Research Article

The Intersection Probability of Brownian Motion and SLE_κ

Shizhong Zhou^{1,2} and Shiyi Lan²

¹Department of Statistics and Financial Mathematics, School of Science, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China

²School of Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, China

Correspondence should be addressed to Shizhong Zhou; zhoushizhong666@sina.com

Received 29 May 2015; Accepted 4 August 2015

Academic Editor: Andrei D. Mironov

Copyright © 2015 S. Zhou and S. Lan. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By using excursion measure Poisson kernel method, we obtain a second-order differential equation of the intersection probability of Brownian motion and SLE_κ . Moreover, we find a transformation such that the second-order differential equation transforms into a hypergeometric differential equation. Then, by solving the hypergeometric differential equation, we obtain the explicit formula of the intersection probability for the trace of the chordal SLE_κ and planar Brownian motion started from distinct points in an upper half-plane $\bar{H}$.

1. Introduction

Shortly after Schramm [1] in 2000 introduced the percolation exploration path and SLE curve, Lawler et al. [2–5] proved in succession the intersection exponents of Brownian motion in the physics literature: half-plane exponents, plane exponents, and two-side exponents. That is, they proved that the intersection exponents of the two independent planar Brownian motions started from distinct points. Naturally, one can ask what the intersection exponents (or probability) of Brownian motion and SLE_κ are in the half-plane. Not only is this question significant itself, but it also can probe more into the property of SLE_κ . Kozdron [6, 7] derived the intersection probability of Brownian motion and SLE_2 . The aim of this paper is to derive the intersection probability of Brownian motion and SLE_κ in a half-plane.

Chordal SLE_κ . The chordal Schramm-Loewner evolution SLE_κ [8, 9] are growth process defined via conformal maps [10, 11] which are solutions of the ordinary differential equation as follows:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad (1)$$

where $g_0(z) = z$ and W_t is a one-dimensional Brownian motion. For any fixed $T > 0$, $z_0 \in \bar{H}$, and all z in

a neighborhood of z_0 , either there exists a solution of (1) for $t \in [0, T]$ or, for some $t_0 \in (0, T]$, there exists a solution for $t \in [0, t_0]$ such that $\lim_{t \nearrow t_0} g_t(z) = W_{t_0}$. Let Ω_T denote the (open) set of $z \in \bar{H}$ such that the first holds and K_T the set of $z \in \bar{H}$ such that the second is true. We call the process $(g_t, t \geq 0)$ the chordal Schramm-Loewner evolution process with parameter κ (abbreviated as SLE_κ). It is easy to see that the hulls of chordal SLE_κ are the hulls of a continuous path:

$$\gamma(t) := \lim_{z \rightarrow 0} g_t^{-1}(z + \sqrt{\kappa}W_t), \quad (2)$$

which is called the trace of the SLE_κ process.

Brownian Excursions Poisson Kernel. For a simply connected domain $D \subset \mathbb{C}$ and when ∂D is locally analytic at points x and y , the Brownian excursion Poisson kernel is defined as the normal derivative of the usual Poisson kernel. Since Brownian excursion measure is the scaling limit of simple random walk excursion measure, the excursion Poisson kernel is also the mass of the Brownian excursion measure. More precisely speaking, the excursion Poisson kernel is defined as

$$H_{\partial D}(x, y) := \lim_{\varepsilon \rightarrow 0} \frac{H_D(x + \varepsilon n_x, y)}{\varepsilon}, \quad (3)$$

where n_x is the unit normal at x pointing into the domain D and $H_D(z, y)$ is the usual Poisson kernel. Brownian excursion

Poisson kernel satisfied the conformal covariance property as follows.

