Research Article

Existence Results for the Periodic Thomas-Fermi-Dirac-von Weizsäcker Equations

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Received 18 October 2014; Accepted 23 December 2014

Academic Editor: Hagen Neidhardt

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We consider the Thomas-Fermi-Dirac-von Weizsäcker equation $-\Delta u + V(x)u + (u^2 \star (1/|x|))u = \lambda|u|^{p-2}u - |u|^{q-2}u$, $u \in H^1(\mathbb{R}^3)$, where $\lambda > 0$ is a parameter, $2 < p < q < 6$, $V \in L^\infty(\mathbb{R}^3)$ is 1-periodic in x_j , for $j = 1, 2, 3$, and 0 is in a spectral gap of the operator $-\Delta + V$. Using a new infinite-dimensional linking theorem, we prove that, for sufficiently small $\lambda > 0$, this equation has a nontrivial solution.

1. Introduction and Statement of Results

In this paper, we consider the following equation:

$$\begin{aligned} -\Delta u + V(x)u + \left(u^2 \star \frac{1}{|x|}\right)u \\ = \lambda|u|^{p-2}u - |u|^{q-2}u, \quad u \in H^1(\mathbb{R}^3), \end{aligned} \quad (1)$$

where $2 < p < q < 6$, $\lambda > 0$ is a parameter,

$$u^2 \star \frac{1}{|x|} = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy, \quad (2)$$

and $H^1(\mathbb{R}^3)$ is the standard Sobolev space with norm $\|u\|_{H^1(\mathbb{R}^3)} = (\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx)^{1/2}$.

The potential function $V \in L^\infty(\mathbb{R}^N)$ is 1-periodic in x_j for $j = 1, 2, 3$. Under this assumption, $\sigma(L)$, the spectrum of the operator

$$L = -\Delta + V: L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3) \quad (3)$$

is a purely continuous spectrum that is bounded below and consists of closed disjoint intervals ([1, Theorem XIII.100]). Thus, the complement $\mathbb{R} \setminus \sigma(L)$ consists of open intervals called spectral gaps. We assume the following:

(v) $V \in L^\infty(\mathbb{R}^N)$ is 1-periodic in x_j for $j = 1, 2, 3$ and 0 is in a spectral gap $(-\alpha, \beta)$ of $-\Delta + V$, where $0 < \alpha, \beta < +\infty$.

A solution u of (1) is called nontrivial if $u \neq 0$. Our main result is as follows.

Theorem 1. Suppose that $2 < p < q < 6$ and (v) is satisfied. Then, there exists $\lambda_0 > 0$ such that, for any $0 < \lambda < \lambda_0$, the problem (1) has a nontrivial solution.

Equation (1) arises in the study of the Thomas-Fermi-Dirac-von Weizsäcker (TFDW) model for atoms and molecules with no external potential, where $\rho = u^2$ is the electron density. In [2, section VIII], Lieb studied the existence and symmetric and asymptotic properties of solutions to

$$\begin{aligned} -A\Delta\psi + (V + \alpha)\psi + (\psi^2 \star |x|^{-1})\psi + \gamma\psi^{2p-1} - C_e\psi^{5/3} \\ = -\mu\psi \quad \text{in } \mathbb{R}^3 \end{aligned} \quad (4)$$

for various choices of V . Here, A, α, γ, C_e , and μ are constants and $p > 4/3$. The term $-C_e\psi^{5/3}$ is called the Dirac term

with strength C_c . In [3], Le Bris considered the minimizing problem

$$\inf_{\psi \in H^1(\mathbb{R}^3), \int_{\mathbb{R}^3} \psi^2 dx = \lambda} E(\psi) \quad (5)$$

with

$$\begin{aligned} E(\psi) = & \int_{\mathbb{R}^3} |\nabla \psi|^2 dx + \int_{\mathbb{R}^3} V(x) \psi^2 dx \\ & + c_1 \int_{\mathbb{R}^3} \psi^{10/3} dx - c_2 \int_{\mathbb{R}^3} \psi^{8/3} dx \\ & + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\psi^2(x) \psi^2(y)}{|x-y|} dx dy, \end{aligned} \quad (6)$$

where c_1 and c_2 are positive constant, and V is the Coulomb potential created by the atomic nuclei. Le Bris proved that there exists a $\lambda_c > 0$, such that, for $0 < \lambda < \lambda_c$, the problem admits a minimizer. Moreover, if c_2 is small enough, then the minimizer is unique. Using the Lagrange multiplier, it is easy to see that the minimizer of (5) is a solution to

$$\begin{aligned} -\Delta \psi + V(x) \psi + (\psi^2 * |x|^{-1}) \psi + c_1 \psi^{7/3} - c_2 \psi^{5/3} \\ = \mu \psi \quad \text{in } \mathbb{R}^3 \end{aligned} \quad (7)$$

for some μ . However, in a recent paper [4] by Lu and Otto, the authors proved that for sufficiently large λ , the variational problem (5) with $V = 0$ does not have a minimizer. Equation (1) with V a periodic function is used to describe a Hartree model for crystals (see [5, 6]). Moreover, when $\lambda = 0$, (1) is often referred to the Thomas-Fermi-von Weizsäcker model in the literature. One can see [7] and references therein. Finally, we should mention that the so-called Schrödinger-Poisson-Slater equation

$$-\Delta u + V(x) u + \left(u^2 * \frac{1}{|x|} \right) u = \lambda |u|^{p-2} u, \quad \text{in } \mathbb{R}^3 \quad (8)$$

is also related to (1) and has attracted much attention in recent years (see [5, 7–22]).

