Research Article

Local Strong Solution for a Class of Shear Thickening Fluids with Non-Newtonian Potential and Heat-Conducting

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The aim of this paper is to discuss the model for a class of shear thickening fluids with non-Newtonian potential and heat-conducting. Existence and uniqueness of local strong solutions for the model are proved. In this paper, there exist two difficulties we have to overcome. One is the strong nonlinearity of the system. The other is that the state function is not fixed.

1. Introduction

With the development of the technology, we find that there are a lot of fluids which belong to non-Newtonian fluids in the process of product and nature. The classical non-Newtonian fluids are macromolecule melt and macromolecule liquor and all kinds of slurry and suspend liquor, paint, dope, palette, and biology fluids, for example, in the body of people and animals, the blood, the synovia of arthrosis cavity, lymph liquor, cell liquor, and brain liquor, which are provided with the property of non-Newtonian fluids. So the non-Newtonian fluids exist widely in nature (see [1–6]).

In the absence of gravitational potential term, the mathematical model of non-Newtonian fluids reduces to the Navier-Stokes equations. There have been many results concerning the existence and nonexistence of solutions for such equations (such as [7–10]). In [11], Xin studied the blowup of smooth solutions to the compressible Navier-Stokes equations in arbitrary space dimensions with initial density of compact support. Choe and Kim in [12, 13] proved an existence result when Ω is either a bounded domain or the whole space with the compatibility condition. For other results one can refer to [14–17] and the references cited therein.

In thermodynamics, a state function, function of state, state quantity, or state variable is a property of a system that depends only on the current state of the system, not on the way in which the system acquired that state. A state function

describes the equilibrium state of a system. For example, internal energy e , temperature θ , density ρ , pressure π , and so on are state quantities because they describe quantitatively an equilibrium state of a thermodynamic system, irrespective of how the system arrived in that state. If two quantities are known, then we can get other quantities. For example, temperature θ and density ρ are known, and we can get internal energy e , pressure π , and so on.

In mathematics, a state function $z = f(x, y)$ has the following properties:

- (i) A state function $z = f(x, y)$ has continuous first partial derivatives.
- (ii) There are the tiny variable dz and the limited variable $\int dz$. The integral value depends on the size of the integral upper limit and lower limit and has nothing to do with other factors. Thus,

$$\int_1^2 dz = f(x_2, y_2) - f(x_1, y_1) = z_2 - z_1. \quad (1)$$

- (iii) Consider

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}. \quad (2)$$

The following are some examples of state equations:

- (i) Polytopic gases: $\phi = A\rho^\gamma$ (A is a constant and γ is gas thermal insulation coefficient and is a constant) (see [15–17]).

(ii) State equation of ideal gas: $\psi = R\rho\theta$ (R is a constant) (see [18]).

(iii) Isentropic process: $\varphi = \theta$ (see [18]).

In this paper, we consider generalized state equation. Energy increases with the increase of temperature. In mathematics, $\pi = \varphi(\theta, \rho)$ and $\varphi_1 = \partial\varphi/\partial\theta > 0$ (we define $\varphi_1 = \partial\varphi/\partial\theta$, $\varphi_2 = \partial\varphi/\partial\rho$, $\psi_1 = \partial\psi/\partial\theta$, and $\psi_2 = \partial\psi/\partial\rho$). Ignore the molecular potential energy and all of the gas fluid, for example, methane, propane, and butane from the oil.

In this paper, we consider a class of heat-conducting non-Newtonian fluids of Navier-Stokes system:

$$\rho_t + (\rho u)_x = 0, \quad (x, t) \in \Omega_T,$$

$$(\rho e)_t + (\rho u e)_x - \left[(\theta_x^2 + \mu)^{(p-2)/2} \theta_x \right]_x + \pi u_x = |u_x|^2, \quad (x, t) \in \Omega_T,$$

$$(\rho u)_t + (\rho u^2)_x - \left[(u_x^2 + \mu_2)^{(q-2)/2} u_x \right]_x + \pi_x = 0, \quad (3) \quad (x, t) \in \Omega_T,$$

$$e = \varphi(\theta, \rho), \quad (x, t) \in \Omega_T,$$

$$\pi = \psi(\theta, \rho), \quad (x, t) \in \Omega_T,$$

with the initial and boundary conditions

$$\begin{aligned} (\rho, u, \theta)|_{t=0} &= (\rho_0, u_0, \theta_0), \quad x \in [0, 1], \\ u(0, t) &= u(1, t) = 0, \quad t \in [0, T], \\ \theta(0, t) &= \theta(1, t) = 0, \quad t \in [0, T], \end{aligned} \quad (4)$$

where $\Omega_T = I \times (0, T)$, $I = (0, 1)$, the initial density $\rho_0 \geq 0$, $p, q > 2$, and $\mu_1, \mu_2 > 0$ are given constants. The unknown variables ρ, u , and θ denote the density, velocity, temperature, and the non-Newtonian gravitational potential, respectively.

Definition 1. The triple (ρ, u, θ) is called a strong solution to the initial boundary value problem (3)-(4), if the following conditions are satisfied:

(i) Consider

$$\begin{aligned} 0 &\leq \rho \in C([0, T]; H^1(I)), \\ u, \theta &\in C([0, T]; H_0^1(I)) \cap L^\infty(0, T; H^2(I)), \\ \rho_t &\in C([0, T]; L^2(I)), \\ u_t, \theta_t &\in L^2(0, T; H_0^1(I)), \\ \sqrt{\rho}u_t, \sqrt{\rho}\theta_t &\in L^\infty(0, T; L^2(I)). \end{aligned} \quad (5)$$

(ii) For all $\chi \in C([0, T]; H^1(I))$, $\chi_t \in L^\infty(0, T; L^2(I))$, for a.e. $t \in (0, T)$, we have

$$\begin{aligned} \int_0^1 \rho \chi(x, t) dx - \int_0^t \int_0^1 (\rho \chi_t + \rho u \chi_x)(x, s) dx ds \\ = \int_0^1 \rho_0 \chi(x, 0) dx. \end{aligned} \quad (6)$$

(iii) For all $\phi \in C([0, T]; H_0^1(I)) \cap L^\infty(0, T; H^2(I))$, $\phi_t \in L^2(0, T; H_0^1(I))$, for a.e. $t \in (0, T)$, we have

$$\begin{aligned} \int_0^1 \rho e \phi(x, t) dx - \int_0^t \int_0^1 \left(\rho e \phi_t \right. \\ \left. + \left(\rho u e - (\theta_x^2 + \mu_1)^{(p-2)/2} \theta_x \right) \phi_x \right)(x, s) dx ds \\ + \int_0^t \int_0^1 -(|u_x|^2 + \pi u_x) \phi(x, s) dx ds \\ = \int_0^1 \rho_0 e_0 \phi(x, 0) dx. \end{aligned} \quad (7)$$

(iv) For all $\phi \in C([0, T]; H_0^1(I)) \cap L^\infty(0, T; H^2(I))$, $\phi_t \in L^2(0, T; H_0^1(I))$, for a.e. $t \in (0, T)$, we have

$$\begin{aligned} \int_0^1 \rho u \phi(x, t) dx - \int_0^t \int_0^1 \left\{ \rho u \phi_t + \rho u^2 \phi_x \right. \\ \left. - (u_x^2 + \mu_2)^{(q-2)/2} u_x \phi_x + \pi_x \phi \right\}(x, s) dx ds \\ = \int_0^1 \rho_0 u_0 \phi(x, 0) dx. \end{aligned} \quad (8)$$

1.1. Main Results

Theorem 2. Assume that (ρ_0, u_0, θ_0) satisfies the following conditions:

$$\begin{aligned} 0 < \delta \leq \rho_0 \in H^1(I), u_0, \theta_0 \in H_0^1(I) \cap H^2(I), \\ 0 < \delta \leq \varphi_1 \end{aligned} \quad (9)$$

and further that there exists a constant $c_0 > 0$, such that

$$1 + |\rho_0|_{H^1} + |\theta_0|_{H^1} + |u_0|_{H^1} \leq c_0. \quad (10)$$

Then there exist a time $T^* \in (0, +\infty)$ and a unique strong solution (ρ, u, θ) to (3)-(4) such that

$$\begin{aligned} \rho &\in C([0, T^*]; H^1(I)), \\ \rho_t &\in C([0, T^*]; L^2(I)), \\ u_t, \theta_t &\in L^2(0, T^*; H_0^1(I)), \\ u, \theta &\in C([0, T^*]; H_0^1(I)) \\ &\cap L^\infty(0, T^*; H^2(I)), \\ \sqrt{\rho}u_t, \sqrt{\rho}\theta_t &\in L^\infty(0, T^*; L^2(I)). \end{aligned} \quad (11)$$

The rest of this paper is organized as follows. In Section 2, we present some elementary lemmas. In Section 3, we devote ourselves to the study of the problem with positive density. In Section 4 we give the proof of the main theorem.

2. Preliminaries

We first give some known facts for later use.

Lemma 3 (see [15]). Assume that $f = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^1$ is bounded and open, $f \in C^{2+\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. Then

$$\|f'\|_{L^\infty(\Omega)} \leq d^{1/2}(\Omega) \|f''\|_{L^2(\Omega)}, \quad (12)$$

where $d(\Omega)$ denotes the length of Ω .

Lemma 4 (see [15]). Let H be a Hilbert space with a scalar product $(\cdot, \cdot)_H$ and let X be a Banach space such that

$$X \hookrightarrow H \cong H^* \hookrightarrow X^*, \quad (13)$$

and X is dense in H , $p > 1$. Then

$$W \equiv \left\{ u \in L^p(I; X); \frac{du}{dt} \in L^{p'}(I; X^*) \right\} \hookrightarrow C(I; H). \quad (14)$$

Lemma 5. $\pi = \varphi(\theta, \rho)$ and $e = \phi(\theta, \rho)$ is state function. According to the mathematical properties of state function, we can get that $\exists M > 0$,

$$\begin{aligned} |\varphi| + |\varphi_1| + |\varphi_2| + |\varphi_{11}| + |\varphi_{22}| &\leq M, \\ |\psi| + |\psi_1| + |\psi_2| + |\psi_{11}| + |\psi_{22}| &\leq M. \end{aligned} \quad (15)$$

3. Existence of Solutions with Positive Density

3.1. A Priori Estimates for Smooth Solutions. We first construct approximate solutions of problem (3)-(4). By using the iterative scheme, inductively, consider the following:

(i) Firstly define $u^0 = 0$, $\theta^0 = 0$.

(ii) Assuming that u^{k-1} , θ^{k-1} were defined for $k \geq 1$, we can obtain $(\rho^k, u^k, \theta^k, e^k, \pi^k)$ as the unique smooth solution to the following system:

$$\rho_t^k + u^{k-1} \rho_x^k + u_x^{k-1} \rho^k = 0, \quad (x, t) \in \Omega_T, \quad (16)$$

$$\pi^k = \phi(\theta^k, \rho^k), \quad (x, t) \in \Omega_T, \quad (17)$$

$$e^k = \varphi(\theta^k, \rho^k), \quad (x, t) \in \Omega_T, \quad (18)$$

$$\begin{aligned} \rho^k \varphi_1^k \theta_t^k + \rho^k \varphi_1^k u^{k-1} \theta_x^k - (\rho^k)^2 \varphi_2^k u_x^{k-1} + H_p^{\mu_1} \theta^k \\ + \psi^k u_x^{k-1} - (u_x^{k-1})^2 = 0, \quad (x, t) \in \Omega_T, \end{aligned} \quad (19)$$

$$\rho^k u_t^k + \rho^k u^{k-1} u_x^k - H_q^{\mu_2} \theta^k + \psi_x = 0, \quad (x, t) \in \Omega_T, \quad (20)$$

$$\rho^k(x, 0) = \rho_0,$$

$$u^k(x, 0) = u_0,$$

$$\theta^k(x, 0) = \theta_0, \quad (21)$$

$$u^k(0, t) = u^k(1, t) = 0,$$

$$\theta^k(0, t) = \theta^k(1, t) = 0,$$

where $m_0 = \int_0^1 \rho_0(x) dx > 0$ is the initial mass, for a given function ω and numbers $s > 2$, $\mu > 0$,

$$H_s^\mu \omega = - \left[((\omega_x)^2 + \mu)^{(s-2)/2} \omega_x \right]_x. \quad (22)$$

With this process, the nonlinear coupled system has been deduced into a sequence of decoupled problems and each problem admits a smooth solution. Moreover, we find ρ^k from (16), (21) with the smooth function u^{k-1} ; that is,

$$\begin{aligned} \rho_t^k + u^{k-1} \rho_x^k + u_x^{k-1} \rho^{k-1} &= 0, \\ \rho^k|_{t=0} &= \rho_0. \end{aligned} \quad (23)$$

Obviously there is a unique solution ρ^k to the above initial value problem and also by a standard argument, we could get

$$\begin{aligned} \rho^k(x, t) &\geq \delta \exp \left(- \int_0^{T^*} |u_x^{k-1}(\cdot, s)|_{L^\infty} ds \right) > 0, \\ \forall t &\in (0, T^*). \end{aligned} \quad (24)$$

Then by (19) and (21) we get θ^k . Lastly, with ρ^k , θ^k , we obtain u^k from (20) and (21).