Lemma 1 ([7, Proposition 2.11]). *Suppose $f : D \rightarrow \bar{D}$ is a conformal transformation where $\bar{D} \subset \mathbb{C}$ is a simply connected Jordan domain and $\partial\bar{D}$ is locally analytic at points $f(x)$ and $f(y)$; then*

$$H_{\partial\bar{D}}(x, y) = |f'(x)| |f'(y)| H_{\partial\bar{D}}(f(x), f(y)). \quad (4)$$

Particularly, when $D = U$ (the unit disc),

$$\begin{aligned} H_{\partial U}(x, y) &= \frac{1}{\pi |y - x|^2} \\ &= \frac{1}{2\pi (1 - \cos(\arg y - \arg x))}. \end{aligned} \quad (5)$$

And when $D = H$ (the upper half-plane),

$$H_{\partial U}(x, y) = \frac{1}{\pi (y - x)^2}. \quad (6)$$

2. Intersection Probability

In this section, our main goal is now to compute the intersection probability of SLE_κ and Brownian motion started from distinct points in a half-plane H . In order to determine the probability, our strategy for establishing this result will be as follows. By using the methods of the excursion measure Poisson kernel, we will first determine an explicit differential equation for $P\{\gamma[0, \infty) \cap \beta[0, t_\beta] = \emptyset\}$, which is a hypergeometric differential equation. Then, we obtain the general solution of the differential equation.

2.1. The Explicit Differential Equation of Intersection Probability. We note that if W_t is a standard one-dimensional Brownian motion, $-W_t$ is also a standard one-dimensional Brownian motion. Hence, $g_t(z)$ satisfies the ordinary differential equation as follows:

$$\partial_t g_t(z) = \frac{2/\kappa}{g_t(z) + W_t}, \quad g_0(z) = 0. \quad (7)$$

Letting $X_t = g_t(x) + W_t$ and $Y_t = g_t(y) + W_t$, we have the following.

Theorem 2. *The intersection probability for the trace of the chordal SLE_κ and planar Brownian motion started from distinct points in a half-plane H satisfies*

$$\begin{aligned} P\{\gamma[0, \infty) \cap \beta[0, t_\beta] = \emptyset\} \\ = E\left[\exp\left\{-\int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s}\right)^2 ds\right\}\right]. \end{aligned} \quad (8)$$

Proof. Suppose that H_t is the slit-plane $H_t = H \setminus \gamma(0, t]$ for any $0 < t < \infty$. This implies that

$$P\{\gamma[0, t] \cap \beta[0, t_\beta] = \emptyset\} = E\left[\frac{H_{\partial H_t}(x, y)}{H_{\partial H}(x, y)}\right]. \quad (9)$$

Letting $t \rightarrow \infty$, this yields

$$\begin{aligned} P\{\gamma[0, \infty) \cap \beta[0, t_\beta] = \emptyset\} &= \lim_{t \rightarrow \infty} E\left[\frac{H_{\partial H_t}(x, y)}{H_{\partial H}(x, y)}\right] \\ &= E\left[\lim_{t \rightarrow \infty} \frac{H_{\partial H_t}(x, y)}{H_{\partial H}(x, y)}\right]. \end{aligned} \quad (10)$$

By using expression (7) and conformal covariance, we can obtain

$$\begin{aligned} H_{\partial H_t}(x, y) &= g'_t(x) g'_t(y) H_{\partial H}(g_t(x), g_t(y)), \\ \frac{H_{\partial H_t}(x, y)}{H_{\partial H}(x, y)} &= \frac{g'_t(x) g'_t(y) H_{\partial H}(g_t(x), g_t(y))}{H_{\partial H}(x, y)} \\ &= g'_t(x) g'_t(y) \left(\frac{x - y}{g_t(x) - g_t(y)}\right)^2. \end{aligned} \quad (11)$$

We define as

$$J_t = g'_t(x) g'_t(y) \left(\frac{x - y}{g_t(x) - g_t(y)}\right)^2, \quad (12)$$

$$J_\infty = \lim_{t \rightarrow \infty} J_t.$$

Hence, it yields

$$\begin{aligned} P\{\gamma[0, \infty) \cap \beta[0, t_\beta] = \emptyset\} &= \lim_{t \rightarrow \infty} E[J_t] \\ &= E\left[\lim_{t \rightarrow \infty} J_t\right] = E[J_\infty]. \end{aligned} \quad (13)$$