The variational functional for (1) is given by

$$\begin{aligned} J_\lambda(u) = & \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx \\ & + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \\ & - \frac{\lambda}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{1}{q} \int_{\mathbb{R}^3} |u|^q dx, \quad u \in H^1(\mathbb{R}^3). \end{aligned} \quad (9)$$

In other words, the critical points of J_λ are solutions to (1). Hence, it is natural to use critical point theory to obtain solutions to (1). Under assumption (v), the quadratic form $\int_{\mathbb{R}^3} (|\nabla u|^2 + V u^2) dx$ has infinite-dimensional positive and negative spaces. It can be shown that for sufficiently small $\lambda > 0$, $-J_\lambda$ has a global infinite-dimensional linking geometry

(for its definition, see (6.4) of Willem [23]). However, the difficulty rises when the classical infinite-dimensional linking theorem (see [23]) is used to obtain a critical point to J_λ . This theorem requires the functional to satisfy some upper semicontinuous assumption (see (6.3) of [23]). However, because the nonlinearity

$$\frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy + \frac{1}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} |u|^p dx \quad (10)$$

is neither positive definite nor negative definite in $H^1(\mathbb{R}^3)$ whenever $\lambda > 0$, $-J_\lambda$ does not satisfy the upper semicontinuous assumption. To overcome this difficulty, we use a new infinite-dimensional linking theorem in [24] to obtain a $(\bar{C})_c$ sequence (see Definition 6) for $-J_\lambda$. We can prove that a $(\bar{C})_c$ sequence for $-J_\lambda$ is bounded in $H^1(\mathbb{R}^3)$ if $\lambda > 0$ is sufficiently small. This result is new and original. Finally, through the concentration-compactness principle and the $(\bar{C})_c$ sequence, a nontrivial solution to (1) is obtained. Our method can be used to study more general equation like

$$-\Delta u + V(x) u + \left(u^2 * \frac{1}{|x|} \right) u = f(x, u), \quad u \in H^1(\mathbb{R}^3). \quad (11)$$

Notation. $B_r(a)$ denotes an open ball of radius r and center a . For a Banach space X , we denote the dual space of X by X' and denote strong and weak convergence in X by $\rightarrow$ and $\rightharpoonup$, respectively. For $\varphi \in C^1(X; \mathbb{R})$, we denote the Fréchet derivative of φ at u by $\varphi'(u)$. The Gateaux derivative of φ is denoted by $\langle \varphi'(u), v \rangle$, for all $u, v \in X$. $L^q(\mathbb{R}^3)$ and $L^q_{\text{loc}}(\mathbb{R}^3)$ denote the standard L^q space and the locally q -integrable function space, respectively ($1 \leq q \leq \infty$). Let Ω be a domain in $\mathbb{R}^N$ ($N \geq 1$). $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support in Ω . We use $O(h)$ and $o(h)$ to mean $|O(h)| \leq C|h|$ and $o(h)/|h| \rightarrow 0$, respectively.

2. Variational Setting for (1)

Let $\mathcal{D}^{1,2}(\mathbb{R}^3)$ be the Hilbert space

$$\mathcal{D}^{1,2}(\mathbb{R}^3) = \left\{ u \in L^6(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\nabla u|^2 dx < \infty \right\} \quad (12)$$

with inner product

$$(u, v)_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \nabla u \nabla v dx. \quad (13)$$

For $u \in H^1(\mathbb{R}^3)$, by the Lax-Milgram theorem, the equation

$$-\Delta \phi = 4\pi u^2 \quad \text{in } \mathbb{R}^3 \quad (14)$$

has a unique solution $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ (see Proposition 2.2 of [10]). And, by Theorem 2.2.1 of [25], ϕ_u can be expressed by

$$\phi_u(x) = \left(u^2 * \frac{1}{|y|} \right)(x) = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy, \quad x \in \mathbb{R}^3. \quad (15)$$

The function ϕ_u has the following properties.

Lemma 2. (i) *There exists a positive constant C such that, for any $u \in H^1(\mathbb{R}^3)$,*

$$\|\phi_u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \leq C \|u\|_{H^1(\mathbb{R}^3)}^2, \quad (16)$$

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx &= 4\pi \int_{\mathbb{R}^3} \phi_u u^2 dx \\ &= 4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \\ &\leq C \|u\|_{L^{12/5}(\mathbb{R}^3)}^4 \leq C \|u\|_{H^1(\mathbb{R}^3)}^4. \end{aligned} \quad (17)$$

(ii) *For any $u \in H^1(\mathbb{R}^3)$, $\phi_u \geq 0$ in $\mathbb{R}^3$.*

Proof. The inequality

$$4\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \leq C \|u\|_{L^{12/5}(\mathbb{R}^3)}^4 \quad (18)$$

can be derived from the the Hardy-Littlewood-Sobolev inequality (see (1.1) of [26]) by choosing $n = 3$, $\lambda = 1$, $p = t = 6/5$, and $f = g = u^2$ there. For a proof of the other properties of ϕ_u in this lemma, one can see [8, 10, 12]. $\square$

By this lemma, the functional J_λ in (9) is well defined in $H^1(\mathbb{R}^3)$. And a direct computation shows that the derivative of J_λ is

$$\begin{aligned} \langle J'_\lambda(u), v \rangle &= \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x) uv) dx \\ &\quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x) v(x) u^2(y)}{|x-y|} dx dy \\ &\quad - \lambda \int_{\mathbb{R}^3} |u|^{p-2} uv dx + \int_{\mathbb{R}^3} |u|^{q-2} uv dx, \\ &\quad \forall u, v \in H^1(\mathbb{R}^3). \end{aligned} \quad (19)$$

It is easy to verify that J_λ is a C^1 functional in $H^1(\mathbb{R}^3)$. Moreover, we have the following.

Lemma 3. *The following statements are equivalent:*

- (i) $u \in H^1(\mathbb{R}^3)$ is a solution of (1);
- (ii) u is a critical point of J_λ .

Under the assumption (v), there is a standard variational setting for the quadratic form $\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx$. One can see section 6.4 of [23]. But for the convenience of the reader, we state it here.

