variable, we get from (29) that, for each $u \in Z_k$,

$$\begin{aligned}
J_\lambda(u) & \geq \frac{1}{4} T(u)^{2\alpha} \int_{\mathbb{R}^2} |\nabla v|^2 dx \\
& \quad + \frac{1}{4} T(u)^{2\alpha-2} \int_{\mathbb{R}^2} \omega v^2 dx \\
& \quad - \frac{C}{2} T(u)^{2p\alpha-2} \int_{\mathbb{R}^2} |v|^{2p} dx \\
& \geq \frac{1}{4} T(u)^{2\alpha-2} - \frac{C}{2} T(u)^{2p\alpha-2} \beta_{k, T(u)},
\end{aligned} \tag{30}$$

where

$$\beta_{k,t} := \sup_{v \in W_{k,t}} \int_{\mathbb{R}^2} |v|^{2p\alpha-2} dx, \tag{31}$$

$$W_{k,t} = \{v \in H_r^1(\mathbb{R}^2) \mid \|v\| = 1, v_t = t^\alpha v(t \cdot) \in Z_k\}.$$

We claim that, for each $t > 0$, $\beta_{k,t} \rightarrow 0$ as $k \rightarrow \infty$. To see this, suppose that $\beta_{k,t} \rightarrow \beta_t > 0$ as $k \rightarrow \infty$. Choose $v_{k,t} \in W_{k,t}$ satisfying

$$\int_{\mathbb{R}^2} |v_{k,t}|^p dx > \frac{\beta_{k,t}}{2}. \tag{32}$$

Since $W_{k,t}$ is the unit sphere of a linear subspace $L_{t^{-1}}(Z_k)$ of $H_r^1(\mathbb{R}^2)$ with codimension $k-1$, we deduce $\{v_{k,t}\}$ converges to 0 weakly in $H_r^1(\mathbb{R}^2)$ and strongly in $L^p(\mathbb{R}^2)$, up to a subsequence. This however contradicts the fact that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} |v_{k,t}|^p dx > \frac{\beta_t}{2} > 0 \quad (33)$$

and the claim is true. We take $r_n = (8n)^{1/(2\alpha-2)}$. Then, for any $u \in Z_k$ satisfying $T(u) = r_n$,

$$J_\lambda(u) \geq 2n - \frac{C}{2} r_n^{2p\alpha-2} \beta_{k,r_n}. \quad (34)$$

For each n , by taking sufficiently large $\{k_n\}$ satisfying

$$\beta_{k_n,r_n} < \frac{2n}{C r_n^{2p\alpha-2}}, \quad (35)$$

we can see that the proof of (i) is complete.

Next we show (ii). Lemma 5 says that, for each $u \in H_r^1(\mathbb{R}^2)$, $J_\lambda(u_t) \rightarrow -\infty$ as $t \rightarrow \infty$. Also, we see from Lemma 6 and (20) that the set $\{u \in Y_k \mid T(u) = \rho\}$ is closed and bounded in finite dimensional space so it is compact. Combining these two facts with the compactness of $[1/2, 1]$, we can deduce easily (ii) holds. $\square$

Define

$$\begin{aligned} B_n &:= \{u \in Y_{k_n} \mid T(u) \leq \rho_n\}, \\ S_n &:= \{u \in Z_{k_n} \mid T(u) = r_n\}, \end{aligned} \quad (36)$$

with k_n, ρ_n , and r_n given in Lemma 8. Let Γ_n be the set of continuous functions $\gamma : B_n \rightarrow H_r^1(\mathbb{R}^2)$ satisfying $\gamma(u) = u$ on $\partial B_n = \{u \in B_n \mid T(u) = \rho_n\}$. By $\gamma(B_n)$, we denote the set $\{\gamma(u) \in H_r^1(\mathbb{R}^2) \mid u \in B_n\}$.

Lemma 9 (intersection property). *For any $n \in \mathbb{N}$, the intersection $\gamma(B_n) \cap S_n \neq \emptyset$ for every $\gamma \in \Gamma_n$.*

Proof. Choose and fix arbitrary n and $\gamma \in \Gamma_n$. Define

$$U := \{u \in \text{int}(B_n) \mid T(\gamma(u)) < r_n\}, \quad (37)$$

where $\text{int}(B_n)$ denotes the interior of B_n in Y_{k_n} . Then U is a symmetric open neighborhood of 0 since T is a continuous even map by Lemma 6 and γ is a continuous odd map. Equation (20) in Lemma 6 says B_n is bounded so that U is also bounded. We claim that $T(\gamma(\partial U)) = r_n$. From the continuity of T and γ , it holds that $T(\gamma(\partial U)) \leq r_n$. Suppose that there is some $u_0 \in \partial U$ such that $T(\gamma(u_0)) < r_n$. Then there is a neighborhood of V of u_0 in Y_{k_n} such that $T(\gamma(V)) < r_n$. Choose some $v_0 \in V \setminus U$. From the definition of U , $v_0 \notin \text{int}(B_n)$. Since $v_0 \in \partial U \subset B_n$, we see that $v_0 \in \partial B_n$. Then, from the definition of γ , we have $T(\gamma(v_0)) = T(v_0) = \rho_n > r_n$, which is a contradiction. This shows the claim is true.