Moreover, assume that for a given T_1 , $0 < T_1 \leq T$

$$\begin{aligned} \sup_{0 \leq t \leq T_1} |u_{xx}^{k-1}|_{L^2} + \int_0^{T_1} |u_{xt}^{k-1}|_{L^2}^2 ds &\leq c_1, \\ \sup_{0 \leq t \leq T_1} |u_{xx}^{k-1}|_{L^2} &\leq c_2. \end{aligned} \quad (25)$$

For some, $c_2 > c_1 > c_0 > 1$ will be fixed later. We will prove that

$$\begin{aligned} \sup_{0 \leq t \leq T_1} |u_x^k|_{L^2} + \int_0^{T_1} |u_{xt}^k|_{L^2}^2 ds &\leq c_1, \\ \sup_{0 \leq t \leq T_1} |u_{xx}^k|_{L^2} &\leq c_2, \end{aligned} \quad (26)$$

and some more useful uniform estimate about (ρ^k, u^k, θ^k) .

(i) Estimation for ρ^k .

Then multiplying (16) by ρ^k and integrating it over $(0, 1)$ with respect to x , we have

$$\frac{1}{2} \frac{d}{dt} \int_0^1 |\rho^k|^2 ds + \int_0^1 (\rho^{k-1} u^{k-1})_x \rho^k dx = 0. \quad (27)$$

Integrating by parts, using the Sobolev inequality and Lemma 3, we deduce that

$$\frac{d}{dt} |\rho^k(t)|_{L^2}^2 \leq |u_{xx}^{k-1}|_{L^2} |\rho^k|_{L^2}^2. \quad (28)$$

Then, considering differential equation (16) with respect to x and multiplying it by ρ_x^k , integrating it over $(0, 1)$ on x , and using the Sobolev inequality, we have

$$\begin{aligned} & \frac{d}{dt} \int_0^1 |\rho_x^k|^2 dx \\ &= - \int_0^1 \left[\frac{3}{2} u_x^{k-1} (\rho_x^k)^2 + \rho^k \rho_x^k u_{xx}^{k-1} \right] (t) dx \\ &\leq \frac{3}{2} \left[\|u_x^{k-1}\|_{L^\infty} \|\rho_x^k\|_{L^2}^2 + \|\rho^k\|_{L^\infty} \|u_{xx}^{k-1}\|_{L^2} \right] \\ &\leq 3 \|\rho_x^k\|_{L^2}^2 \|u_{xx}^{k-1}\|_{L^2}. \end{aligned} \quad (29)$$

From (28) and (29), by Gronwall's inequality, it follows that

$$\sup_{0 \leq t \leq T} |\rho^k(t)|_{H^1}^2 \leq |\rho_0^k|_{H^1}^2 \exp \left\{ 4 \int_0^t \|u_{xx}^{k-1}\|_{L^2} ds \right\}, \quad (30)$$

and, taking $T_2 = c_2^{-1}$, we get

$$\begin{aligned} \sup_{0 \leq t \leq T} |\rho^k(t)|_{H^1} &\leq |\rho_0^k|_{H^1} \left[\exp \left\{ 4 \int_0^t \|u_{xx}^{k-1}\|_{L^2} ds \right\} \right]^{1/2} \\ &\leq C c_0. \end{aligned} \quad (31)$$

And using (16) we can also obtain

$$|\rho_t^k(t)|_{L^2} \leq |\rho_x^{k-1}|_{L^2} \|u_x^{k-1}\|_{L^2} + |\rho^{k-1}|_{L^2} \|u_{xx}^{k-1}\|_{L^2} \leq C c_1^2, \quad (32)$$

where C is a positive constant.

(ii) Estimation for $\theta^k(t)$.

Differentiating (19) with respect to t , multiplying it by θ_t^k , and integrating it over $(0, 1)$ on x , we derive

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^k \varphi_1^k |\theta_t^k|^2 dx \\ &+ \int_0^1 \left[(\theta_x^2 + \mu_1)^{(p-2)/2} \theta_x \right]_t \theta_{xt}^k dx \\ &= \frac{1}{2} \int_0^1 [\rho^k \varphi_1^k]_t |\theta_t^k|^2 dx - \int_0^1 [\rho^k \varphi_1^k u^{k-1} \theta_x^k]_t \theta_t^k dx \\ &- \int_0^1 [(\rho^k)^2 \varphi_2^k u_x^{k-1}]_t \theta_t^k dx - \int_0^1 [\psi^k u_x^{k-1}]_t \theta_t^k dx \\ &+ \int_0^1 (u_x^{k-1})_t^2 \theta_t^k dx = \sum_{j=1}^5 I_j. \end{aligned} \quad (33)$$

Using the Sobolev inequality, Young's inequality, Lemma 3, and (25), (30) and (31), we obtain

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^1 [\rho^k \varphi_1^k]_t |\theta_t^k|^2 dx \\ &= \frac{1}{2} \int_0^1 \rho_t^k \varphi_1^k |\theta_t^k|^2 dx + \frac{1}{2} \int_0^1 \rho^k \varphi_{11}^k \theta_t^k |\theta_t^k|^2 dx \\ &+ \frac{1}{2} \int_0^1 \rho^k \varphi_{12}^k \rho_t^k |\theta_t^k|^2 dx = \sum_{j=1}^3 I_{1j}. \end{aligned} \quad (34)$$

Here,

$$\begin{aligned} I_{11} &= \frac{1}{2} \int_0^1 \rho_t^k \varphi_1^k |\theta_t^k|^2 dx \leq C \left| \int_0^1 \rho_t^k |\theta_t^k|^2 dx \right| \\ &\leq C \left| \int_0^1 \rho^k u^{k-1} \theta_t^k \rho_{xt}^k dx \right| \\ &\leq C \|\rho^k\|_{L^\infty}^{1/2} \|\sqrt{\rho^k} \theta_t^k\|_{L^2} \|u^{k-1}\|_{L^\infty} \|\theta_{xt}^k\|_{L^2} \\ &\leq C c_0^{1/2} c_1 \|\sqrt{\rho^k} \theta_t^k\|_{L^2} \|\theta_{xt}^k\|_{L^2} \\ &\leq \eta \|\theta_{xt}^k\|_{L^2}^2 + C_\eta c_1^3 \|\sqrt{\rho^k} \theta_t^k\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned} I_{12} &= \frac{1}{2} \int_0^1 \rho^k \varphi_{11}^k \theta_t^k |\theta_t^k|^2 dx \leq C \left| \int_0^1 \rho^k \theta_t^k |\theta_t^k|^2 dx \right| \\ &\leq C \|\theta_t^k\|_{L^\infty} \|\sqrt{\rho^k} \theta_t^k\|_{L^2}^2 \leq \eta \|\theta_{xt}^k\|_{L^2}^2 + C_\eta \|\sqrt{\rho^k} \theta_t^k\|_{L^2}^4, \end{aligned} \quad (35)$$

$$\begin{aligned} I_{13} &= \frac{1}{2} \int_0^1 \rho^k \varphi_{12}^k \rho_t^k |\theta_t^k|^2 dx \\ &\leq C \|\rho^k\|_{L^\infty} \left| \int_0^1 \rho_t^k |\theta_t^k|^2 dx \right| \\ &\leq C c_0^{3/2} c_1 \|\sqrt{\rho^k} \theta_t^k\|_{L^2} \|\theta_{xt}^k\|_{L^2} \\ &\leq \eta \|\theta_{xt}^k\|_{L^2}^2 + C_\eta c_1^5 \|\sqrt{\rho^k} \theta_t^k\|_{L^2}^2. \end{aligned}$$

In the same way, we have

$$\begin{aligned} I_2 &= - \int_0^1 [\rho^k \varphi_1^k u^{k-1} \theta_x^k]_t \theta_t^k dx \\ &= - \int_0^1 \rho_t^k \varphi_1^k u^{k-1} \theta_x^k \theta_t^k dx - \int_0^1 \rho^k \varphi_{11}^k \theta_t^k u^{k-1} \theta_x^k \theta_t^k dx \\ &- \int_0^1 \rho^k \varphi_{12}^k \rho_t^k u^k \theta_x^k \theta_t^k dx - \int_0^1 \rho^k \varphi_1^k u_t^{k-1} \theta_x^k \theta_t^k dx \\ &- \int_0^1 \rho^k \varphi_1^k u^{k-1} \theta_{xt}^k \theta_t^k dx = \sum_{j=1}^5 I_{2j}. \end{aligned} \quad (36)$$

Here,

$$\begin{aligned} I_{21} &= - \int_0^1 \rho_t^k \varphi_1^k u^{k-1} \theta_x^k \theta_t^k dx \leq C \left| \int_0^1 \rho_t^k u^{k-1} \theta_x^k \theta_t^k dx \right| \\ &\leq C \|\rho_t^k\|_{L^2} \|u^{k-1}\|_{L^\infty} \|\theta_x^k\|_{L^2} \|\theta_t^k\|_{L^\infty} \\ &\leq C \|\rho_t^k\|_{L^2} \|u_x^{k-1}\|_{L^2} \|\theta_x^k\|_{L^2} \|\theta_{xt}^k\|_{L^2} \leq C c_1^3 \|\theta_x^k\|_{L^2} \|\theta_{xt}^k\|_{L^2} \\ &\leq \eta \|\theta_{xt}^k\|_{L^2}^2 + C_\eta c_1^6 \|\theta_x^k\|_{L^2}^2, \end{aligned}$$

$$\begin{aligned}
I_{22} &= - \int_0^1 \rho^k \varphi_{11}^k \theta_t^k u^{k-1} \theta_x^k \theta_t^k dx \\
&\leq C \left| \int_0^1 \rho^k \theta_t^k u^{k-1} \theta_x^k \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty}^{1/2} \left| u^{k-1} \right|_{L^\infty} \left| \theta_t^k \right|_{L^\infty} \left| \theta_x^k \right|_{L^2} \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^3 \left| \theta_x^k \right|_{L^2}^2 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2, \\
I_{23} &= - \int_0^1 \rho^k \varphi_{12}^k \rho_t^k u^{k-1} \theta_x^k \theta_t^k dx \\
&\leq C \left| \int_0^1 \rho^k \rho_t^k u^{k-1} \theta_x^k \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty} \left| \rho_t^k \right|_{L^2} \left| u^{k-1} \right|_{L^\infty} \left| \theta_x^k \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^8 \left| \theta_x^k \right|_{L^2}^2, \\
I_{24} &= - \int_0^1 \rho^k \varphi_{11}^k u_t^{k-1} \theta_x^k \theta_t^k dx \leq C \left| \int_0^1 \rho^k u_t^{k-1} \theta_x^k \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty}^{1/2} \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2} \left| u_t^{k-1} \right|_{L^\infty} \left| \theta_x^k \right|_{L^2} \\
&\leq C c_0 \left| \theta_x^k \right|_{L^2}^2 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 + \frac{1}{2} \left| u_{xt}^{k-1} \right|_{L^2}^2, \\
I_{25} &= - \int_0^1 \rho^k \varphi_{11}^k u^{k-1} \theta_{xt}^k \theta_t^k dx \\
&\leq C \left| \rho^k \right|_{L^\infty}^{1/2} \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2} \left| u^{k-1} \right|_{L^\infty} \left| \theta_{xt}^k \right|_{L^2} \\
&\leq C \left| \rho^k \right|_{L^\infty}^{1/2} \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2} \left| u^{k-1} \right|_{L^\infty} \left| \theta_{xt}^k \right|_{L^2} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^6 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2.
\end{aligned} \tag{37}$$