Let L be the operator defined by (3). We denote by $|L|^{1/2}$ the square root of the absolute value of L . The domain of $|L|^{1/2}$ is the space

$$X := H^1(\mathbb{R}^3). \quad (20)$$

On X , we choose the inner product $(u, v) = \int_{\mathbb{R}^3} |L|^{1/2} u \cdot |L|^{1/2} v dx$ and the corresponding norm $\|u\| = \sqrt{(u, u)}$. Since

0 lies in a gap of the essential spectrum of L , there exists an orthogonal decomposition $X = Y \oplus Z$ such that Y and Z are the positive and negative spaces corresponding to the spectral decomposition of L . Since V is 1-periodic for all variables, they are invariant under the action of $\mathbb{Z}^3$, that is, for any $u \in Y$ or $u \in Z$ and for any $\mathbf{k} = (n_1, n_2, n_3) \in \mathbb{Z}^3$, $u(\cdot - \mathbf{k})$ is also in Y or Z . Furthermore,

$$\begin{aligned} \forall u \in Y, \quad \int_{\mathbb{R}^3} (|\nabla u|^2 + V u^2) dx &= (u, u) = \|u\|^2, \\ \forall u \in Z, \quad \int_{\mathbb{R}^3} (|\nabla u|^2 + V u^2) dx &= -(u, u) = -\|u\|^2. \end{aligned} \quad (21)$$

Let $Q : X \rightarrow Z$, $P : X \rightarrow Y$ be the orthogonal projections. By (21),

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + V u^2) dx = \|Pu\|^2 - \|Qu\|^2, \quad \forall u \in X. \quad (22)$$

Moreover, by $X = Y \oplus Z$, we have

$$u = Pu + Qu, \quad \|u\|^2 = \|Pu\|^2 + \|Qu\|^2, \quad \forall u \in X. \quad (23)$$

By (22) and (9),

$$\begin{aligned} -J_\lambda(u) &= \frac{1}{2} \|Qu\|^2 - \frac{1}{2} \|Pu\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \\ &\quad + \frac{\lambda}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q dx, \quad u \in X. \end{aligned} \quad (24)$$

Moreover, by (19),

$$\begin{aligned} \langle -J'_\lambda(u), v \rangle &= (Qu, v) - (Pu, v) \\ &\quad - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u(x) v(x) u^2(y)}{|x-y|} dx dy \\ &\quad + \lambda \int_{\mathbb{R}^3} |u|^{p-2} uv dx \\ &\quad - \int_{\mathbb{R}^3} |u|^{q-2} uv dx, \quad \forall u, v \in X. \end{aligned} \quad (25)$$

3. A Global Linking Geometry for $-J_\lambda$

Let $\{e_k\}$ be a total orthonormal sequence in Y and

$$\|u\| = \max \left\{ \|Qu\|, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} |(Pu, e_k)| \right\}. \quad (26)$$

For $R > r > 0$ and $u_0 \in Z$ with $\|u_0\| = 1$, set

$$\begin{aligned} N &= \{u \in Z \mid \|u\| = r\}, \\ M &= \{u + tu_0 \mid u \in Y, t \geq 0, \|u + tu_0\| \leq R\}, \\ \partial M &= \{u \in Y \mid \|u\| \leq R\} \\ &\quad \cup \{u + tu_0 \mid u \in Y, t > 0, \|u + tu_0\| = R\}. \end{aligned} \quad (27)$$

Definition 4. Let $\phi \in C^1(X, \mathbb{R})$. ϕ' is called weakly sequentially continuous if $u \in X$ and $\{u_n\} \subset X$ are such that $u_n \rightharpoonup u$; then, for any $\varphi \in X$, $\langle \phi'(u_n), \varphi \rangle \rightarrow \langle \phi'(u), \varphi \rangle$.

Lemma 5. The functional $-J_\lambda$ satisfies the following.

- (a) $-J'_\lambda$ is weakly sequentially continuous and, for every $\lambda > 0$,

$$\sup_M (-J_\lambda) < +\infty. \quad (28)$$

- (b) There exist $\delta > 0$, $R > r > 0$, $u_0 \in Z$ with $\|u_0\| = 1$ and $\lambda'_0 > 0$ such that, for $0 < \lambda < \lambda'_0$,

$$\inf_N (-J_\lambda) > \max \left\{ \sup_{\partial M} (-J_\lambda), \sup_{\|u\| \leq \delta} (-J_\lambda(u)) \right\}. \quad (29)$$

Proof. (a) Let $u \in X$ and $\{u_n\} \subset X$ be such that $u_n \rightharpoonup u$ as $n \rightarrow \infty$. It follows that

$$(Qu_n, v) \rightarrow (Qu, v), \quad (Pu_n, v) \rightarrow (Pu, v), \quad (30)$$

$$n \rightarrow \infty, \forall v \in X.$$

Since $u_n \rightharpoonup u$ implies $u_n \rightarrow u$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for any $1 \leq s < 6$, we get that, for any $v \in C^\infty_0(\mathbb{R}^3)$, as $n \rightarrow \infty$,

$$\int_{\mathbb{R}^3} |u_n|^{p-2} u_n v \, dx \rightarrow \int_{\mathbb{R}^3} |u|^{p-2} u v \, dx, \quad (31)$$

$$\int_{\mathbb{R}^3} |u_n|^{q-2} u_n v \, dx \rightarrow \int_{\mathbb{R}^3} |u|^{q-2} u v \, dx.$$

By Lemma 2, we have that $\{\phi_{u_n}\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Therefore, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ as $n \rightarrow \infty$. It follows that $\phi_{u_n} \rightarrow \phi_u$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for any $1 \leq s < 6$. This yields that, for any $v \in C^\infty_0(\mathbb{R}^3)$, as $n \rightarrow \infty$,

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n v \, dx \rightarrow \int_{\mathbb{R}^3} \phi_u u v \, dx. \quad (32)$$

From (25), (30), (31), and (32), we deduce that, as $n \rightarrow \infty$,

$$\langle -J'_\lambda(u_n), v \rangle \rightarrow \langle -J'_\lambda(u), v \rangle, \quad \forall v \in C^\infty_0(\mathbb{R}^3). \quad (33)$$

Hence, $-J'_\lambda$ is weakly sequentially continuous. Moreover, it is easy to see that $-J_\lambda$ maps bounded sets into bounded sets; hence $\sup_M(-J_\lambda) < +\infty$.