Recalling $X_t = g_t(x) + W_t$ and $Y_t = g_t(y) + W_t$, we have

$$\begin{aligned} dX_t &= \frac{2}{\kappa X_t} dt + dW_t, \\ dY_t &= \frac{2}{\kappa Y_t} dt + dW_t. \end{aligned} \quad (14)$$

We also note that

$$\begin{aligned} \frac{\partial}{\partial t} [\log g'_t(x)] &= \frac{\partial_t \partial_x g_t(x)}{g'_t(x)} = \frac{\partial_x \partial_t g_t(x)}{g'_t(x)} \\ &= \frac{\partial_x (2/\kappa X_t)}{g'_t(x)} = -\frac{2}{\kappa (X_t)^2}, \\ \frac{\partial}{\partial t} [\log g'_t(y)] &= -\frac{2}{\kappa (Y_t)^2}. \end{aligned} \quad (15)$$

Then, we can obtain

$$\begin{aligned} \frac{\partial}{\partial t} [\log g_t(x) - \log g_t(y)] &= \frac{(\partial/\partial t) [g_t(x) - g_t(y)]}{g_t(x) - g_t(y)} \\ &= \frac{(\partial/\partial t) g_t(x) - (\partial/\partial t) g_t(y)}{(g_t(x) + W_t) - (g_t(y) + W_t)} = \frac{2/\kappa X_t - 2/\kappa Y_t}{X_t - Y_t} \\ &= -\frac{2}{\kappa X_t Y_t}. \end{aligned} \quad (16)$$

Because of

$$J_0 = g'_0(x) g'_0(y) \left(\frac{x-y}{g_0(x) - g_0(y)} \right)^2 = 1, \quad (17)$$

we finally obtain

$$\begin{aligned} J_t &= J_0 \exp \left[\int_0^t \frac{\partial}{\partial s} (\log J_s) ds \right] \\ &= \exp \left[- \int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 ds \right]. \end{aligned} \quad (18)$$

$$\begin{aligned} P \{ \gamma [0, \infty) \cap \beta [0, t_\beta] = \emptyset \} \\ = \begin{cases} \frac{\Gamma(4/\kappa) \Gamma(4/\kappa + \hat{b}_\kappa)}{\Gamma(1 + \hat{b}_\kappa) \Gamma(8/\kappa - 1)} \left(\frac{x}{y} \right)^{1/2 - 2/\kappa + \hat{b}_\kappa/2} {}_2F_1 \left(1 - \frac{4}{\kappa}, 1 - \frac{4}{\kappa} + \hat{b}_\kappa, 1 + \hat{b}_\kappa; \frac{x}{y} \right), & 0 < \kappa < 8, \\ 1, & \kappa \geq 8, \end{cases} \end{aligned} \quad (20)$$

where $\hat{b}_\kappa = \sqrt{9\kappa^2 - 8\kappa + 16}/\kappa$.

Proof. In [8], the authors obtained the following deep theorem.

In the range $\kappa \in (0, 4]$, there exists a random simple curve $\gamma : [0, \infty) \rightarrow \bar{H}$ with $\gamma(0) = 0$ and $\gamma(0, \infty) \subset H$. In the range $\kappa \in (4, 8)$, there exists a random simple curve $\gamma : [0, \infty) \rightarrow \bar{H}$. These curves have double points and they hit $\mathbb{R}$, but they never cross themselves. In the range $\kappa \in [8, \infty)$, there exists a random simple curve $\gamma : [0, \infty) \rightarrow \bar{H}$. These curves are the space-filling curves, but they never cross themselves.

When $\kappa \in [8, \infty)$, these random curves are the space-filling curves. It is easy to obtain that

$$P \{ \gamma [0, \infty) \cap \beta [0, t_\beta] = \emptyset \} = 1. \quad (21)$$

In order to prove Theorem 2, therefore, it is sufficient that we only need to prove the theorem when $\kappa \in (0, 8)$.