Following the same method, we obtain

$$\begin{aligned}
I_3 &= - \int_0^1 \left[(\rho^k)^2 \varphi_2^k u_x^{k-1} \right]_t \theta_t^k dx \\
&= - \int_0^1 2\rho^k \rho_t^k \varphi_2^k u_x^{k-1} \theta_t^k dx \\
&\quad - \int_0^1 (\rho^k)^2 \varphi_{12}^k \theta_t^k u_x^{k-1} \theta_t^k dx \\
&\quad - \int_0^1 (\rho^k)^2 \varphi_{22}^k \rho_t^k u_x^{k-1} \theta_t^k dx \\
&\quad - \int_0^1 (\rho^k)^2 \varphi_2^k u_{xt}^{k-1} \theta_t^k dx = \sum_{j=1}^4 I_{3j}.
\end{aligned} \tag{38}$$

Here,

$$\begin{aligned}
I_{31} &= - \int_0^1 2\rho^k \rho_t^k \varphi_2^k u_x^{k-1} \theta_t^k dx \\
&\leq C \left| \int_0^1 \rho^k \rho_t^k u_x^{k-1} \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty} \left| \rho_t^k \right|_{L^2} \left| u_x^{k-1} \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^8, \\
I_{32} &= - \int_0^1 (\rho^k)^2 \varphi_{12}^k \theta_t^k u_x^{k-1} \theta_t^k dx \\
&\leq C \left| \int_0^1 (\rho^k)^2 \theta_t^k u_x^{k-1} \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty} \left| u_x^{k-1} \right|_{L^\infty} \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 \leq C_\eta c_1^2 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2, \\
I_{33} &= - \int_0^1 (\rho^k)^2 \varphi_{22}^k \rho_t^k u_x^{k-1} \theta_t^k dx \\
&\leq C \left| \int_0^1 (\rho^k)^2 \rho_t^k u_x^{k-1} \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty}^2 \left| \rho_t^k \right|_{L^2} \left| u_x^{k-1} \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^{10}, \\
I_{34} &= - \int_0^1 (\rho^k)^2 \varphi_2^k u_{xt}^{k-1} \theta_t^k dx \\
&\leq C \left| \int_0^1 (\rho^k)^2 u_{xt}^{k-1} \theta_t^k dx \right| \\
&\leq C \left| \rho^k \right|_{L^\infty}^2 \left| u_{xt}^{k-1} \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^4 \left| u_{xt}^{k-1} \right|_{L^2}^2.
\end{aligned} \tag{39}$$

In a similar way,

$$\begin{aligned}
I_4 &= - \int_0^1 \left[\psi^k u_x^{k-1} \right]_t \theta_t^k dx \\
&= - \int_0^1 \psi_1^k \theta_t^k u_x^{k-1} \theta_t^k dx - \int_0^1 \psi_2^k \rho_t^k u_x^{k-1} \theta_t^k dx \\
&\quad - \int_0^1 \psi^{k-1} u_{xt}^{k-1} \theta_t^k dx = \sum_{j=1}^3 I_{4j}.
\end{aligned} \tag{40}$$

Here,

$$\begin{aligned}
I_{41} &= - \int_0^1 \psi_1^k \theta_t^k u_x^{k-1} \theta_t^k dx \leq C \left| \int_0^1 \frac{\rho^k}{\rho^k} \theta_t^k u_x^{k-1} \theta_t^k dx \right| \\
&\leq C \left| u_x^{k-1} \right|_{L^\infty} \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 \leq C c_1 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2, \\
I_{42} &= - \int_0^1 \psi_2^k \rho_t^k u_x^{k-1} \theta_t^k dx \leq C \left| \int_0^1 \rho_t^k u_x^{k-1} \theta_t^k dx \right| \\
&\leq C \left| \rho_t^k \right|_{L^2} \left| u_x^{k-1} \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^{10}, \\
I_{43} &= - \int_0^1 \psi^k u_{xt}^{k-1} \theta_t^k dx \leq C \left| \int_0^1 u_{xt}^{k-1} \theta_t^k dx \right| \\
&\leq C \left| u_{xt}^{k-1} \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^2.
\end{aligned} \tag{41}$$

Similarly,

$$\begin{aligned}
I_5 &= \int_0^1 \left(u_x^{k-1} \right)_t^2 \theta_t^k dx \leq 2 \left| u_x^{k-1} \right|_{L^\infty} \left| u_{xt}^{k-1} \right|_{L^2} \left| \theta_t^k \right|_{L^\infty} \\
&\leq 2 \left| u_{xx}^{k-1} \right|_{L^2} \left| u_{xt}^{k-1} \right|_{L^2} \left| \theta_{xt}^k \right|_{L^2} \\
&\leq \eta \left| \theta_{xt}^k \right|_{L^2}^2 + C_\eta c_1^2 \left| u_{xt}^{k-1} \right|_{L^2}^2.
\end{aligned} \tag{42}$$

Substituting I_j ($j = 1, 2, \dots, 5$) into (43), after choosing $\eta = \mu_1^{(p-2)/2}/26$, we get

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left| \sqrt{\rho^k} \varphi_1^k \theta_t^k(t) \right|_{L^2}^2 + \frac{1}{2} \mu_1^{(p-2)/2} \left| \theta_{xt}^k \right|_{L^2}^2(s) ds \\
&\leq C c_1^{10} + C c_1^8 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 + C \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^4 \\
&\quad + C c_1^6 \left| \theta_x^k \right|_{L^2}^2 + C c_1^3 \left| \theta_x^k \right|_{L^2}^2 \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 \\
&\quad + C c_1^4 \left| u_{xt}^{k-1} \right|_{L^2}^2,
\end{aligned} \tag{43}$$

and, integrating it over $(\tau, t) \subset (0, T)$, we have

$$\begin{aligned}
&\frac{1}{2} \left| \sqrt{\rho^k} \varphi_1^k \theta_t^k(t) \right|_{L^2}^2 + \frac{1}{2} \mu_1^{(p-2)/2} \int_\tau^t \left| \theta_{xt}^k \right|_{L^2}^2(s) ds \\
&\leq C c_1^{10} \int_\tau^t \left(1 + \left| \theta_x^k \right|_{L^2}^2 \right) \left(1 + \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 \right) (s) ds \\
&\quad + C c_1^4 \int_\tau^t \left| u_{xt}^{k-1} \right|_{L^2}^2(s) ds + \left| \sqrt{\rho^k} \varphi_1^k \theta_t^k(\tau) \right|_{L^2}^2,
\end{aligned} \tag{44}$$

where C is a positive constant.

We estimate $\left| \sqrt{\rho^k} \varphi_1^k \theta_t^k(\tau) \right|_{L^2}^2$ as follows: multiplying (17) by θ_t^k and integrating it over $(0, 1)$, we obtain

$$\begin{aligned}
&\int_0^1 \rho^k \varphi_1^k \left| \theta_t^k \right|^2 dx \leq \int_0^1 (\varphi_1^k)^{-1} \left[(\rho^k)^3 \left| \varphi_2^k \right|^2 \left| u_x^{k-1} \right|^2 \right. \\
&\quad \left. + (\rho^k)^{-1} \left| \psi^k \right|^2 \left| u_x^{k-1} \right|^2 \right. \\
&\quad \left. + (\rho^k)^{-1} \left| H_p^{\mu_1} \theta^k - (u_x^k)^2 \right|^2 \right] dx + \int_0^1 \rho^k \left| \varphi_1^k \right| \\
&\quad \cdot \left| u^{k-1} \right|^2 \left| \theta_x^k \right|^2 dx.
\end{aligned} \tag{45}$$

Next, we get

$$\begin{aligned}
&\int_0^1 \rho^k \varphi_1^k \left| \theta_t^k \right|^2 dx \leq C \int_0^1 \left(\rho^k \left| u^{k-1} \right|^2 \left| \theta_x^k \right|^2 \right. \\
&\quad \left. + (\rho^k)^3 \left| u_x^{k-1} \right|^2 + (\rho^k)^{-1} \left| u_x^{k-1} \right|^2 \right. \\
&\quad \left. + (\rho^k)^{-1} \left| H_p^{\mu_1} \theta^k - (u_x^k)^2 \right|^2 \right) dx,
\end{aligned} \tag{46}$$

and, according to the smoothness of (ρ^k, u^k, θ^k) , we have

$$\begin{aligned}
&\lim_{\tau \rightarrow 0} \int_0^1 \left(\rho^k \left| u^{k-1} \right|^2 \left| \theta_x^k \right|^2 + (\rho^k)^3 \left| u_x^{k-1} \right|^2 \right. \\
&\quad \left. + (\rho^k)^{-1} \left| u_x^{k-1} \right|^2 \right. \\
&\quad \left. + (\rho^k)^{-1} \left| H_p^{\mu_1} \theta^k - (u_x^k)^2 \right|^2 \right) dx \leq C c_0^5,
\end{aligned} \tag{47}$$

where C is a positive constant.

We have

$$\begin{aligned}
&\frac{d}{dt} \left| \theta_x^k \right|_{L^2}^2 \leq 2 \int_0^1 \theta_x^k \theta_{xt}^k dx \leq 2 \left| \theta_x^k \right|_{L^2} \left| \theta_{xt}^k \right|_{L^2} \\
&\leq \left| \theta_x^k \right|_{L^2}^2 + \left| \theta_{xt}^k \right|_{L^2}^2;
\end{aligned} \tag{48}$$

by Gronwall's inequality, it follows that

$$\left| \theta_x^k \right|_{L^2}^2 \leq \exp \{t\} \left(\left| \theta_x^k(0) \right|_{L^2}^2 + \int_0^t \left| \theta_{xt}^k \right|_{L^2}^2 ds \right) \tag{49}$$

for $0 < t < \min\{T_1, T_2\}$,

$$\left| \theta_x^k \right|_{L^2}^2 \leq C \left| \theta_x^k(0) \right|_{L^2}^2 + C \int_0^t \left| \theta_{xt}^k \right|_{L^2}^2 ds. \tag{50}$$

Due to (44), (47), and (50), we get

$$\begin{aligned}
&\left| \theta_x^k \right|_{L^2}^2 + \left| \sqrt{\rho^k} \theta_t^k(t) \right|_{L^2}^2 + \int_\tau^t \left| \theta_{xt}^k \right|_{L^2}^2(s) ds \\
&\leq C c_1^{10} \int_\tau^t \left(1 + \left| \theta_x^k \right|_{L^2}^2 \right) \left(1 + \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2}^2 \right) ds + C c_1^5.
\end{aligned} \tag{51}$$

Denote $\Gamma(t) = 1 + |\theta_x^k|_{L^2}^2 + |\sqrt{\rho^k} \theta_t^k|_{L^2}^2$; we get

$$\Gamma(t) \leq C c_1^{10} \int_0^t \Gamma(s) ds + C c_1^5; \quad (52)$$

by Gronwall's inequality, $\exists T_3 \leq T_2$ (T_3 has nothing to do with k), it follows that

$$\Gamma(t) \leq C c_1^5. \quad (53)$$

Note that, for a given function ω , by a simple computation, we have

$$\begin{aligned} & \left| \left[\left((\theta_x^k)^2 + \mu_1 \right)^{(p-2)/2} \theta_x^k \right]_t \theta_{xt}^k \right| \\ &= \left((\theta_x^k)^2 + \mu_1 \right)^{(p-4)/2} \left[(r-1) (\theta_x^k)^2 + \mu_1 \right] |\theta_{xt}^k|^2 \\ &\geq \left((\theta_x^k)^2 + \mu_1 \right)^{(p-4)/2} \left[(\theta_x^k)^2 + \mu_1 \right] |\theta_{xt}^k|^2 \\ &= \left((\theta_x^k)^2 + \mu_1 \right)^{(p-2)/2} |\theta_{xt}^k|^2 \geq \mu_1^{(p-2)/2} |\theta_{xt}^k|^2. \end{aligned} \quad (54)$$