- (b) If $u \in Z$, then $Pu = 0$ and $Qu = u$. Using (17), we get from (24) that, for any $u \in Z$,

$$\begin{aligned} -J_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy \\ &\quad - \frac{1}{q} \int_{\mathbb{R}^3} |u|^q \, dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{C}{16\pi} \|u\|^4 - \frac{C_1}{q} \|u\|^q, \end{aligned} \quad (34)$$

where C_1 comes from the Sobolev inequality $\|u\|_{L^q(\mathbb{R}^3)} \leq C_1^{1/q} \|u\|$, for all $u \in X$. It follows that we can choose sufficiently small $r > 0$ such that, for $N = \{u \in Z \mid \|u\| = r\}$,

$$\inf_N (-J_\lambda) > \frac{1}{4} r^2 > 0. \quad (35)$$

Since $q > p$, there exists $\Lambda > 0$ such that

$$\frac{\lambda}{p} |t|^p - \frac{1}{q} |t|^q < 0 \quad \text{if } 0 < \lambda \leq 1, \quad |t| > \Lambda. \quad (36)$$

Let $C' > 0$ be such that

$$\|u\|_{L^2(\mathbb{R}^3)} \leq C' \|u\|, \quad \forall u \in X. \quad (37)$$

From $q > p > 2$, we deduce that there exists $0 < \lambda'_0 < 1$ such that, for any $0 < \lambda \leq \lambda'_0$,

$$\sup_{|t| \leq \Lambda} t^{-2} \left(\frac{\lambda}{p} |t|^p - \frac{1}{q} |t|^q \right) < \frac{1}{4C'^2}. \quad (38)$$

From (36) and (38), we deduce that, for any $0 < \lambda \leq \lambda'_0$ and $u \in X$,

$$\begin{aligned} -J_\lambda(u) &= \frac{1}{2} \|Qu\|^2 - \frac{1}{2} \|Pu\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy \\ &\quad + \int_{\mathbb{R}^3} \left(\frac{\lambda}{p} |u|^p - \frac{1}{q} |u|^q \right) \, dx \\ &\leq \frac{1}{2} \|Qu\|^2 - \frac{1}{2} \|Pu\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy \\ &\quad + \int_{\{|x||u(x)| \leq \Lambda\}} \left(\frac{\lambda}{p} |u|^p - \frac{1}{q} |u|^q \right) \, dx \\ &\leq \frac{1}{2} \|Qu\|^2 - \frac{1}{2} \|Pu\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy \\ &\quad + \frac{1}{4C'^2} \int_{\mathbb{R}^3} u^2 \, dx \\ &\leq \frac{1}{2} \|Qu\|^2 - \frac{1}{2} \|Pu\|^2 \\ &\quad - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy + \frac{1}{4} \|u\|^2 \\ &= \frac{3}{4} \|Qu\|^2 - \frac{1}{4} \|Pu\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy. \end{aligned} \quad (39)$$

From Theorem 1.1 of [27], we deduce that there exists $C_* > 0$ such that, for any $u \in X$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} \, dx \, dy \\ &\geq C_* \left(\int_{\mathbb{R}^3} \frac{u^2}{|x|^{1/2} (1 + |\ln |x||)} \, dx \right)^2. \end{aligned} \quad (40)$$

Let $u_0 \in Z$ be such that $\|u_0\| = 1$. And let $u = v + tu_0 \in Y \oplus \mathbb{R}u_0$. Since there exists a continuous projection

$$L^2(\mathbb{R}^3, |x|^{-1/2} (1 + |\ln |x||)^{-1}) \longrightarrow \mathbb{R}u_0, \quad (41)$$

we deduce that there exists a constant $C_{**} > 0$ such that, for any $u = v + tu_0 \in Y \oplus \mathbb{R}u_0$,

$$\left(\int_{\mathbb{R}^3} \frac{u^2}{|x|^{1/2} (1 + |\ln |x||)} dx \right)^2 \geq C_{**} t^4. \quad (42)$$

Combining (39)–(42), we deduce that there exists a constant $\kappa > 0$ such that, for any $u = tu_0 + v \in Y \oplus \mathbb{R}u_0$,

$$-J_\lambda(u) \leq \frac{3}{4} t^2 - \frac{1}{4} \|v\|^2 - \kappa t^4. \quad (43)$$

It follows that

$$-J_\lambda(u) \longrightarrow -\infty, \quad \text{as } \|u\| \longrightarrow \infty, u \in Y \oplus \mathbb{R}u_0. \quad (44)$$

Moreover, for $0 < \lambda \leq \lambda'_0$ and $u \in Y$, (39) implies

$$-J_\lambda(u) \leq -\frac{1}{4} \|u\|^2 - \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \leq 0. \quad (45)$$

Together with (44), this implies that there exists $R > r$ such that

$$\sup_{\partial M} (-J_\lambda) \leq 0 < \inf_N (-J_\lambda). \quad (46)$$

From (39) and the definition of $\|\cdot\|$ (see (26)), we have, for $0 < \lambda \leq \lambda'_0$,

$$-J_\lambda(u) \leq \frac{3}{4} \|Qu\|^2 \leq \frac{3}{4} \|u\|^2. \quad (47)$$

Choosing $\delta = r/\sqrt{6}$, (47) and (35) give that

$$\sup_{\|u\| \leq \delta} (-J_\lambda(u)) < \inf_N (-J_\lambda). \quad (48)$$

Together with (46), this yields (29). $\square$

4. Boundedness of $(\bar{C})_c$ Sequence of $-J_\lambda$

Definition 6. Let $\phi \in C^1(X, \mathbb{R})$. A sequence $\{u_n\} \subset X$ is called a $(\bar{C})_c$ sequence for ϕ , if

$$\sup_n \phi(u_n) \leq c, \quad (1 + \|u_n\|) \|\phi'(u_n)\|_{X'} \longrightarrow 0, \quad (49)$$

as $n \longrightarrow \infty$.