Recalling

$$\begin{aligned} \tilde{H}^*(x, y) &= P \{ \gamma [0, \infty) \cap \beta [0, t_\beta] = \emptyset \} \\ &= E \left[\exp \left\{ - \int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 ds \right\} \right] = J_t, \end{aligned} \quad (22)$$

$$J_t = \frac{g'_t(x) g'_t(y)}{(g_t(y) - g_t(x))^2},$$

Hence, we have

$$\begin{aligned} P \{ \gamma [0, \infty) \cap \beta [0, t_\beta] = \emptyset \} \\ = E \left[\exp \left\{ - \int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 ds \right\} \right]. \end{aligned} \quad (19)$$

This completes the proof of the theorem. $\square$

2.2. Intersection Probability

Theorem 3. *The intersection probability for the trace of the chordal SLE_κ and planar Brownian motion started from distinct points in a half-plane H satisfies*

by using Markov property, it is easy to see that $J_t \tilde{H}^*(X_t, Y_t)$ is a martingale. That is, M_t is a martingale if we define $M_t := E[J_\infty | F_t]$. Hence, we have

$$\begin{aligned} M_t &= E[J_\infty | F_t] \\ &= E \left[\exp \left\{ - \int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 ds \right\} \mid F_t \right] \\ &= \exp \left\{ - \int_0^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 ds \right\} \\ &\quad \cdot E \left[\exp \left\{ - \int_t^\infty \left(\frac{1}{X_s} - \frac{1}{Y_s} \right)^2 ds \right\} \mid F_t \right] \\ &= J_t \tilde{H}^*(X_t, Y_t). \end{aligned} \quad (23)$$

By using Itô formula [12] at $t = 0$, this implies that

$$\begin{aligned} \frac{\partial^2 \tilde{H}^*}{\partial x \partial y} + \frac{2}{\kappa x} \frac{\partial^2 \tilde{H}^*}{\partial x^2} + \frac{2}{\kappa y} \frac{\partial^2 \tilde{H}^*}{\partial y^2} + \frac{1}{x} \frac{\partial \tilde{H}^*}{\partial x} + \frac{1}{y} \frac{\partial \tilde{H}^*}{\partial y} \\ - \left(\frac{1}{x} - \frac{1}{y} \right)^2 \tilde{H}^* = 0. \end{aligned} \quad (24)$$

Because the intersection probability in our question only depends on the ratio x/y , we define the transformation

$\tilde{H}^*(x, y) = f(x/y)$, where f is some second-order derivative function. Thus, we have

$$\begin{aligned} & \left(\frac{1}{2y^2} + \frac{x^2}{y^4} - \frac{x}{y^3} \right) f'' \left(\frac{x}{y} \right) \\ & + \left(\frac{2}{\kappa xy} - \frac{2}{\kappa xy^3} + \frac{x}{y^3} - \frac{1}{y^2} \right) f' \left(\frac{x}{y} \right) \\ & - \left(\frac{1}{x} - \frac{1}{y} \right)^2 f \left(\frac{x}{y} \right) = 0. \end{aligned} \quad (25)$$

Multiplying by $2y^2$ and letting $u = x/y$, this yields

$$\begin{aligned} & (1 + u^2 - 2u) f''(u) + 2 \left(\frac{2}{\kappa u} - \frac{2u}{\kappa} + u - 1 \right) f'(u) \\ & - 2(u - 1)^2 f(u) = 0. \end{aligned} \quad (26)$$

Multiplying by u^2 and combining terms, we obtain

$$\begin{aligned} & u^2 (1 - u)^2 f''(u) \\ & + 2u (1 - u) \left[\frac{2}{\kappa} + \left(\frac{2}{\kappa} - 1 \right) u \right] f'(u) \\ & - 2(1 - u)^2 f(u) = 0. \end{aligned} \quad (27)$$

Or equivalently (because $0 < u < 1$)

$$\begin{aligned} & u^2 (1 - u) f''(u) + 2u \left[\frac{2}{\kappa} + \left(\frac{2}{\kappa} - 1 \right) u \right] f'(u) \\ & - 2(1 - u) f(u) = 0. \end{aligned} \quad (28)$$

It is easy to know that the second-order differential equation (28) has regular singular points at 0, 1, and ∞ . Hence, it is possible to transform it into a hypergeometric differential equation [13] by writing (28) as

$$f''(u) - 2 \left(\frac{1 - 4/\kappa}{1 - u} - \frac{2/\kappa}{u} \right) f'(u) - \frac{2}{u^2} f(u) = 0. \quad (29)$$

In order to solve (29), our main goal is to find a transformation such that it is a hypergeometric differential equation. We find that the transformation as follows just is our expectation:

$$f(u) = u^\alpha \phi(u), \quad (30)$$

where $\alpha = 1/2 - 2/\kappa + \sqrt{9\kappa^2 - 8\kappa + 16}/2\kappa$.