Firstly, we need to do the following estimates: by (19)

$$\begin{aligned} & \left[(\theta_x^2 + \mu_1)^{(p-2)/2} \theta_x \right]_x \\ &= \rho^k \phi_1^k \theta_t^k + \rho^k \phi_1^k u^{k-1} \theta_x^k - (\rho^k)^2 \phi_2^k u_x^{k-1} + \psi^k u_x^{k-1} \\ &\quad - (u_x^{k-1})^2, \end{aligned} \quad (55)$$

and we have

$$\begin{aligned} & \left| \left[\left((\theta_x^k)^2 + \mu_1 \right)^{(p-2)/2} \theta_x^k \right]_x \right| \\ &= \left((\theta_x^k)^2 + \mu_1 \right)^{(p-4)/2} \left[(p-1) (\theta_x^k)^2 + \mu_1 \right] |\theta_{xx}^k| \\ &\geq \left((\theta_x^k)^2 + \mu_1 \right)^{(p-4)/2} \left[(\theta_x^k)^2 + \mu_1 \right] |\theta_{xx}^k| \\ &= \left((\theta_x^k)^2 + \mu_1 \right)^{(p-2)/2} |\theta_{xx}^k| \geq \mu_1^{(p-2)/2} |\theta_{xx}^k|. \end{aligned} \quad (56)$$

taking it by L^2 -norm, we get

$$\begin{aligned} |\theta_{xx}^k|_{L^2} &\leq \mu_1^{(2-p)/2} \left[|\rho^k \phi_1^k \theta_t^k|_{L^2} + |\rho^k \phi_1^k u^{k-1} \theta_x^k|_{L^2} \right. \\ &\quad \left. + |(\rho^k)^2 \phi_2^k u_x^{k-1}|_{L^2} + |\psi^k u_x^{k-1}|_{L^2} + |(u_x^{k-1})^2|_{L^2} \right] \\ &\leq C \left[|\rho^k|_{L^\infty}^{1/2} |\sqrt{\rho^k} \theta_t^k|_{L^2} + |\rho^k|_{L^\infty} |u^{k-1}|_{L^\infty} |\theta_x^k|_{L^2} \right. \\ &\quad \left. + |\rho^k|_{L^\infty}^2 |u_x^{k-1}|_{L^2} + |u_x^{k-1}|_{L^2} + |u_x^{k-1}|_{L^\infty} |u_x^{k-1}|_{L^2} \right] \end{aligned}$$

$$\begin{aligned} &\leq C \left[|\rho_x^k|_{L^2}^{1/2} |\sqrt{\rho^k} \theta_t^k|_{L^2} + |\rho_x^k|_{L^2} |u_x^{k-1}|_{L^2} |\theta_x^k|_{L^2} \right. \\ &\quad \left. + |\rho_x^k|_{L^2}^2 |u_x^{k-1}|_{L^2} + |u_x^{k-1}|_{L^2} + |u_{xx}^{k-1}|_{L^2} |u_x^{k-1}|_{L^2} \right] \\ &\leq C c_1^4 c_2. \end{aligned} \quad (57)$$

(iii) Estimation for $u^k(t)$.

Differentiating (20) with respect to t , multiplying it by u_t^k , and integrating it over $(0,1)$ on x , we derive

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^k |u_t^k|^2 dx \\ &+ \int_0^1 \left[(u_x^2 + \mu_2)^{(q-2)/2} u_x \right]_t u_{xt}^k dx \\ &= \frac{1}{2} \int_0^1 \rho_t^k |u_t^k|^2 dx \\ &- \int_0^1 [\rho_t^k u^{k-1} u_x^k + \rho^k u_t^{k-1} u_x^k + \rho^k u^{k-1} u_{xt}^k] u_t^k dx \\ &+ \int_0^1 \psi_t^k u_{xt}^k dx = \sum_{j=1}^5 I_j. \end{aligned} \quad (58)$$

Then, using the Sobolev inequality, Young's inequality, Lemma 3, and (25), (30), and (31), we obtain

$$\begin{aligned} I_1 &= \frac{1}{2} \int_0^1 \rho_t^k |u_t^k|^2 dx \leq \int_0^1 \rho^k |u^{k-1}| |u_t^k| |u_{xt}^k| dx \\ &\leq |\rho^k|_{L^\infty}^{1/2} |u^{k-1}|_{L^\infty} |\sqrt{\rho^k} u_t^k|_{L^2} |u_{xt}^k|_{L^2} \\ &\leq \eta_1 |u_{xt}^k|_{L^2}^2 + C_{\eta_1} c_1^3 |\sqrt{\rho^k} u_t^k|_{L^2}^2, \\ I_2 &= - \int_0^1 \rho_t^k u^{k-1} u_x^k u_t^k dx \leq \int_0^1 |\rho_t^k| |u^{k-1}| |u_x^k| |u_t^k| dx \\ &\leq |\rho_t^k|_{L^2} |u^{k-1}|_{L^\infty} |u_x^k|_{L^2} |u_t^k|_{L^\infty} \\ &\leq \eta_1 |u_{xt}^k|_{L^2}^2 + C_{\eta_1} c_1^6 |u_x^k|_{L^2}^2, \\ I_3 &= - \int_0^1 \rho^k u_t^{k-1} u_x^k u_t^k dx \leq \int_0^1 \rho^k |u_t^{k-1}| |u_x^k| |u_t^k| dx \\ &\leq |\rho^k|_{L^\infty}^{1/2} |u_t^{k-1}|_{L^\infty} |u_x^k|_{L^2} |\sqrt{\rho^k} u_t^k|_{L^2} \\ &\leq c_0^{1/2} |u_{xt}^{k-1}|_{L^2} |u_x^k|_{L^2} |\sqrt{\rho^k} u_t^k|_{L^2} \\ &\leq c_1^{-1} |u_{xt}^{k-1}|_{L^2}^2 |\sqrt{\rho^k} u_t^k|_{L^2}^2 + \frac{1}{4} c_0 c_1^{-1} |u_x^k|_{L^2}^2, \end{aligned}$$

$$\begin{aligned}
I_4 &= - \int_0^1 \rho^k u^{k-1} u_{xt}^k u_t^k dx \leq \int_0^1 \rho^k |u^{k-1}| |u_t^k| |u_{xt}^k| dx \\
&\leq |\rho^k|_{L^\infty}^{1/2} |u^{k-1}|_{L^\infty} \left| \sqrt{\rho^k} u_t^k \right|_{L^2} |u_{xt}^k|_{L^2} \\
&\leq \eta_1 |u_{xt}^k|_{L^2}^2 + C_{\eta_1} c_1^3 \left| \sqrt{\rho^k} u_t^k \right|_{L^2}^2, \\
I_5 &= \int_0^1 |\psi_t^k| |u_{xt}^k| dx \\
&= \int_0^1 |\psi_1^k| |\theta_t^k| |u_{xt}^k| dx + \int_0^1 |\psi_2^k| |\rho_t^k| |u_{xt}^k| dx \\
&\leq C \left| \sqrt{\rho^k} \theta_t^k \right|_{L^2} |u_{xt}^k|_{L^2} + C |\rho_t^k|_{L^2} |u_{xt}^k|_{L^2} \\
&\leq 2\eta_1 |u_{xt}^k|_{L^2}^2 + C_{\eta_1} c_1^4 + C_{\eta_1} c_1^5.
\end{aligned} \tag{59}$$

Substituting I_j ($j = 1, 2, \dots, 5$) into (33), after choosing $\eta = \mu_2^{(q-2)/2}/10$

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2 + \frac{1}{2} \mu_2^{(q-2)/2} |u_{xt}^k|_{L^2}^2 \\
&\leq C c_1^3 \left| \sqrt{\rho^k} u_t^k \right|_{L^2}^2 + C c_1^6 |u_x^k|_{L^2}^2 \\
&\quad + c_1^{-1} |u_{xt}^{k-1}|_{L^2} \left| \sqrt{\rho^k} u_t^k \right|_{L^2} + \frac{1}{4} c_0 c_1^{-1} |u_x^k|_{L^2}^2 + C c_1^5.
\end{aligned} \tag{60}$$

We estimate $|\sqrt{\rho^k} u_t^k(\tau)|_{L^2}^2$ as follows: multiplying (20) by u_t^k and integrating it over $(0, 1)$, we obtain

$$\begin{aligned}
&\int_0^1 \rho^k |u_t^k|^2 dx \\
&\leq 2 \int_0^1 \left(\rho^k |u^{k-1}|^2 |u_x^k|^2 + (\rho^k)^{-1} |H_q^{\mu_2} u^k - \psi_x|^2 \right) dx.
\end{aligned} \tag{61}$$

According to the smoothness of (ρ^k, u^k, θ^k) , we have

$$\begin{aligned}
&\lim_{\tau \rightarrow 0} \int_0^1 \left(\rho^k |u^{k-1}|^2 |u_x^k|^2 + (\rho^k)^{-1} |L_\rho u^k + \psi_x|^2 \right) dx \\
&\leq |\rho_0|_{L^\infty} |u_0|_{L^\infty}^2 |u_{0x}|_{L^2}^2 + |g|_{L^2}^2 \leq C c_0^5,
\end{aligned} \tag{62}$$

where C is a positive constant.

In the same way,

$$|u_x^k|_{L^2}^2 \leq C |u_x^k(0)|_{L^2}^2 + C \int_0^t |u_{xt}^k|_{L^2}^2 ds. \tag{63}$$

Integrating (60) over $(0, t) \subset (0, T)$, combining with (62) and (63) we have

$$\begin{aligned}
&|u_x^k|_{L^2}^2 + \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2 + \int_0^t |u_{xt}^k|_{L^2}^2 \\
&\leq C \int_0^t \left(c_1^6 + c_1^{-1} |u_{xt}^{k-1}|_{L^2} \right) \\
&\quad \cdot \left(\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2 + |u_x^k|_{L^2}^2 \right) ds + C c_1^5 t + C c_0^5,
\end{aligned} \tag{64}$$

and for $0 < t < \min\{T_1, T_4\}$, $T_4 \leq c_0^5/c_1^5$, we get

$$\begin{aligned}
&|u_x^k|_{L^2}^2 + \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2 + \int_0^t |u_{xt}^k|_{L^2}^2 \\
&\leq C \int_0^t \left(c_1^6 + c_1^{-1} |u_{xt}^{k-1}|_{L^2} \right) \\
&\quad \cdot \left(\left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2 + |u_x^k|_{L^2}^2 \right) ds + C c_0^5,
\end{aligned} \tag{65}$$

and, by Gronwall's inequality, $\exists T_5 \leq T_4$ (T_5 has nothing to do with k), it follows that

$$|u_x^k|_{L^2}^2 + \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2 + \int_0^t |u_{xt}^k|_{L^2}^2(s) ds \leq C c_0^5 = c_1. \tag{66}$$

Secondly, we need to do the following estimates. By (20)

$$\begin{aligned}
&\left[(u_x^2 + \mu_2)^{(q-2)/2} u_x \right]_x \\
&= \rho^k u_t^k + \rho^k u^{k-1} u_x^{k-1} + \phi_1^{k-1} \theta_x^{k-1} + \phi_2^{k-1} \rho_x^k,
\end{aligned} \tag{67}$$

we have

$$\begin{aligned}
&\left| \left[(u_x^k)^2 + \mu_2 \right]^{(q-2)/2} u_x^k \right|_x \\
&= \left((u_x^k)^2 + \mu_2 \right)^{(q-4)/2} \left[(q-1) (u_x^k)^2 + \mu_2 \right] |u_{xx}^k| \\
&\geq \left((u_x^k)^2 + \mu_2 \right)^{(q-4)/2} \left[(u_x^k)^2 + \mu_2 \right] |u_{xx}^k| \\
&= \left((u_x^k)^2 + \mu_2 \right)^{(q-2)/2} |u_{xx}^k| \geq \mu_2^{(q-2)/2} |u_{xx}^k|,
\end{aligned} \tag{68}$$

and, taking it by L^2 -norm, using Lemma 3, we get

$$\begin{aligned}
&|u_{xx}^k|_{L^2} \leq \mu_2^{(2-q)/2} \left[|\rho^k u_t^k|_{L^2} + |\rho^k u^{k-1} u_x^{k-1}|_{L^2} + |\psi_1^k \theta_x^k|_{L^2} \right. \\
&\quad \left. + |\psi_2^k \rho_x^k|_{L^2} \right] \leq C \left[|\rho^k|_{L^\infty}^{1/2} \left| \sqrt{\rho^k} u_t^k \right|_{L^2} \right. \\
&\quad \left. + |\rho^k|_{L^\infty} |u^{k-1}|_{L^\infty} |u_x^k|_{L^q} + |\theta_x^k|_{L^2} + |\rho_x^k|_{L^2} \right] \\
&\leq C \left[|\rho_x^k|_{L^2}^{1/2} \left| \sqrt{\rho^k} u_t^k \right|_{L^2} + |\rho_x^k|_{L^2} |u_x^{k-1}|_{L^q} |u_x^k|_{L^q} \right. \\
&\quad \left. + |\theta_x^k|_{L^2} + |\rho_x^k|_{L^2} \right] \leq C c_1^3 = c_2 = C c_0^{15},
\end{aligned} \tag{69}$$

where C is a positive constant. For $T_6 = \min[T_1, T_2, T_3, T_4, T_5]$, we get

$$\begin{aligned}
& \sup_{0 \leq s \leq T_6} |u_x^k|_{L^2}(s) + \int_0^{T_6} |u_{xt}^k|_{L^2}^2(s) ds \leq c_1, \\
& \sup_{0 \leq s \leq T_6} |u_{xx}^k|_{L^2}(s) \leq c_2, \\
& \sup_{0 \leq s \leq T_6} |\theta_x^k|_{L^2}^2(s) + \left| \sqrt{\rho^k} \theta_t^k(t) \right|_{L^2}^2(s) + \int_0^{T_6} |\theta_{xt}^k|_{L^2}^2(s) ds \\
& \leq C c_1^5, \\
& \sup_{0 \leq s \leq T_6} |u_x^k|_{L^2}^2(s) + \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2(s) + \int_0^{T_6} |u_{xt}^k|_{L^2}^2(s) ds \quad (70) \\
& \leq c_1, \\
& \sup_{0 \leq s \leq T_6} |\theta_{xx}^k|_{L^2}(s) < C c_1^4 c_2, \\
& \sup_{0 \leq s \leq T_6} |\rho^k(t)|_{H^1}(s) \leq C c_0, \\
& \sup_{0 \leq s \leq T_6} |\rho_t^k(t)|_{L^2}(s) \leq C c_1^2.
\end{aligned}$$