In this section, we will prove that if $\lambda > 0$ is sufficiently small, then a $(\bar{C})_c$ sequence for $-J_\lambda$ is bounded in X .

Since $q > p > 2$, there exists $D > 0$ such that

$$|t|^{q-2} - \lambda |t|^{p-2} > \|V_-\|_{L^\infty(\mathbb{R}^3)} \quad \text{if } 0 < \lambda \leq 1, |t| > \frac{D}{2}, \quad (50)$$

where $V_-(x) = \max\{-V(x), 0\}$, $x \in \mathbb{R}^3$.

Lemma 7. Suppose that $0 < \lambda \leq 1$. If $\{u_n\}$ is a $(\bar{C})_c$ sequence for $-J_\lambda$, then

$$\lim_{n \rightarrow \infty} \int_{\{x | u_n(x) > D\}} |u_n|^6 dx = 0. \quad (51)$$

Proof. Let $v_n = \max\{u_n(x) - D/2, 0\}$. It is easy to verify that $v_n \in X$ and $\|v_n\| \leq \|u_n\|$, for all $n \in \mathbb{N}$. Together with the fact that $\{u_n\}$ is a $(\bar{C})_c$ sequence of $-J_\lambda$, this implies $\langle J'(u_n), v_n \rangle = o(1)$, where $o(1)$ denotes the infinitesimal depending only on n ; that is, $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Choosing $v = v_n$ in (19), by (50) and the fact that ϕ_{u_n} , u_n , and v_n are nonnegative in $\{x | u_n(x) > D/2\}$, we have

$$\begin{aligned} o(1) &= \langle J'(u_n), v_n \rangle \\ &= \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(x) u_n v_n dx \\ &\quad + \int_{\mathbb{R}^3} \phi_{u_n} \cdot u_n \cdot v_n dx \\ &\quad + \int_{\mathbb{R}^3} (|u_n|^{q-2} - \lambda |u_n|^{p-2}) u_n v_n dx \\ &= \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V_+ u_n v_n dx \\ &\quad + \int_{\{x | u_n(x) > D/2\}} \phi_{u_n} \cdot u_n \cdot v_n dx \\ &\quad + \int_{\{x | u_n(x) > D/2\}} (|u_n|^{q-2} - \lambda |u_n|^{p-2} \\ &\quad - V_-(x)) u_n v_n dx \end{aligned} \quad (52)$$

$$\begin{aligned} &\geq \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \geq \bar{C} \left(\int_{\mathbb{R}^3} |v_n|^6 dx \right)^{1/3} \\ &= \bar{C} \left(\int_{\{x | u_n(x) > D/2\}} |u_n|^6 dx \right)^{1/3}, \end{aligned}$$

where $V_+ = V + V_- \geq 0$ in $\mathbb{R}^3$ and $\bar{C}$ is the Sobolev constant. Note that $v_n \geq u_n/2$ on $\{x | u_n(x) > D\}$. Together with (52), this implies

$$\lim_{n \rightarrow \infty} \int_{\{x | u_n(x) > D\}} |u_n|^6 dx = 0. \quad (53)$$

Similarly, we can prove

$$\lim_{n \rightarrow \infty} \int_{\{x | -u_n(x) > D\}} |u_n|^6 dx = 0. \quad (54)$$

The result of this lemma follows from these two limits. $\square$

Lemma 8. There exists $\lambda''_0 > 0$ such that if $0 < \lambda < \lambda''_0$ and $\{u_n\}$ is a $(\bar{C})_c$ sequence for $-J_\lambda$, then $\{u_n\}$ is bounded in X .

Proof. By $(1 + \|u_n\|) \|\phi'(u_n)\|_{X'} \rightarrow 0$, we have

$$\langle -J'_\lambda(u_n), Qu_n \rangle = o(1), \quad \langle -J'_\lambda(u_n), Pu_n \rangle = o(1). \quad (55)$$

Choosing $v = Qu_n$ and $v = Pu_n$ in (25), (55) implies that

$$\begin{aligned}\|Qu_k\|^2 &= \int_{\mathbb{R}^3} u_n \phi_{u_n} \cdot Qu_n dx \\ &\quad - \int_{\mathbb{R}^3} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n) \cdot Qu_n dx + o(1), \\ \|Pu_n\|^2 &= - \int_{\mathbb{R}^3} u_n \phi_{u_n} \cdot Pu_n dx \\ &\quad + \int_{\mathbb{R}^3} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n) \cdot Pu_n dx + o(1).\end{aligned}\quad (56)$$

By these two equalities and $\|u_n\|^2 = \|Pu_n\|^2 + \|Qu_n\|^2$ (see (23)), we obtain

$$\begin{aligned}\|u_n\|^2 &= \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n} \cdot (Qu_n - Pu_n) dx \\ &\quad - \int_{\mathbb{R}^3} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n) \\ &\quad \cdot (Qu_n - Pu_n) dx + o(1).\end{aligned}\quad (57)$$

Let

$$\chi_{n,1}(x) = \begin{cases} 1, & \text{if } |u_n(x)| \leq D, \\ 0, & \text{if } |u_n(x)| > D \end{cases} \quad (58)$$

and $\chi_{n,2} = 1 - \chi_{n,1}$, where D comes from (50). Then, $u_n^2 = (u_n \chi_{n,1})^2 + (u_n \chi_{n,2})^2$ and

$$\phi_{u_n} = \phi_{u_n \chi_{n,1}} + \phi_{u_n \chi_{n,2}}. \quad (59)$$

Since $\phi_{u_n \chi_{n,1}}$ is a solution of the equation

$$-\Delta \phi = 4\pi (u_n \chi_{n,1})^2 \quad \text{in } \mathbb{R}^3, \quad (60)$$

by $(u_n \chi_{n,1})^2 \leq D^2$ in $\mathbb{R}^3$ and the standard elliptic estimates (see, [28, Theorem 8.17]), we get that there exists a positive constant C_2 that is independent of n and $y \in \mathbb{R}^3$, such that, for any $y \in \mathbb{R}^3$,

$$\begin{aligned}\|\phi_{u_n \chi_{n,1}}\|_{L^\infty(B_1(y))} &\leq C_2 \left(\int_{B_2(y)} |u_n \chi_{n,1}|^4 dx \right)^{1/2} \\ &\leq C_2 D^2 \left(\int_{B_2(y)} dx \right)^{1/2}.\end{aligned}\quad (61)$$