By (30), we have

$$\begin{aligned} & f'(u) = \alpha u^{\alpha-1} \phi(u) + u^\alpha \phi'(u), \\ & f''(u) = \alpha(\alpha - 1) u^{\alpha-2} \phi(u) + 2\alpha u^{\alpha-1} \phi'(u) \\ & + u^\alpha \phi''(u). \end{aligned} \quad (31)$$

By substituting (30) and (31) into (29), we finally obtain

$$\begin{aligned} & u(1 - u) \phi''(u) + \left[\left(1 + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa} \right) \right. \\ & \left. - \left(3 - \frac{8}{\kappa} + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa} \right) u \right] \phi'(u) + \left(\frac{4}{\kappa} \right. \\ & \left. - 1 \right) \left(1 - \frac{4}{\kappa} + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa} \right) \phi(u) = 0. \end{aligned} \quad (32)$$

It is easy to see that (32) is now a well-known hypergeometric differential equation whose general solution is given by

$$\begin{aligned} \phi(u) = & C_1 {}_2F_1 \left(1 - \frac{4}{\kappa}, 1 - \frac{4}{\kappa} + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}, 1 \right. \\ & \left. + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}; u \right) + C_2 u^{-\sqrt{9\kappa^2 - 8\kappa + 16}/\kappa} {}_2F_1 \left(1 \right. \\ & \left. - \frac{4}{\kappa}, 1 - \frac{4}{\kappa} - \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}, 1 \right. \\ & \left. - \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}; u \right). \end{aligned} \quad (33)$$

Hence, we finally obtain that the general solution of (28) is

$$\begin{aligned} f(u) = & C_1 u^\alpha {}_2F_1 \left(1 - \frac{4}{\kappa}, 1 - \frac{4}{\kappa} + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}, 1 \right. \\ & \left. + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}; u \right) + C_2 u^{\alpha - \sqrt{9\kappa^2 - 8\kappa + 16}/\kappa} {}_2F_1 \left(1 \right. \\ & \left. - \frac{4}{\kappa}, 1 - \frac{4}{\kappa} - \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}, 1 \right. \\ & \left. - \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}; u \right). \end{aligned} \quad (34)$$

Furthermore, physical considerations imply that $f(u) \rightarrow 0$ as $u \rightarrow 0^+$ (or $y \rightarrow \infty$) and $f(u) \rightarrow 1$ as $u \rightarrow 1^-$ (or $y \rightarrow x$). This yields

$$\begin{aligned} & C_2 = 0, \\ & C_1 = \frac{\Gamma(4/\kappa) \Gamma\left(\left(4 + \sqrt{9\kappa^2 - 8\kappa + 16}\right)/\kappa\right)}{\Gamma\left(1 + \sqrt{9\kappa^2 - 8\kappa + 16}/\kappa\right) \Gamma(8/\kappa - 1)}. \end{aligned} \quad (35)$$

Thus,

$$\begin{aligned} f(u) = & \frac{\Gamma(4/\kappa) \Gamma\left(\left(4 + \sqrt{9\kappa^2 - 8\kappa + 16}\right)/\kappa\right)}{\Gamma\left(1 + \sqrt{9\kappa^2 - 8\kappa + 16}/\kappa\right) \Gamma(8/\kappa - 1)} \\ & \cdot u^\alpha {}_2F_1 \left(1 \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{4}{\kappa}, 1 - \frac{4}{\kappa} + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}, 1 \\
& + \frac{\sqrt{9\kappa^2 - 8\kappa + 16}}{\kappa}; u \Big).
\end{aligned} \tag{36}$$