3.2. Convergence. In this subsection, the approximate solutions (ρ^k, u^k, θ^k) strongly converging to the solution of (3)-(4) will be proved. To this end, we define

$$\begin{aligned}
\bar{\rho}^{k+1} &= \rho^{k+1} - \rho^k, \\
\bar{u}^{k+1} &= u^{k+1} - u^k, \\
\bar{\theta}^{k+1} &= \theta^{k+1} - \theta^k,
\end{aligned} \quad (71)$$

and then we can easily verify that $\bar{\rho}^{k+1}, \bar{u}^{k+1}, \bar{\theta}^{k+1}$ satisfy the system of equations

$$\bar{\rho}_t^{k+1} + (\bar{\rho}^{k+1} u^k)_x + (\rho^k \bar{u}^k)_x = 0, \quad (72)$$

$$\begin{aligned}
& \rho^{k+1} \varphi_1^{k+1} \theta_t^{k+1} + \rho^{k+1} \varphi_1^{k+1} u^k \theta_x^{k+1} - (\rho^{k+1})^2 \varphi_2^{k+1} u_x^k \\
& + H_p^{\mu_1} \theta^{k+1} + \psi^{k+1} u_x^k - (u_x^k)^2 - [\rho^k \varphi_1^k \theta_t^k \\
& + \rho^k \varphi_1^k u^{k-1} \theta_x^k - (\rho^k)^2 \varphi_2^k u_x^{k-1} + H_p^{\mu_1} \theta^k + \psi^k u_x^{k-1} \\
& - (u_x^{k-1})^2] = 0, \quad (x, t) \in \Omega_T,
\end{aligned} \quad (73)$$

$$\begin{aligned}
& \rho^{k+1} \bar{u}_t^{k+1} + \rho^{k+1} u^k \bar{u}_x^{k+1} + (H_q^{\mu_2} u^{k+1} - H_q^{\mu_2} u^k) \\
& + (\psi^{k+1})_x - (\psi^k)_x = \bar{\rho}^{k+1} (-u_t^k - u^{k-1} u_x^k) \\
& - \rho^{k+1} \bar{u}^k u_x^k,
\end{aligned} \quad (74)$$

with the initial and boundary conditions

$$\begin{aligned}
\bar{u}^{k+1} &= 0, \\
\bar{\theta}^{k+1} &= 0, \\
& \text{on } \partial\Omega \times (0, T), \\
\bar{\rho}^{k+1}(x, 0) &= 0, \\
\bar{u}^{k+1}(x, 0) &= 0, \\
\bar{\theta}^{k+1}(x, 0) &= 0, \\
& \text{in } \Omega.
\end{aligned} \quad (75)$$

Multiplying (72) by $\bar{\rho}^{k+1}$ and integrating it over $(0, 1)$, we obtain

$$\begin{aligned}
\frac{d}{dt} |\bar{\rho}^{k+1}|_{L^2}^2 &\leq C |\bar{\rho}^{k+1}|_{L^2}^2 |u_x^k|_{L^\infty} + |\rho^k|_{H^1} |\bar{u}_x^k|_{L^2} |\bar{\rho}^{k+1}|_{L^2} \\
&\leq B_\eta^k(t) |\bar{\rho}^{k+1}|_{L^2}^2 + \eta |\bar{u}_x^k|_{L^2}^2,
\end{aligned} \quad (76)$$

where $B_\eta^k(t) = C |u_{xx}^k|_{L^2} + C_\eta |\rho^k|_{H^1}^2$, for all $t < T_1$ and $k \geq 1$.

Multiplying (73) by $\bar{\theta}^{k+1}$ and integrating it over $(0, 1)$, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^{k+1} \varphi_1^{k+1} |\bar{\theta}^{k+1}|^2 dx \\
& + \int_0^1 (H_p^{\mu_1} \theta^{k+1} - H_p^{\mu_1} \theta^k) \bar{\theta}^{k+1} dx \\
& = \frac{1}{2} \int_0^1 [\rho^{k+1} \varphi_1^{k+1}]_t |\bar{\theta}^{k+1}|^2 dx \\
& - \int_0^1 \rho^{k+1} \varphi_1^{k+1} \theta_t^k \bar{\theta}^{k+1} dx + \int_0^1 \rho^k \varphi_1^k \theta_t^k \bar{\theta}^{k+1} dx \\
& - \int_0^1 [\rho^{k+1} \varphi_1^{k+1} u^k \theta_x^{k+1} - \rho^k \varphi_1^k u^{k-1} \theta_x^k] \bar{\theta}^{k+1} dx \\
& - \int_0^1 [\psi^{k+1} u_x^k - \psi^k u_x^{k-1}] \bar{\theta}^{k+1} dx \\
& + \int_0^1 [(u_x^k)^2 - (u_x^{k-1})^2] \bar{\theta}^{k+1} dx \\
& - \int_0^1 [(\rho^{k+1})^2 \varphi_2^{k+1} u_x^k - (\rho^k)^2 \varphi_2^k u_x^{k-1}] \bar{\theta}^{k+1} dx \\
& = \sum_{j=1}^6 I_j.
\end{aligned} \quad (77)$$

Using the Sobolev inequality, Young's inequality, and Lemma 3, we obtain

$$\begin{aligned}
 I_1 &= \frac{1}{2} \int_0^1 \left[\rho^{k+1} \varphi_1^{k+1} \right]_t \left| \bar{\theta}^{k+1} \right|^2 dx \\
 &= \frac{1}{2} \int_0^1 \rho_t^{k+1} \varphi_1^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \\
 &\quad + \frac{1}{2} \int_0^1 \rho^{k+1} \varphi_{11}^{k+1} \theta_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \\
 &\quad + \frac{1}{2} \int_0^1 \rho^{k+1} \varphi_{12}^{k+1} \rho_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx = \sum_{j=1}^3 I_{1j}.
 \end{aligned} \tag{78}$$

Here,

$$\begin{aligned}
 I_{11} &= \frac{1}{2} \int_0^1 \rho_t^{k+1} \varphi_1^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \\
 &\leq C \left| \int_0^1 \rho_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \right| \\
 &\leq C \left| \int_0^1 \rho^{k+1} u^k \bar{\theta}^{k+1} \bar{\theta}_x^{k+1} dx \right| \\
 &\leq C \left| \rho^{k+1} \right|_{L^\infty}^{1/2} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2} \left| u^k \right|_{L^\infty} \left| \bar{\theta}_x^{k+1} \right|_{L^2} \\
 &\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \rho_x^{k+1} \right|_{L^2} \left| u_x^k \right|_{L^2}^2 \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2, \\
 I_{12} &= \frac{1}{2} \int_0^1 \rho^{k+1} \varphi_{11}^{k+1} \theta_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \\
 &\leq C \left| \int_0^1 \rho^{k+1} \theta_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \right| \\
 &\leq C \left| \theta_t^{k+1} \right|_{L^\infty} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2 \\
 &\leq C \left| \theta_{xt}^{k+1} \right|_{L^2} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2, \\
 I_{13} &= \frac{1}{2} \int_0^1 \rho^{k+1} \varphi_{12}^{k+1} \rho_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \\
 &\leq C \left| \rho^{k+1} \right|_{L^\infty} \left| \int_0^1 \rho_t^{k+1} \left| \bar{\theta}^{k+1} \right|^2 dx \right| \\
 &\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \rho_x^{k+1} \right|_{L^2}^3 \left| u_x^k \right|_{L^2}^2 \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2.
 \end{aligned} \tag{79}$$

In the same way, we have

$$\begin{aligned}
 I_2 &= - \int_0^1 \rho^{k+1} \varphi_1^{k+1} \theta_t^k \bar{\theta}^{k+1} dx + \int_0^1 \rho^k \varphi_1^k \theta_t^k \bar{\theta}^{k+1} dx \\
 &= - \int_0^1 \bar{\rho}^{k+1} \varphi_1^{k+1} \theta_t^k \bar{\theta}^{k+1} dx + \int_0^1 \rho^k \bar{\varphi}_1^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \\
 &\leq C \left| \int_0^1 \bar{\rho}^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \right| + C \left| \int_0^1 \rho^k \bar{\rho}^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \right| \\
 &\quad + C \left| \int_0^1 \rho^k \bar{\theta}^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \right| = \sum_{j=1}^3 I_{2j}.
 \end{aligned} \tag{80}$$

Here,

$$\begin{aligned}
 I_{21} &= C \left| \int_0^1 \bar{\rho}^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \right| \leq C \left| \bar{\rho}^{k+1} \right|_{L^2} \left| \theta_{xt}^k \right|_{L^2} \left| \bar{\theta}_x^{k+1} \right|_{L^2} \\
 &\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \theta_{xt}^k \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2, \\
 I_{22} &= C \left| \int_0^1 \rho^k \bar{\rho}^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \right| \\
 &\leq C \left| \rho_x^k \right|_{L^2} \left| \bar{\rho}^{k+1} \right|_{L^2} \left| \theta_{xt}^k \right|_{L^2} \left| \bar{\theta}_x^{k+1} \right|_{L^2} \\
 &\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^k \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2, \\
 I_{23} &= C \left| \int_0^1 \rho^k \bar{\theta}^{k+1} \theta_t^k \bar{\theta}^{k+1} dx \right| \\
 &\leq C \left| \rho_x^k \right|_{L^2} \left| \theta_{xt}^k \right|_{L^2} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2.
 \end{aligned} \tag{81}$$

Following the same method, we obtain

$$\begin{aligned}
 I_3 &= - \int_0^1 \left[\rho^{k+1} \varphi_1^{k+1} u^k \theta_x^{k+1} - \rho^k \varphi_1^k u^{k-1} \theta_x^k \right] \bar{\theta}^{k+1} dx \\
 &= - \int_0^1 \bar{\rho}^{k+1} \varphi_1^{k+1} u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \\
 &\quad - \int_0^1 \rho^k \bar{\varphi}_1^{k+1} u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \\
 &\quad - \int_0^1 \rho^k \varphi_1^k u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \\
 &\quad - \int_0^1 \rho^k \varphi_1^k u^{k-1} \bar{\theta}_x^{k+1} \bar{\theta}^{k+1} dx = \sum_{j=1}^4 I_{3j}.
 \end{aligned} \tag{82}$$

Here,

$$\begin{aligned}
 I_{31} &= - \int_0^1 \bar{\rho}^{k+1} \varphi_1^{k+1} u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \\
 &\leq C \left| \bar{\rho}^{k+1} \right|_{L^2} \left| u_x^k \right|_{L^2} \left| \theta_{xt}^{k+1} \right|_{L^2} \left| \bar{\theta}_x^{k+1} \right|_{L^2} \\
 &\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| u_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2,
 \end{aligned}$$

$$\begin{aligned}
I_{32} &= - \int_0^1 \rho^k \bar{\varphi}_1^{k+1} u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \\
&\leq C \left| \int_0^1 \rho^k \bar{\rho}^{k+1} u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \right| \\
&\quad + C \left| \int_0^1 \rho^k \bar{\theta}^{k+1} u^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \right| \\
&\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| u_x^k \right|_{L^2}^2 \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \\
&\quad + C \left| u_x^k \right|_{L^2} \left| \rho_x^k \right|_{L^2} \left| \theta_{xt}^{k+1} \right|_{L^2} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2, \\
I_{33} &= \int_0^1 \rho^k \varphi_1^k \bar{u}^k \theta_x^{k+1} \bar{\theta}^{k+1} dx \\
&\leq C \left| \bar{u}_x^k \right|_{L^2} \left| \rho_x^k \right|_{L^2} \left| \theta_{xt}^{k+1} \right|_{L^2} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2} \\
&\leq C \left| \bar{u}_x^k \right|_{L^2}^2 + C \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2, \\
I_{34} &= \int_0^1 \rho^k \varphi_1^k u^{k-1} \bar{\theta}_x^{k+1} \bar{\theta}^{k+1} dx \\
&\leq C \left| \rho^k \right|_{L^\infty} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2} \left| u^{k-1} \right|_{L^\infty} \left| \bar{\theta}_x^{k+1} \right|_{L^2} \\
&\leq \eta \left| \bar{\theta}_x^k \right|_{L^2}^2 + C_\eta \left| \rho_x^k \right|_{L^2}^2 \left| u_x^{k-1} \right|_{L^2}^2 \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2.
\end{aligned} \tag{83}$$