Now, consider a map $P_n \circ \gamma : \partial U \rightarrow Y_{k_n-1}$, where P_n is the projection map from $H_r^1(\mathbb{R}^2) \rightarrow Y_{k_n-1}$. Then the well-known Borsuk-Ulam theorem applies to see the continuous odd map $P_n \circ \gamma$ has a vanishing point $u \in \partial U$; that is, $P_n(\gamma(u)) = 0$. This means that $\gamma(u) \in Z_{k_n}$. Therefore $\gamma(u) \in \gamma(B_n) \cap S_n$. The proof is complete. $\square$

Now, we are ready to prove the existence of infinitely many approximate solution sequences of J . For each $\lambda \in [1/2, 1]$, we define infinitely many minimax levels as follows:

$$C_n(\lambda) := \min_{\gamma \in \Gamma_n} \max_{u \in B_n} J_\lambda(\gamma(u)). \quad (38)$$

It follows from Lemmas 8 and 9 that $C_n(\lambda) \geq n$ for all $\lambda \in [1/2, 1]$.

Proposition 10. *For every fixed $n \in \mathbb{N}$, there exists an approximate solution sequence $\{(\lambda_j, u_j)\}$ of J such that $J_{\lambda_j}(u_j) = C_n(\lambda_j)$.*

Proof. We invoke Theorem 2. From Lemmas 8 and 9, it holds that

$$C_n(\lambda) \geq n > 0 \geq \max_{v \in \partial B_n} J_\lambda(v). \quad (39)$$

Let us check J_λ enjoys property (H). Let $\{\lambda_j\} \in [1/2, 1]$ be a sequence strictly increasing to some $\lambda_0 \in [1/2, 1]$ and $\{v_j\} \in H_r^1(\mathbb{R}^2)$ a sequence such that

$$\begin{aligned} &-J_{\lambda_0}(v_j), \\ &J_{\lambda_j}(v_j), \\ &\frac{J_{\lambda_j}(v_j) - J_{\lambda_0}(v_j)}{\lambda_0 - \lambda_j} < C. \end{aligned} \quad (40)$$

We need to show (i) $\{v_j\}$ is bounded in $H_r^1(\mathbb{R}^2)$ and (ii), for given $\varepsilon > 0$, there exists $N > 0$ satisfying

$$J(\lambda_0, v_j) \leq J(\lambda_j, v_j) + \varepsilon \quad \forall n \geq N. \quad (41)$$

We first show (i). We see from (40) that

$$\frac{1}{2} \|v_j\|^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{h_{v_j}^2(|x|)}{|x|^2} v_j^2 dx < 2C, \quad (42)$$

$$-C < \frac{1}{2} \int_{\mathbb{R}^2} V(u_j^2) dx < C,$$

which shows $\{v_j\}$ is bounded in $H_r^1(\mathbb{R}^2)$. Also, for given $\varepsilon > 0$,

$$\begin{aligned} J(\lambda_0, v_j) - J(\lambda_j, v_j) &= (\lambda_j - \lambda_0) \frac{1}{2} \int_{\mathbb{R}^2} V(u_j^2) dx \\ &\leq (\lambda_0 - \lambda_j) C \leq \varepsilon \end{aligned} \quad (43)$$

if j is sufficiently large. This shows (ii). Therefore, there exists a subset $M \subset [1/2, 1]$ with full measure in $[1/2, 1]$ that, for every $\lambda \in M$, there exists a bounded (PS) sequence $\{u_{\lambda,k}\}_{k=1}^\infty$ of J_λ at level $C_n(\lambda)$. Arguing similarly to Step 2 of Proposition 7, we also deduce $\{u_{\lambda,k}\}_{k=1}^\infty$ converges, up to a subsequence, to some critical point u_λ of J_λ with $J_\lambda(u_\lambda) = C_n(\lambda)$. Since M has full measure in $[1/2, 1]$, this completes the proof. $\square$

Completion of the Proof of Theorem 1. Now we complete the proof of Theorem 1. We first choose and fix arbitrary $n \in \mathbb{N}$. Let $\{u_j\}$ be an approximate solution sequence of J_{λ_j} , given by Proposition 10. Take $\gamma \in \Gamma_n$ satisfying

$$\max_{u \in B_n} J(\gamma(u)) \leq C_n(1) + \frac{1}{2}. \quad (44)$$

It follows from the compactness of B_n that

$$\begin{aligned} C_n(\lambda_j) &\leq \max_{u \in B_n} J_{\lambda_j}(\gamma(u)) \\ &\leq \max_{u \in B_n} J(\gamma(u)) \\ &\quad + (1 - \lambda_j) \max_{u \in B_n} \int_{\mathbb{R}^2} \frac{1}{2} V(\gamma(u)^2) dx \\ &\leq C_n(1) + \frac{1}{2} + \frac{1}{2} = C_n(1) + 1 \end{aligned} \quad (45)$$

for sufficiently large j . Then Proposition 7 applies to see u_j converges, up to a subsequence, to some u_0 which is a critical point of J . Recall that $J_{\lambda_j}(u_j) = C_n(\lambda_j) \geq n$. By taking a limit $j \rightarrow \infty$, we deduce $J(u_0) \geq n$. Since n is arbitrary, this shows the existence of infinitely many critical points of J . This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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