Let $C_3 = (\int_{B_2(0)} dx)^{1/2}$. By (61),

$$\|\phi_{u_n \chi_{n,1}}\|_{L^\infty(\mathbb{R}^3)} \leq C_2 C_3 D^2. \quad (62)$$

Together with $\phi_{u_n \chi_{n,1}} \geq 0$ in $\mathbb{R}^3$ (see Lemma 2(ii)), this implies

$$0 \leq \phi_{u_n \chi_{n,1}} \leq C_2 C_3 D^2 \quad \text{in } \mathbb{R}^3. \quad (63)$$

By (57), (59), and (63), we have

$$\begin{aligned}\|u_n\|^2 &= \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n \chi_{n,1}} \cdot (Qu_n - Pu_n) dx \\ &\quad + \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n \chi_{n,2}} \cdot (Qu_n - Pu_n) dx \\ &\quad - \int_{\{x||u_n(x)| \leq D\}} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n) \\ &\quad \cdot (Qu_n - Pu_n) dx \\ &\quad - \int_{\{x||u_n(x)| > D\}} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n) \\ &\quad \cdot (Qu_n - Pu_n) dx + o(1) \\ &\leq \|\phi_{u_n \chi_{n,1}}\|_{L^2(\mathbb{R}^3)} \|Qu_n - Pu_n\|_{L^2(\mathbb{R}^3)} \\ &\quad + \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n \chi_{n,2}} \cdot (Qu_n - Pu_n) dx \\ &\quad + \left(\int_{\{x||u_n(x)| \leq D\}} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \\ &\quad \cdot \|Qu_n - Pu_n\|_{L^2(\mathbb{R}^3)} \\ &\quad + \lambda \left(\int_{\{x||u_n(x)| > D\}} |u_n|^p dx \right)^{(p-1)/p} \\ &\quad \cdot \|Qu_n - Pu_n\|_{L^p(\mathbb{R}^3)} \\ &\quad + \left(\int_{\{x||u_n(x)| > D\}} |u_n|^q dx \right)^{(q-1)/q} \\ &\quad \cdot \|Qu_n - Pu_n\|_{L^q(\mathbb{R}^3)}.\end{aligned}\quad (64)$$

Since $\|Qu_n - Pu_n\|_{L^2(\mathbb{R}^3)} \leq 2C' \|u_n\|$ (C' is the constant coming from (37)), $\|Qu_n - Pu_n\|_{L^p(\mathbb{R}^3)} \leq C'' \|u_n\|$, and $\|Qu_n - Pu_n\|_{L^q(\mathbb{R}^3)} \leq C'' \|u_n\|$, where C'' is a positive constant independent of n and λ , we have

$$\begin{aligned}\|u_n\|^2 &\leq 2C' \|u_n \phi_{u_n \chi_{n,1}}\|_{L^2(\mathbb{R}^3)} \|u_n\| \\ &\quad + \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n \chi_{n,2}} \cdot (Qu_n - Pu_n) dx \\ &\quad + 2C' \left(\int_{\{x||u_n(x)| \leq D\}} (\lambda |u_n|^{p-2} u_n \right. \\ &\quad \left. - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \|u_n\| \\ &\quad + \lambda C'' \left(\int_{\{x||u_n(x)| > D\}} |u_n|^p dx \right)^{(p-1)/p} \|u_n\| \\ &\quad + C'' \left(\int_{\{x||u_n(x)| > D\}} |u_n|^q dx \right)^{(q-1)/q} \|u_n\|.\end{aligned}\quad (65)$$

By Lemma 7, we deduce that $\int_{\{|x||u_n(x)|>D\}} |u_n|^p dx = o(1)$ and $\int_{\{|x||u_n(x)|>D\}} |u_n|^q dx = o(1)$ if $\lambda \leq 1$. Together with (63), this implies that, for $0 < \lambda \leq 1$,

$$\begin{aligned} \|u_n\|^2 &\leq 2C'D(C_2C_3)^{1/2} \|u_n\phi_{u_n\chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)} \|u_n\| \\ &\quad + \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n\chi_{n,2}} \cdot (Qu_n - Pu_n) dx \\ &\quad + 2C' \left(\int_{\{|x||u_n(x)|\leq D\}} (\lambda |u_n|^{p-2} u_n \right. \\ &\quad \left. - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \|u_n\| \\ &\quad + o(1) \cdot \|u_n\|. \end{aligned} \quad (66)$$

Using the Hardy-Littlewood-Sobolev inequality (see, (1.1) of [26]), we get that

$$\begin{aligned} &\int_{\mathbb{R}^3} u_n \cdot \phi_{u_n\chi_{n,2}} \cdot (Qu_n - Pu_n) dx \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_n\chi_{n,2}(y))^2 \cdot u_n(x) \cdot (Qu_n - Pu_n)(x)}{|x-y|} dx dy \\ &\leq C_{HLS} \| (u_n\chi_{n,2})^2 \|_{L^{6/5}(\mathbb{R}^3)} \|u_n \cdot (Qu_n - Pu_n)\|_{L^{6/5}(\mathbb{R}^3)} \\ &\leq C_{HLS} \| (u_n\chi_{n,2})^2 \|_{L^{6/5}(\mathbb{R}^3)} \|u_n\|_{L^{12/5}(\mathbb{R}^3)} \\ &\quad \cdot \|Qu_n - Pu_n\|_{L^{12/5}(\mathbb{R}^3)} \\ &\leq C''' \left(\int_{\{|x||u_n(x)|>D\}} |u_n|^{12/5} dx \right)^{5/6} \|u_n\|^2, \end{aligned} \quad (67)$$

where C_{HLS} and C''' are positive constants independent of n and λ . By Lemma 7, we have

$$\int_{\{|x||u_n(x)|>D\}} |u_n|^{12/5} dx = o(1) \quad \text{if } 0 < \lambda \leq 1. \quad (68)$$