In a similar way, we have

$$\begin{aligned}
I_4 &= - \int_0^1 \left[\psi^{k+1} u_x^k - \psi^k u_x^{k-1} \right] \bar{\theta}^{k+1} dx \\
&= - \int_0^1 \bar{\psi}^{k+1} u_x^k \bar{\theta}^{k+1} dx - \int_0^1 \psi^k \bar{u}_x^k \bar{\theta}^{k+1} dx = \sum_{j=1}^2 I_{4j}.
\end{aligned} \tag{84}$$

Here,

$$\begin{aligned}
I_{41} &= - \int_0^1 \bar{\psi}^{k+1} u_x^k \bar{\theta}^{k+1} dx \\
&\leq C \left| \int_0^1 \bar{\rho}^{k+1} u_x^k \bar{\theta}^{k+1} dx \right| + C \left| \int_0^1 \bar{\theta}^{k+1} u_x^k \bar{\theta}^{k+1} dx \right| \\
&\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| u_x^k \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \\
&\quad + C \left| u_x^k \right|_{L^2} \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2, \\
I_{42} &= - \int_0^1 \psi^k \bar{u}_x^k \bar{\theta}^{k+1} dx \leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \bar{u}_x^k \right|_{L^2}^2.
\end{aligned} \tag{85}$$

Similarly,

$$\begin{aligned}
I_5 &= \int_0^1 \left[(u_x^k)^2 - (u_x^{k-1})^2 \right] \bar{\theta}^{k+1} dx \\
&\leq \left| \bar{u}_x^k \right|_{L^2} \left| \bar{\theta}_x^{k+1} \right|_{L^2} \left(\left| u_x^k \right|_{L^2} + \left| u_x^{k-1} \right|_{L^2} \right) \\
&\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left(\left| u_x^k \right|_{L^2}^2 + \left| u_x^{k-1} \right|_{L^2}^2 \right) \left| \bar{u}_x^k \right|_{L^2}^2.
\end{aligned} \tag{86}$$

Using the same method, we have

$$\begin{aligned}
I_6 &= - \int_0^1 \left[(\rho^{k+1})^2 \varphi_2^{k+1} u_x^k - (\rho^k)^2 \varphi_2^k u_x^{k-1} \right] \bar{\theta}^{k+1} dx \\
&= - \int_0^1 \bar{\rho}^{k+1} (\rho^{k+1} + \rho^k) \varphi_2^{k+1} u_x^k \bar{\theta}^{k+1} dx \\
&\quad - \int_0^1 (\rho^k)^2 \bar{\varphi}_2^{k+1} u_x^k \bar{\theta}^{k+1} dx \\
&\quad - \int_0^1 (\rho^k)^2 \varphi_2^k \bar{u}_x^k \bar{\theta}^{k+1} dx = \sum_{j=1}^3 I_{6j}.
\end{aligned} \tag{87}$$

Here,

$$\begin{aligned}
I_{61} &= \int_0^1 \bar{\rho}^{k+1} (\rho^{k+1} + \rho^k) \varphi_2^{k+1} u_x^k \bar{\theta}^{k+1} dx \\
&\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 \\
&\quad + C_\eta \left(\left| \rho_x^{k+1} \right|_{L^2} + \left| \rho_x^k \right|_{L^2} \right) \left| u_x^k \right|_{L^2} \left| \bar{\rho}^{k+1} \right|_{L^2}^2, \\
I_{62} &= \int_0^1 (\rho^k)^2 \bar{\varphi}_2^{k+1} u_x^k \bar{\theta}^{k+1} dx \\
&= C \left| \int_0^1 (\rho^k)^2 \bar{\rho}^{k+1} u_x^k \bar{\theta}^{k+1} dx \right| \\
&\quad + C \left| \int_0^1 (\rho^k)^2 \bar{\theta}^{k+1} u_x^k \bar{\theta}^{k+1} dx \right| \\
&\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| u_x^k \right|_{L^2}^2 \left| \rho_x^k \right|_{L^2}^4 \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \\
&\quad + C \left| u_x^k \right|_{L^2} \left| \rho_x^k \right|_{L^2}^2 \left| \sqrt{\rho^{k+1} \bar{\theta}^{k+1}} \right|_{L^2}^2, \\
I_{63} &= \int_0^1 (\rho^k)^2 \varphi_2^k \bar{u}_x^k \bar{\theta}^{k+1} dx \\
&\leq \eta \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \rho_x^k \right|_{L^2}^4 \left| \bar{u}_x^k \right|_{L^2}^2.
\end{aligned} \tag{88}$$

Substituting I_j ($j = 1, 2, \dots, 6$) into (105), after choosing $\eta = \mu_1^{(p-2)/2}/26$, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 + \frac{1}{2} \mu_1^{(p-2)/2} \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 \leq \left| \bar{u}_x^k \right|_{L^2}^2 \\
& \cdot C \left(1 + \left| u_x^k \right|_{L^2}^2 + \left| u_x^{k-1} \right|_{L^2}^2 + \left| \rho_x^k \right|_{L^2}^4 \right) + \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \\
& \cdot C \left[\left| \theta_{xt}^k \right|_{L^2}^2 + \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^k \right|_{L^2}^2 + \left| u_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 \right. \\
& + \left| u_x^k \right|_{L^2}^2 \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 + \left| u_x^k \right|_{L^2}^2 \\
& + \left(\left| \rho_x^{k+1} \right|_{L^2} + \left| \rho_x^k \right|_{L^2} \right) \left| u_x^k \right|_{L^2} + \left| u_x^k \right|_{L^2}^2 \left| \rho_x^k \right|_{L^2}^4 \left. \right] \\
& + \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 \cdot C \left(\left| \rho_x^{k+1} \right|_{L^2} \left| u_x^k \right|_{L^2}^2 + \left| \theta_{xt}^{k+1} \right|_{L^2}^2 \right. \\
& + \left| \rho_x^{k+1} \right|_{L^2}^3 \left| u_x^k \right|_{L^2}^2 + \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^k \right|_{L^2}^2 + \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 \\
& + \left| \rho_x^k \right|_{L^2}^2 \left| \theta_{xt}^{k+1} \right|_{L^2}^2 + \left| \rho_x^k \right|_{L^2}^2 \left| u_x^{k-1} \right|_{L^2}^2 + \left| u_x^k \right|_{L^2}^2 \\
& + \left| u_x^k \right|_{L^2}^2 \left| \rho_x^k \right|_{L^2}^2 \left. \right). \tag{89}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
& \frac{\sqrt{\delta}}{2} \frac{d}{dt} \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 + \frac{1}{2} \mu_1^{(p-2)/2} \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 \\
& \leq \bar{C} \left| \bar{u}_x^k \right|_{L^2}^2 + D^k(t) \left(\left| \bar{\rho}^{k+1} \right|_{L^2}^2 + \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 \right). \tag{90}
\end{aligned}$$

There $D^k(t) \in L^1(0, T_1)$, $\int_0^1 D^k(t) \leq \bar{C} + \bar{C}t$.

Multiplying (74) by $\bar{u}^{k+1}$ and integrating it over $(0, 1)$, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^{k+1} \left| \bar{u}^{k+1} \right|^2 dx \\
& + \int_0^1 \left(H_q^{\mu_2} u^{k+1} - H_q^{\mu_2} u^k \right) \bar{u}^{k+1} dx \\
& = - \int_0^1 \left(\psi_x^{k+1} - \psi_x^k \right) \bar{u}^{k+1} dx \\
& + \int_0^1 \bar{\rho}^{k+1} \left(-u_t^k - u^{k-1} u_x^k \right) \bar{u}^{k+1} dx \\
& - \int_0^1 \rho^{k+1} \bar{u}^k u_x^k \bar{u}^{k+1} dx. \tag{91}
\end{aligned}$$

We use the Sobolev inequality, Young's inequality, and Lemma 3, and we obtain

$$\begin{aligned}
I_1 &= - \int_0^1 \left(\psi_x^{k+1} - \psi_x^k \right) \bar{u}^{k+1} dx = - \int_0^1 \bar{\psi}^{k+1} \bar{u}^{k+1} dx \\
&\leq C \left| \int_0^1 \bar{\rho}^{k+1} \bar{u}^{k+1} dx \right| + C \left| \int_0^1 \bar{\theta}^{k+1} \bar{u}^{k+1} dx \right| \\
&\leq \eta \left| \bar{u}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| \bar{\rho}^{k+1} \right|_{L^2}^2 + \eta \left| \bar{u}_x^{k+1} \right|_{L^2}^2 \\
&\quad + C_\eta \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2, \\
I_2 &= - \int_0^1 \bar{\rho}^{k+1} u_t^k \bar{u}^{k+1} dx \leq \left| \bar{\rho}^{k+1} \right|_{L^2} \left| u_t^k \right|_{L^\infty} \left| \bar{u}_x^{k+1} \right|_{L^2} \\
&\leq \eta \left| \bar{u}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| u_{xt}^k \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2, \\
I_3 &= - \int_0^1 \bar{\rho}^{k+1} u^{k-1} u_x^k \bar{u}^{k+1} dx \\
&\leq \left| \bar{\rho}^{k+1} \right|_{L^2} \left| u^{k-1} \right|_{L^\infty} \left| u_x^k \right|_{L^2} \left| \bar{u}^{k+1} \right|_{L^\infty} \\
&\leq \left| \bar{\rho}^{k+1} \right|_{L^2} \left| u_x^{k-1} \right|_{L^2} \left| u_x^k \right|_{L^2} \left| \bar{u}_x^{k+1} \right|_{L^2} \\
&\leq \eta \left| \bar{u}_x^{k+1} \right|_{L^2}^2 + C_\eta \left| u_x^{k-1} \right|_{L^2}^2 \left| u_x^k \right|_{L^2}^2 \left| \bar{\rho}^{k+1} \right|_{L^2}^2, \\
I_4 &= - \int_0^1 \rho^{k+1} \bar{u}^k u_x^k \bar{u}^{k+1} dx \\
&\leq C \left| \rho^{k+1} \right|_{L^\infty}^{1/2} \left| \sqrt{\rho^{k+1}} \bar{u}^{k+1} \right|_{L^2} \left| u_x^k \right|_{L^2} \left| \bar{u}_x^k \right|_{L^2} \\
&\leq \varepsilon \left| \bar{u}_x^k \right|_{L^2}^2 + C_\varepsilon \left| \rho_x^{k+1} \right|_{L^2} \left| u_x^k \right|_{L^2}^2 \left| \sqrt{\rho^{k+1}} \bar{u}^{k+1} \right|_{L^2}^2. \tag{92}
\end{aligned}$$

Substituting I_j ($j = 1, 2, \dots, 4$) into (91), after choosing $\eta = \mu_2^{(q-2)/2}/8$,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^{k+1} \left| \bar{u}^{k+1} \right|^2 dx + \frac{1}{2} \mu_2^{(q-2)/2} \left| \bar{u}_x^{k+1} \right|_{L^2}^2 \\
& \leq \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \cdot C \left(1 + \left| u_{xt}^k \right|_{L^2}^2 + \left| u_x^{k-1} \right|_{L^2}^2 \left| u_x^k \right|_{L^2}^2 \right) \\
& \quad + \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 \cdot C_\varepsilon \left| \bar{u}_x^k \right|_{L^2}^2 \\
& \quad + C_\varepsilon \left| \rho_x^{k+1} \right|_{L^2} \left| u_x^k \right|_{L^2}^2 \left| \sqrt{\rho^{k+1}} \bar{u}^{k+1} \right|_{L^2}^2. \tag{93}
\end{aligned}$$