This completes the proof of Theorem 3. $\square$

$$\begin{aligned}
& P \{ \gamma [0, \infty) \cap \beta [0, t_\beta] = \emptyset \} \\
& = \begin{cases} \frac{\Gamma(4/\kappa) \Gamma(4/\kappa + \widehat{b}_\kappa)}{\Gamma(1 + \widehat{b}_\kappa) \Gamma(8/\kappa - 1)} \phi(\omega_2)^{1/2-2/\kappa+\widehat{b}_\kappa/2} {}_2F_1 \left(1 - \frac{4}{\kappa}, 1 - \frac{4}{\kappa} + \widehat{b}_\kappa, 1 + \widehat{b}_\kappa; \phi(\omega_2) \right), & 0 < \kappa < 8, \\ 1, & \kappa \geq 8, \end{cases}
\end{aligned} \tag{37}$$

where $\widehat{b}_\kappa = \sqrt{9\kappa^2 - 8\kappa + 16}/\kappa$ and $\phi : D \rightarrow H$ is the conformal transformation with $\phi(\omega_1) = 0$, $\phi(\omega_3) = 1$, and $\phi(\omega_4) = \infty$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11161004) and the Natural Science Foundation of Guangxi (no. 2013GXNSFAA019015).

References

- [1] O. Schramm, "Scaling limits of loop-erased random walks and uniform spanning trees," *Israel Journal of Mathematics*, vol. 118, pp. 221–288, 2000.
- [2] G. F. Lawler, O. Schramm, and W. Werner, "One-arm exponent for critical 2D percolation," *Electronic Journal of Probability*, vol. 7, no. 2, pp. 1–13, 2002.
- [3] G. F. Lawler, O. Schramm, and W. Werner, "Values of Brownian intersection exponents, I: half-plane exponents," *Acta Mathematica*, vol. 187, no. 2, pp. 237–273, 2001.
- [4] G. F. Lawler, O. Schramm, and W. Werner, "Values of Brownian intersection exponents. II. Plane exponents," *Acta Mathematica*, vol. 187, no. 2, pp. 275–308, 2001.
- [5] G. F. Lawler, O. Schramm, and W. Werner, "Values of Brownian intersection exponents III: two-sided exponents," *Annales de l'Institut Henri Poincaré*, vol. 38, no. 1, pp. 109–123, 2002.
- [6] M. J. Kozdron, "On the scaling limit of simple random walk excursion measure in the plane," *ALEA—Latin American Journal of Probability and Mathematical Statistics*, vol. 2, pp. 125–155, 2006.
- [7] M. J. Kozdron, "The scaling limit of Fomin's identity for two paths in the plane," *J. C. R. Math. Rep. Acad. SCI. Canada*, vol. 29, no. 3, pp. 65–80, 2007.

By using the knowledge of the conformal transformation, For any bounded, simply connected planar domain, we have a similar theorem.

Theorem 4. Let $D \subset \mathbb{C}$ be a bounded, simply connected domain and let $\omega_1, \omega_2, \omega_3$, and ω_4 be four points ordered counterclockwise around ∂D . The intersection probability for the trace of the chordal SLE_κ from ω_1 to ω_4 in D and planar Brownian motion started from ω_2 to ω_3 in D satisfies

- [8] G. F. Lawler, *Conformally Invariant Processes in the Plane*, vol. 114 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2005.
- [9] S. Rohde and O. Schramm, "Basic properties of SLE," *Annals of Mathematics*, vol. 161, no. 2, pp. 883–924, 2005.
- [10] R. Langlands, P. Pouliot, and Y. Saint-Aubin, "Conformal invariance in two-dimensional percolation," *Bulletin American Mathematical Society: New Series*, vol. 30, no. 1, pp. 1–61, 1994.
- [11] C. Pommerenke, *Boundary Behaviour of Conformal Maps*, vol. 299 of *Grundlehren der Mathematischen Wissenschaften*, Springer, New York, NY, USA, 1992.
- [12] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Springer, 2004.
- [13] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Function with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, DC, USA, 1972.