Combining (68) and (67) with (66) yields that, for $0 < \lambda \leq 1$,

$$\begin{aligned} \|u_n\|^2 &\leq 2C'D(C_2C_3)^{1/2} \|u_n\phi_{u_n\chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)} \|u_n\| \\ &\quad + o(1) \cdot \|u_n\|^2 \\ &\quad + 2C' \left(\int_{\{|x||u_n(x)|\leq D\}} (\lambda |u_n|^{p-2} u_n \right. \\ &\quad \left. - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \|u_n\| \\ &\quad + o(1) \cdot \|u_n\|. \end{aligned} \quad (69)$$

Let $0 < \epsilon < D$. Then, for $0 < \lambda \leq 1$,

$$\begin{aligned} &\left(\int_{\{|x||u_n(x)|\leq \epsilon\}} (\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \\ &\leq \left(\int_{\{|x||u_n(x)|\leq \epsilon\}} (|u_n|^{p-1} + |u_n|^{q-1})^2 dx \right)^{1/2} \\ &\leq 2 \left(\int_{\{|x||u_n(x)|\leq \epsilon\}} (|u_n|^{2p-2} + |u_n|^{2q-2}) dx \right)^{1/2} \\ &\leq 4(\epsilon^{p-2} + \epsilon^{q-2}) \left(\int_{\{|x||u_n(x)|\leq \epsilon\}} |u_n|^2 dx \right)^{1/2} \\ &\leq 4C'(\epsilon^{p-2} + \epsilon^{q-2}) \|u_n\|. \end{aligned} \quad (70)$$

Combining (70) with (69) yields that, for $0 < \lambda \leq 1$,

$$\begin{aligned} \|u_n\|^2 &\leq 2C'D(C_2C_3)^{1/2} \|u_n\phi_{u_n\chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)} \|u_n\| \\ &\quad + o(1) \cdot \|u_n\|^2 + 4C'^2(\epsilon^{p-2} + \epsilon^{q-2}) \|u_n\|^2 \\ &\quad + 2C' \left(\int_{\{|x||\epsilon < |u_n(x)|\leq D\}} (\lambda |u_n|^{p-2} u_n \right. \\ &\quad \left. - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \|u_n\| \\ &\quad + o(1) \cdot \|u_n\|. \end{aligned} \quad (71)$$

Let $\epsilon_0 \in (0, \min\{D, D^{1/(q-p)}, 1\})$ be such that $4C'^2(\epsilon_0^{p-2} + \epsilon_0^{q-2}) < 1/2$. Then, (71) implies that

$$\begin{aligned} \|u_n\|^2 &\leq 4C'D(C_2C_3)^{1/2} \|u_n\phi_{u_n\chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)} \|u_n\| \\ &\quad + o(1) \cdot \|u_n\|^2 + o(1) \cdot \|u_n\| \\ &\quad + 4C' \left(\int_{\{|x||\epsilon_0 < |u_n(x)|\leq D\}} (\lambda |u_n|^{p-2} u_n \right. \\ &\quad \left. - |u_n|^{q-2} u_n)^2 dx \right)^{1/2} \|u_n\|. \end{aligned} \quad (72)$$

From $\sup_n (-J_\lambda(u_n)) \leq c$ and $(1 + \|u_n\|) \|J'_\lambda(u_n)\|_{X'} \rightarrow 0$, we obtain

$$\begin{aligned} o(1) + c &\geq -J_\lambda(u_n) + \frac{1}{2} \langle J'_\lambda(u_n), u_n \rangle \\ &= \frac{1}{4} \int_{\mathbb{R}^3} u_n^2 \phi_{u_n} dx \\ &\quad - \int_{\mathbb{R}^3} \left(\lambda \left(\frac{1}{2} - \frac{1}{p} \right) |u_n|^p - \left(\frac{1}{2} - \frac{1}{q} \right) |u_n|^q \right) dx. \end{aligned} \quad (73)$$

It follows that

$$\begin{aligned}
& \frac{1}{4} \int_{\mathbb{R}^3} u_n^2 \phi_{u_n} dx \\
& + \int_{\{x | |u_n(x)| > \lambda^{1/(q-p)}\}} \left(\left(\frac{1}{2} - \frac{1}{q} \right) |u_n(x)|^q \right. \\
& \quad \left. - \lambda \left(\frac{1}{2} - \frac{1}{p} \right) |u_n|^p \right) dx \\
& \leq c + \int_{\{x | |u_n(x)| \leq \lambda^{1/(q-p)}\}} \left(\lambda \left(\frac{1}{2} - \frac{1}{p} \right) |u_n|^p \right. \\
& \quad \left. - \left(\frac{1}{2} - \frac{1}{q} \right) |u_n|^q \right) dx \quad (74) \\
& + o(1) \\
& \leq c + C_4 \lambda^{(q-2)/(q-p)} \int_{\mathbb{R}^3} |u_n|^2 dx + o(1),
\end{aligned}$$

where C_4 is a positive constant depending only on p and q . Because $(1/2 - 1/q)|t|^q - \lambda(1/2 - 1/p)|t|^p > 0$ if $|t| \geq \lambda^{1/(q-p)}$, we get that, for $0 < \lambda \leq \epsilon_0^{q-p}$,

$$\begin{aligned}
& \int_{\{x | D \geq |u_n(x)| > \epsilon_0\}} \left(\left(\frac{1}{2} - \frac{1}{q} \right) |u_n|^q - \lambda \left(\frac{1}{2} - \frac{1}{p} \right) |u_n|^p \right) dx \\
& \leq \int_{\{x | |u_n(x)| > \lambda^{1/(q-p)}\}} \left(\left(\frac{1}{2} - \frac{1}{q} \right) |u_n|^q - \lambda \left(\frac{1}{2} - \frac{1}{p} \right) |u_n|^p \right) dx. \quad (75)
\end{aligned}$$