Thus, we get

$$\begin{aligned}
& \frac{d}{dt} \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \\
& \leq C \left| \bar{\rho}^{k+1} \right|_{L^2}^2 \left| u_x^k \right|_{L^\infty} + \left| \rho^k \right|_{H^1} \left| \bar{u}_x^k \right|_{L^2} \left| \bar{\rho}^{k+1} \right|_{L^2} \\
& \leq B_\eta^k(t) \left| \bar{\rho}^{k+1} \right|_{L^2}^2 + \eta \left| \bar{u}_x^k \right|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 B_\eta^k(t) \leq \widetilde{C}t, \\
& \frac{1}{2} \frac{d}{dt} \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 + \frac{1}{2} \mu_1^{(p-2)/2} \int_0^1 \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 dx \\
& \leq \widetilde{C} \left| \bar{u}_x^k \right|_{L^2}^2 + D^k(t) \left(\left| \bar{\rho}^{k+1} \right|_{L^2}^2 + \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 \right), \\
& \int_0^1 D^k(t) \leq \widetilde{C} + \widetilde{C}t, \\
& \frac{1}{2} \frac{d}{dt} \int_0^1 \rho^{k+1} \left| \bar{u}^{k+1} \right|^2 dx + \frac{1}{2} \mu_2^{(q-2)/2} \left| \bar{u}_x^{k+1} \right|_{L^2}^2 \\
& \leq E_\varepsilon^k(t) \left(\left| \bar{\rho}^{k+1} \right|_{L^2}^2 + \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 + \left| \sqrt{\rho^{k+1}} \bar{u}^{k+1} \right|_{L^2}^2 \right) \\
& \quad + \varepsilon \left| \bar{u}_x^k \right|_{L^2}^2.
\end{aligned} \tag{94}$$

There $E_\varepsilon^k(t) \in L^1(0, T_1)$, $\int_0^1 E_\varepsilon^k(t) \leq \widetilde{C} + \widetilde{C}_\varepsilon t$. Using (94), denote

$$\begin{aligned}
\Psi^{k+1} &= \left| \bar{\rho}^{k+1} \right|_{L^2}^2 + \frac{\varepsilon}{2} \left| \sqrt{\rho^{k+1}} \bar{\theta}^{k+1} \right|_{L^2}^2 \\
& \quad + \frac{1}{2} \left| \sqrt{\rho^{k+1}} \bar{u}^{k+1} \right|_{L^2}^2.
\end{aligned} \tag{95}$$

There $F_\varepsilon^k(t) \in L^1(0, T_1)$, $\int_0^1 F_\varepsilon^k(t) \leq \widetilde{C} + \widetilde{C}_\varepsilon t$.

Thus, we get

$$\begin{aligned}
& \frac{d}{dt} \Psi^{k+1}(t) + \frac{\varepsilon}{2\sqrt{\delta}} \mu_1^{(p-2)/2} \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2 \\
& \quad + \frac{1}{2} \mu_2^{(q-2)/2} \left| \bar{u}_x^{k+1} \right|_{L^2}^2 \\
& \leq F_\varepsilon^k(t) \cdot \Psi^{k+1}(t) + \widetilde{C} \cdot \varepsilon \cdot \left| \bar{u}_x^k \right|_{L^2}^2.
\end{aligned} \tag{96}$$

By Gronwall's inequality, noting that $\Psi^{k+1}(t) = 0$, we get

$$\begin{aligned}
& \Psi^{k+1}(t) + \frac{\varepsilon}{2\sqrt{\delta}} \mu_1^{(p-2)/2} \int_0^t \left| \bar{\theta}_x^{k+1} \right|_{L^2}^2(s) ds \\
& \quad + \frac{1}{2} \mu_2^{(q-2)/2} \int_0^t \left| \bar{u}_x^{k+1} \right|_{L^2}^2(s) ds \leq \exp \left\{ \int_0^t F_\varepsilon^k(t) dt \right\} \\
& \quad \cdot \int_0^t \widetilde{C} \cdot \varepsilon \cdot \left| \bar{u}_x^k \right|_{L^2}^2(s) ds \leq \exp \left\{ \widetilde{C} + \widetilde{C}_\varepsilon t \right\} \\
& \quad \cdot \int_0^t \widetilde{C} \cdot \varepsilon \cdot \left| \bar{u}_x^k \right|_{L^2}^2(s) ds.
\end{aligned} \tag{97}$$

We can obtain

$$\begin{aligned}
& \widetilde{C} \cdot \varepsilon \cdot \exp \left\{ \widetilde{C} \right\} = \frac{1}{4}, \\
& \exp \left\{ \widetilde{C}_\varepsilon \cdot T_7 \right\} = \mu_2^{(q-2)/2},
\end{aligned} \tag{98}$$

and thus we can get

$$\begin{aligned}
& \sum_{k=1}^K \Psi^{k+1}(t) + \sum_{k=1}^K \int_0^{T^*} \left(\left| \bar{\theta}_x^{k+1}(s) \right|_{L^2}^2 + \left| \bar{u}_x^{k+1}(s) \right|_{L^2}^2 \right) ds \\
& \leq \widetilde{C} < +\infty.
\end{aligned} \tag{99}$$

There, $T^* = \min\{T_6, T_7\}$.

Therefore, we conclude that the full sequence (ρ^k, u^k, θ^k) converges to a limit (ρ, u, θ) in the following strong sense:

$$\begin{aligned}
\rho^k &\longrightarrow \rho \quad \text{in } L^\infty(0, T^*; L^2), \\
u^k &\longrightarrow u \quad \text{in } L^\infty(0, T^*; L^2) \cap L^2(0, T^*; H_0^1), \\
\theta^k &\longrightarrow \theta \quad \text{in } L^\infty(0, T^*; L^2) \cap L^2(0, T^*; H_0^1),
\end{aligned} \tag{100}$$

for $T^* = \min[T_1, T_2, T_3, T_4, T_5, T_6, T_7]$, and we get

$$\begin{aligned}
& \sup_{0 \leq s \leq T^*} \left| u_x \right|_{L^2}(s) + \int_0^{T^*} \left| u_{xt} \right|_{L^2}^2(s) ds \leq c_1, \\
& \sup_{0 \leq s \leq T^*} \left| u_{xx} \right|_{L^2}(s) \leq c_2, \\
& \sup_{0 \leq s \leq T^*} \left| \theta_x \right|_{L^2}^2(s) + \left| \sqrt{\rho^k} \theta_t^k(t) \right|_{L^2}^2(s) \\
& \quad + \int_0^{T^*} \left| \theta_{xt} \right|_{L^2}^2(s) ds \leq C c_1^5, \\
& \sup_{0 \leq s \leq T^*} \left| u_x \right|_{L^2}^2(s) + \left| \sqrt{\rho^k} u_t^k(t) \right|_{L^2}^2(s) \\
& \quad + \int_0^{T^*} \left| u_{xt} \right|_{L^2}^2(s) ds \leq c_1, \\
& \sup_{0 \leq s \leq T^*} \left| \theta_{xx} \right|_{L^2}(s) < C c_1^4 c_2, \\
& \sup_{0 \leq s \leq T^*} \left| \rho(t) \right|_{H^1}(s) \leq C c_0, \\
& \sup_{0 \leq s \leq T^*} \left| \rho_t(t) \right|_{L^2}(s) \leq C c_1^2.
\end{aligned} \tag{101}$$

4. Proof of the Main Theorem

4.1. Existence ($k \rightarrow \infty$). In this part, we need to prove that (ρ, u, θ) is a solution of problem

$$\begin{aligned}
\rho_t + (\rho u)_x &= 0, \quad (x, t) \in \Omega_T, \\
(\rho e)_t + (\rho u e)_x - \left[(\theta_x^2 + \mu)^{(p-2)/2} \theta_x \right]_x + \pi u_x \\
&= \left| u_x \right|^2, \quad (x, t) \in \Omega_T,
\end{aligned}$$

$$\begin{aligned}
(\rho u)_t + (\rho u^2)_x - \left[(u_x^2 + \mu_2)^{(q-2)/2} u_x \right]_x + \pi_x &= 0, \\
(x, t) &\in \Omega_T, \\
\pi &= \varphi(\theta, \rho), \quad (x, t) \in \Omega_T, \\
e &= \psi(\theta, \rho), \quad (x, t) \in \Omega_T,
\end{aligned} \tag{102}$$

with the initial and boundary conditions

$$\begin{aligned}
\rho|_{t=0} &= \rho_0 \geq 0, \\
u|_{t=0} &= u_0, \\
\theta|_{t=0} &= \theta_0, \\
u(0, t) &= u(1, t) = 0, \\
\theta(0, t) &= \theta(1, t) = 0
\end{aligned} \tag{103}$$

and also satisfy the integral identity after the $k \rightarrow \infty$ process. Finally, we need to show that the solution satisfies the regular properties in Definition 1. Since the abovementioned can be obtained by the same method as in [15], we omit it for brevity.

4.2. Uniqueness. Let (ρ^1, u^1, θ^1) , (ρ^2, u^2, θ^2) be two solutions of problem (3)-(4). After substituting into (7), respectively, we define

$$\begin{aligned}
\bar{\rho} &= \rho^1 - \rho^2, \\
\bar{u} &= u^2 - u^1, \\
\bar{\theta} &= \theta^1 - \theta^2,
\end{aligned} \tag{104}$$

and then we can obtain

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^1 \rho^1 \varphi_1^1 |\bar{\theta}|^2 dx + \int_0^1 (H_p^{\mu_1} \theta^1 - H_p^{\mu_2} \theta^2) \bar{\theta} dx \\
&= \frac{1}{2} \int_0^1 (\rho^1 \varphi_1^1)_t |\bar{\theta}|^2 dx - \int_0^1 \rho^1 \varphi_1^1 \theta_t^2 \bar{\theta} dx \\
&\quad + \int_0^1 \rho^2 \varphi_1^2 \theta_t^2 \bar{\theta} dx \\
&\quad - \int_0^1 [\rho^1 \varphi_1^1 u^1 \theta_x^1 - \rho^2 \varphi_1^2 u^2 \theta_x^2] \bar{\theta} dx \\
&\quad - \int_0^1 [\psi^1 u_x^1 - \psi^2 u_x^2] \bar{\theta} dx \\
&\quad + \int_0^1 [(u_x^1)^2 - (u_x^2)^2] \bar{\theta} dx \\
&\quad + \int_0^1 [(\rho^1)^2 \varphi_2^1 u_x^1 - (\rho^2)^2 \varphi_2^2 u_x^2] \bar{\theta} dx = \sum_{j=1}^6 I_j,
\end{aligned} \tag{105}$$

since

$$\begin{aligned}
&\int_0^t \int_0^1 (|\theta_x^1|^{p-2} \theta_x^1 - |\theta_x^2|^{p-2} \theta_x^2) \bar{\theta}_x dx \\
&\geq \mu_1^{(p-2)/2} \int_0^t \|\bar{\theta}_x\|_{L^2}^2 ds.
\end{aligned} \tag{106}$$

Using the Sobolev inequality, Young's inequality, and Lemma 3, we obtain

$$\begin{aligned}
I_1 &= \frac{1}{2} \int_0^1 [\rho^1 \varphi_1^1]_t |\bar{\theta}|^2 dx \\
&= \frac{1}{2} \int_0^1 \rho_t^1 \varphi_1^1 |\bar{\theta}|^2 dx + \frac{1}{2} \int_0^1 \rho^1 \varphi_{11}^1 \theta_t^1 |\bar{\theta}|^2 dx \\
&\quad + \frac{1}{2} \int_0^1 \rho^1 \varphi_{12}^1 \rho_t^1 |\bar{\theta}|^2 dx = \sum_{j=1}^3 I_{1j}.
\end{aligned} \tag{107}$$

Here,

$$\begin{aligned}
I_{11} &= \frac{1}{2} \int_0^1 \rho_t^1 \varphi_1^1 |\bar{\theta}|^2 dx \leq C \left| \int_0^1 \rho_t^1 |\bar{\theta}|^2 dx \right| \\
&= C \left| \int_0^1 \rho^1 u^1 \bar{\theta} \bar{\theta}_x dx \right| \\
&\leq C |\rho^1|_{L^\infty}^{1/2} \left| \sqrt{\rho^1} \bar{\theta}^1 \right|_{L^2} |u^1|_{L^\infty} |\bar{\theta}_x|_{L^2} \\
&\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\rho_x^1|_{L^2} |u_x^1|_{L^2}^2 \left| \sqrt{\rho^1} \bar{\theta}^1 \right|_{L^2}^2, \\
I_{12} &= \frac{1}{2} \int_0^1 \rho^1 \varphi_{11}^1 \theta_t^1 |\bar{\theta}|^2 dx \leq C \left| \int_0^1 \rho^1 \theta_t^1 |\bar{\theta}|^2 dx \right| \\
&\leq C |\theta_t^1|_{L^\infty} \left| \sqrt{\rho^1} \bar{\theta}^1 \right|_{L^2}^2 \leq C |\theta_{xt}^1|_{L^2} \left| \sqrt{\rho^1} \bar{\theta}^1 \right|_{L^2}^2, \\
I_{13} &= \frac{1}{2} \int_0^1 \rho^1 \varphi_{12}^1 \rho_t^1 |\bar{\theta}|^2 dx \\
&\leq C |\rho^1|_{L^\infty} \left| \int_0^1 \rho_t^1 |\bar{\theta}|^2 dx \right| \\
&\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\rho_x^1|_{L^2}^3 |u_x^1|_{L^2}^2 \left| \sqrt{\rho^1} \bar{\theta}^1 \right|_{L^2}^2.
\end{aligned} \tag{108}$$

In the same way,

$$\begin{aligned}
I_2 &= - \int_0^1 \rho^1 \varphi_1^1 \theta_t^2 \bar{\theta} dx + \int_0^1 \rho^2 \varphi_1^2 \theta_t^2 \bar{\theta} dx \\
&= - \int_0^1 \bar{\rho} \varphi_1^1 \theta_t^2 \bar{\theta} dx + \int_0^1 \rho^2 \bar{\varphi}_1^1 \theta_t^2 \bar{\theta} dx \\
&\leq C \left| \int_0^1 \bar{\rho}^1 \theta_t^2 \bar{\theta} dx \right| + C \left| \int_0^1 \rho^2 \bar{\rho} \theta_t^2 \bar{\theta} dx \right| \\
&\quad + C \left| \int_0^1 \rho^2 \bar{\theta} \theta_t^2 \bar{\theta} dx \right| = \sum_{j=1}^3 I_{2j}.
\end{aligned} \tag{109}$$

Here,

$$\begin{aligned}
 I_{21} &= C \left| \int_0^1 \bar{\rho}^1 \theta_t^2 \bar{\theta}^1 dx \right| \leq C |\bar{\rho}|_{L^2} |\theta_{xt}^2|_{L^2} |\bar{\theta}_x|_{L^2} \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\theta_{xt}^2|_{L^2}^2 |\bar{\rho}|_{L^2}^2, \\
 I_{22} &= C \left| \int_0^1 \rho^2 \bar{\rho}^1 \theta_t^2 \bar{\theta}^1 dx \right| \\
 &\leq C |\rho_x^2|_{L^2} |\bar{\rho}|_{L^2} |\theta_{xt}^2|_{L^2} |\bar{\theta}_x|_{L^2} \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\rho_x^2|_{L^2}^2 |\theta_{xt}^2|_{L^2}^2 |\bar{\rho}|_{L^2}^2, \\
 I_{23} &= C \left| \int_0^1 \rho^k \bar{\theta} \theta_t^2 \bar{\theta} dx \right| \leq C |\rho_x^2|_{L^2} |\theta_{xt}^2|_{L^2} \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2.
 \end{aligned} \tag{110}$$

By the same token,

$$\begin{aligned}
 I_3 &= - \int_0^1 [\rho^1 \varphi_1^1 u^1 \theta_x^1 - \rho^2 \varphi_1^2 u^2 \theta_x^2] \bar{\theta} dx \\
 &= - \int_0^1 \bar{\rho} \varphi_1^1 u^1 \theta_x^1 \bar{\theta} dx - \int_0^1 \rho^2 \bar{\varphi}_1 u^1 \theta_x^1 \bar{\theta} dx \\
 &\quad - \int_0^1 \rho^2 \varphi_1^2 \bar{u} \theta_x^2 \bar{\theta} dx - \int_0^1 \rho^2 \varphi_1^2 u^2 \bar{\theta}_x \bar{\theta} dx = \sum_{j=1}^4 I_{3j}.
 \end{aligned} \tag{111}$$

Here,

$$\begin{aligned}
 I_{31} &= - \int_0^1 \bar{\rho} \varphi_1^1 u^1 \theta_x^1 \bar{\theta} dx \leq C |\bar{\rho}|_{L^2} |u_x^1|_{L^2} |\theta_{xt}^1|_{L^2} |\bar{\theta}_x|_{L^2} \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |u_x^1|_{L^2}^2 |\theta_{xt}^1|_{L^2}^2 |\bar{\rho}|_{L^2}^2, \\
 I_{32} &= - \int_0^1 \rho^2 \bar{\varphi}_1 u^1 \theta_x^1 \bar{\theta} dx \\
 &\leq C \left| \int_0^1 \rho^2 \bar{\rho} u^1 \theta_x^1 \bar{\theta} dx \right| + C \left| \int_0^1 \rho^2 \bar{\theta} u^1 \theta_x^1 \bar{\theta} dx \right| \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |u_x^1|_{L^2}^2 |\rho_x^2|_{L^2}^2 |\theta_{xt}^1|_{L^2}^2 |\bar{\rho}|_{L^2}^2 \\
 &\quad + C |u_x^1|_{L^2} |\rho_x^2|_{L^2} |\theta_{xt}^1|_{L^2} \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2, \\
 I_{33} &= \int_0^1 \rho^2 \varphi_1^2 \bar{u} \theta_x^2 \bar{\theta} dx \\
 &\leq C |\bar{u}_x|_{L^2} |\rho_x^2|_{L^2} |\theta_{xt}^2|_{L^2} \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2} \\
 &\leq C |\bar{u}_x|_{L^2}^2 + C |\rho_x^2|_{L^2}^2 |\theta_{xt}^2|_{L^2}^2 \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2,
 \end{aligned}$$

$$\begin{aligned}
 I_{34} &= \int_0^1 \rho^2 \varphi_1^2 u^2 \bar{\theta}_x \bar{\theta} dx \left(\frac{\rho^1}{\rho^1} \right) \\
 &\leq C |\rho^2|_{L^\infty} \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2} |u^2|_{L^\infty} |\bar{\theta}_x|_{L^2} \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\rho_x^2|_{L^2}^2 |u_x^2|_{L^2}^2 \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2.
 \end{aligned} \tag{112}$$

In the similar way, we have

$$\begin{aligned}
 I_4 &= - \int_0^1 [\psi^1 u_x^1 - \psi^2 u_x^2] \bar{\theta} dx \\
 &= - \int_0^1 \bar{\psi} u_x^1 \bar{\theta} dx - \int_0^1 \psi^2 \bar{u} \bar{\theta} dx = \sum_{j=1}^2 I_{4j}.
 \end{aligned} \tag{113}$$

Here,

$$\begin{aligned}
 I_{41} &= - \int_0^1 \bar{\psi}^1 u_x^1 \bar{\theta} dx \\
 &\leq C \left| \int_0^1 \bar{\rho}^1 u_x^1 \bar{\theta}^1 dx \right| + C \left| \int_0^1 \bar{\theta} u_x^1 \bar{\theta} dx \right| \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |u_x^1|_{L^2}^2 |\bar{\rho}|_{L^2}^2 + C |u_x^1|_{L^2} \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2,
 \end{aligned} \tag{114}$$

$$I_{42} = - \int_0^1 \psi^2 \bar{u} \bar{\theta}^1 dx \leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\bar{u}_x|_{L^2}^2.$$

Similarly,

$$\begin{aligned}
 I_5 &= \int_0^1 [(u_x^1)^2 - (u_x^2)^2] \bar{\theta} dx \\
 &\leq |\bar{u}_x|_{L^2} |\bar{\theta}_x|_{L^2} (|u_x^1|_{L^2} + |u_x^2|_{L^2}) \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta (|u_x^1|_{L^2}^2 + |u_x^2|_{L^2}^2) |\bar{u}_x|_{L^2}^2.
 \end{aligned} \tag{115}$$

Using the same method, we obtain

$$\begin{aligned}
 I_6 &= \int_0^1 [(\rho^1)^2 \varphi_2^1 u_x^1 - (\rho^2)^2 \varphi_2^2 u_x^2] \bar{\theta} dx \\
 &= - \int_0^1 \bar{\rho} (\rho^1 + \rho^2) \varphi_2^1 u_x^1 \bar{\theta} dx - \int_0^1 (\rho^2)^2 \bar{\varphi}_2 u_x^2 \bar{\theta} dx \\
 &\quad - \int_0^1 (\rho^2)^2 \varphi_2^2 \bar{u} \bar{\theta} dx = \sum_{j=1}^3 I_{6j}.
 \end{aligned} \tag{116}$$

Here,

$$\begin{aligned}
 I_{61} &= \int_0^1 \bar{\rho} (\rho^2 + \rho^1) \varphi_2^1 u_x^k \bar{\theta} dx \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta (|\rho_x^1|_{L^2}^2 + |\rho_x^2|_{L^2}^2) |u_x^1|_{L^2}^2 |\bar{\rho}|_{L^2}^2, \\
 I_{62} &= \int_0^1 (\rho^2)^2 \bar{\varphi}_2 u_x^1 \bar{\theta} dx \\
 &= C \left| \int_0^1 (\rho^2)^2 \bar{\rho} u_x^1 \bar{\theta} dx \right| + C \left| \int_0^1 (\rho^2)^2 \bar{\theta} u_x^1 \bar{\theta} dx \right| \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |u_x^1|_{L^2}^2 |\rho_x^2|_{L^2}^4 |\bar{\rho}|_{L^2}^2 \\
 &\quad + C |u_x^1|_{L^2}^2 |\rho_x^2|_{L^2}^2 \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2, \\
 I_{63} &= \int_0^1 (\rho^2)^2 \varphi_2^2 \bar{u}_x \bar{\theta}^{k+1} dx \\
 &\leq \eta |\bar{\theta}_x|_{L^2}^2 + C_\eta |\rho_x^2|_{L^2}^4 |\bar{u}_x|_{L^2}^2.
 \end{aligned} \tag{117}$$

Thus, we have

$$\begin{aligned}
 &\frac{\sqrt{\delta}}{2} \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2 + \frac{\mu_1^{(p-2)/2}}{2} \int_0^t |\bar{\theta}_x|_{L^2}^2 ds \\
 &\leq \int_0^t E(t) \left(|\bar{\rho}|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{u} \right|_{L^2}^2 \right) ds \\
 &\quad + \bar{C} \int_0^t |\bar{u}_x^k|_{L^2}^2 ds.
 \end{aligned} \tag{118}$$

There $E(t) \in L^1(0, T^*)$, $\int_0^1 E(t) \leq \bar{C} + \bar{C}t$.

In the same way,

$$\begin{aligned}
 &\frac{1}{2} \left| \sqrt{\rho^1} \bar{u} \right|_{L^2}^2 + \frac{\mu_2^{(p-2)/2}}{2} \int_0^t |\bar{u}_x|_{L^2}^2 ds \\
 &\leq \int_0^t F(t) \left(|\bar{\rho}|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{u} \right|_{L^2}^2 \right) ds.
 \end{aligned} \tag{119}$$

There $F(t) \in L^1(0, T^*)$, $\int_0^1 F(t) \leq \bar{C} + \bar{C}t$. Consider

$$|\bar{\rho}^1|_{L^2}^2 \leq \int_0^t B(t) (|\bar{\rho}|_{L^2}^2) ds + \eta \int_0^t |\bar{u}_x^k|_{L^2}^2 ds. \tag{120}$$

There $B(t) \in L^1(0, T^*)$, $\int_0^1 B(t) \leq \bar{C}t$.

With (119) multiplied by η , combining with (118) and (120),

$$(\bar{C} + 1) \eta = \frac{\mu_2^{(p-2)/2}}{4}. \tag{121}$$

We can get

$$\begin{aligned}
 &|\bar{\rho}|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{u} \right|_{L^2}^2 + \int_0^t |\bar{\theta}_x|_{L^2}^2 ds \\
 &\quad + \int_0^t |\bar{u}_x|_{L^2}^2 ds \\
 &\leq \int_0^t G(t) \left(|\bar{\rho}|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{\theta} \right|_{L^2}^2 + \left| \sqrt{\rho^1} \bar{u} \right|_{L^2}^2 \right) ds.
 \end{aligned} \tag{122}$$

There $G(t) \in L^1(0, T^*)$, $\int_0^1 G(t) \leq \bar{C}t + \bar{C}$.

By Gronwall's inequality, it yields

$$\begin{aligned}
 u^1 &= u^2, \\
 \rho^1 &= \rho^2, \\
 \theta^1 &= \theta^2.
 \end{aligned} \tag{123}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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