Then, by (74) and $\int_{\mathbb{R}^3} u_n^2 \phi_{u_n} dx \geq \int_{\mathbb{R}^3} u_n^2 \phi_{u_n \chi_{n,1}} dx = \|u_n \phi_{u_n \chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)}^2$, we get that, for $0 < \lambda \leq \epsilon_0^{q-p}$,

$$\begin{aligned}
& \frac{1}{4} \|u_n \phi_{u_n \chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)}^2 \\
& + \int_{\{x | D \geq |u_n(x)| > \epsilon_0\}} \left(\left(\frac{1}{2} - \frac{1}{q} \right) |u_n|^q \right. \\
& \quad \left. - \lambda \left(\frac{1}{2} - \frac{1}{p} \right) |u_n|^p \right) dx \quad (76) \\
& \leq c + C_4 \lambda^{(q-2)/(q-p)} \int_{\mathbb{R}^3} |u_n|^2 dx + o(1) \\
& \leq c + C'^2 C_4 \lambda^{(q-2)/(q-p)} \|u_n\|^2 + o(1).
\end{aligned}$$

It is easy to verify that there exists a positive constant C_5 depending only on p, q, D , and ϵ_0 such that, if $0 < \lambda \leq \epsilon_0^{q-p}/2$ and $\epsilon_0 \leq |t| \leq D$, then

$$C_5 \left(\lambda |t|^{p-2} t - |t|^{q-2} t \right)^2 \leq \left(\frac{1}{2} - \frac{1}{q} \right) |t|^q - \lambda \left(\frac{1}{2} - \frac{1}{p} \right) |t|^p. \quad (77)$$

Together with (76), this yields that, for $0 < \lambda \leq \epsilon_0^{q-p}/2$,

$$\begin{aligned}
& \frac{1}{4} \|u_n \phi_{u_n \chi_{n,1}}^{1/2}\|_{L^2(\mathbb{R}^3)}^2 \\
& + C_5 \int_{\{x | D \geq |u_n(x)| > \epsilon_0\}} \left(\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n \right)^2 dx \quad (78) \\
& \leq c + C'^2 C_4 \lambda^{(q-2)/(q-p)} \|u_n\|^2 + o(1).
\end{aligned}$$

Combining (78) with (72), we deduce that there exists $\lambda_0'' \in (0, \epsilon_0^{q-p}/2)$ that is small enough such that $\{\|u_n\|\}$ is bounded if $0 < \lambda < \lambda_0''$. $\square$

5. Proof of Theorem 1

Proof of Theorem 1. Let $\lambda_0 = \min\{\lambda_0', \lambda_0''\}$, where λ_0' and λ_0'' are the constants in Lemmas 5 and 8, respectively. By Lemmas 5 and 8, and Theorem 1.3 of [24], we deduce that, for any $0 < \lambda < \lambda_0$, there exists a bounded $(\overline{C})_c$ sequence $\{u_n\}$ for $-J_\lambda$ with $\inf_n \|u_n\| > 0$. Up to a subsequence, either

- (a) $\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_n|^2 dx = 0$, or
- (b) there exist $\varrho > 0$ and $a_n \in \mathbb{Z}^3$ such that $\int_{B_1(a_n)} |u_n|^2 dx \geq \varrho$.

If (a) occurs, the Lions lemma (see [23, Theorem 1.21]) implies that u_n satisfies $u_n \rightarrow 0$ in $L^s(\mathbb{R}^3)$ for any $2 < s < 6$. It follows that

$$\begin{aligned}
& \int_{\mathbb{R}^3} \left(\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n \right) \cdot Q u_n dx \rightarrow 0, \\
& \int_{\mathbb{R}^3} \left(\lambda |u_n|^{p-2} u_n - |u_n|^{q-2} u_n \right) \cdot P u_n dx \rightarrow 0. \quad (79)
\end{aligned}$$

As (67), we have

$$\begin{aligned}
& \int_{\mathbb{R}^3} u_n \cdot \phi_{u_n} \cdot (Q u_n - P u_n) dx \\
& = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(y) \cdot u_n(x) \cdot (Q u_n - P u_n)(x)}{|x - y|} dx dy \\
& \leq C_{HLS} \|u_n^2\|_{L^{6/5}(\mathbb{R}^3)} \|u_n \cdot (Q u_n - P u_n)\|_{L^{6/5}(\mathbb{R}^3)} \\
& \leq C_{HLS} \|u_n^2\|_{L^{6/5}(\mathbb{R}^3)} \|u_n\|_{L^{12/5}(\mathbb{R}^3)} \|Q u_n - P u_n\|_{L^{12/5}(\mathbb{R}^3)} \\
& = C_{HLS} \|u_n\|_{L^{12/5}(\mathbb{R}^3)}^3 \|Q u_n - P u_n\|_{L^{12/5}(\mathbb{R}^3)} \rightarrow 0, \quad n \rightarrow \infty. \quad (80)
\end{aligned}$$

Combining (79) and (80) with (57) yields $\|u_n\| \rightarrow 0$. This contradicts $\inf_n \|u_n\| > 0$. Therefore, case (a) cannot occur. As case (b) therefore occurs, $w_n = u_n(\cdot + a_n)$ satisfies $w_n \rightarrow u_0 \neq 0$. From $(1 + \|w_n\|) - J'_\lambda(w_n)\|_{X'} = (1 + \|u_n\|) - J'_\lambda(u_n)\|_{X'} \rightarrow 0$ and the weakly sequential continuity of $-J'_\lambda$, we have that $-J'_\lambda(u_0) = 0$. Therefore, u_0 is a nontrivial solution of (1). This completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank the anonymous referees for their comments and suggestions on the paper. Shaowei Chen was supported by Science Foundation of Huaqiao University and Promotion Program for Young and Middle-Aged Teacher in Science and Technology Research of Huaqiao University (ZQN-PY119).

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