Research Article

The Periodic Boundary Value Problem for the Weakly Dissipative μ -Hunter-Saxton Equation

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We study the periodic boundary value problem for the weakly dissipative μ -Hunter-Saxton equation. We establish the local well-posedness in Besov space $B_{2,1}^{3/2}$, which extends the previous regularity range to the critical case.

1. Introduction

In this paper we consider the weakly dissipative μ -Hunter-Saxton (μ HS) equation with periodic boundary condition [1]:

$$\begin{aligned} \mu(u)_t - u_{xxt} + 2\mu(u)u_x + \lambda(\mu(u) - u_{xx}) \\ = 2u_x u_{xx} + uu_{xxx}, \quad t \geq 0, \quad x \in \mathbb{S}, \end{aligned} \quad (1)$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{S}, \quad (2)$$

where $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$, $\mu(u) = \int_{\mathbb{S}} u dx$, $\lambda \geq 0$, and the term $\lambda(\mu(u) - u_{xx})$ models energy dissipation.

When $\lambda = 0$ and $\mu(u) = 0$, (1) is the Hunter-Saxton equation [2]

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0. \quad (3)$$

The Hunter-Saxton (HS) equation is an asymptotic equation for rotators in liquid crystals and modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal [2]. The HS equation has a bi-Hamiltonian structure [3, 4] and is completely integrable [5]. The initial value problem of the HS equation also has been studied extensively; see [2, 6, 7]. We refer to [8–12] for the weak solution to the HS equation. When $\mu(u) = 0$, the global existence and blow-up phenomena for the dissipative HS equation can be found in [13].

When $\lambda = 0$, (1) becomes the μ -HS equation

$$\mu(u)_t - u_{xxt} + 2\mu(u)u_x = 2u_x u_{xx} + uu_{xxx}, \quad (4)$$

which is derived and studied in [14]. Khesin et al. [14] showed that the μ -HS equation describes the geodesic flow on $D^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product

$$(u, v) = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x v_x dx. \quad (5)$$

Moreover, if interactions of rotators and an external magnetic field are allowed, then the μ -HS equation can be viewed as a natural generalization of the rotator equation. The periodic peaked solution and the multipeakons solution to μ -HS equation were showed in [14, 15].

Another important model which possesses some similar structures to the μ -HS equation is the Camassa-Holm (CH) equation:

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \quad (6)$$

The Camassa-Holm (CH) equation was derived independently by Fokas and Fuchssteiner in [16] and by Camassa and Holm in [17]. Fokas and Fuchssteiner derived (6) in studying completely integrable generalizations of the KdV equation with bi-Hamiltonian structures, while Camassa and Holm proposed (6) to describe the unidirectional propagation of shallow water waves over a flat bottom. In [18], Constantin and Lannes proved that the CH equation, as a model for the propagation of shallow water waves, is valid approximations to the governing equations for water waves. The relation

between the CH and HS equation is that the HS equation can be viewed as the high-frequency limit of the CH equation [5]. In [17], Camassa and Holm discovered the bi-Hamiltonian structure of CH, which ensures the existence of an infinite number of conservation laws. The integrability of CH (as an infinite-dimensional Hamiltonian system) was studied in [19, 20]. Moreover, the CH equation is such an equation that exhibits both phenomena of soliton interaction (peaked soliton solutions) and wave breaking (the solution remains bounded while its slope becomes unbounded in finite time [21]). The global existence of strong and weak solution and blow-up phenomena for the CH equation can be found in [21–25].

In general, it is almost impossible to avoid energy dissipation in the real world. Therefore, it is reasonable to study the model with energy dissipation. For example, the weakly dissipative CH equation and weakly dissipative DP equation have been studied in [26] and [27–29], respectively.

Recently, Liu and Yin [1] suggested the weakly dissipative μ -HS equation. They established the local well-posedness in Sobolev spaces H^s ($s > 3/2$), obtained the global existence, and studied the blow-up scenario. Motivated by the previous work, in this paper, we aim to prove the local well-posedness for (1) (2) in the critical Besov space $B_{2,1}^{3/2}$.

Integrating (1) with respect to $x \in \mathbb{S}$ yields $(d/dt)\mu(u) = -\lambda\mu(u)$, which implies that

$$\mu(u) = \mu(u_0)e^{-\lambda t} = \mu_0 e^{-\lambda t}, \quad (7)$$

where $\mu_0 := \mu(u_0) = \int_{\mathbb{S}} u_0(x) dx$.

Now we denote $A = (\mu - \partial_x^2)$ and reformulate (1), (2) as the nonlocal form:

$$\begin{aligned} \partial_t u + u \partial_x u + F(u) &= 0, \quad t \geq 0, \quad x \in \mathbb{S}, \\ u(0, x) &= u_0(x), \quad x \in \mathbb{S}, \end{aligned} \quad (8)$$

where $F(u) = F_1(u) + F_2(u) + F_3(u)$ with

$$\begin{aligned} F_1(u) &= 2\partial_x A^{-1}(\mu_0 e^{-\lambda t} u), \\ F_2(u) &= \frac{1}{2}\partial_x A^{-1}(u_x^2), \\ F_3(u) &= \lambda u. \end{aligned} \quad (9)$$

For the inverse operator A^{-1} , the relation $v = A^{-1}w$ is given explicitly by [14, 30],

$$\begin{aligned} v(x) &= \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12} \right) \mu(w) + \int_0^1 \int_0^y w(s) ds dy \\ &\quad - \int_0^x \int_0^y w(s) ds dy \\ &\quad + \int_0^1 \int_0^y \int_0^s w(r) dr ds dy. \end{aligned} \quad (10)$$

In terms of Green's function, we see that $v = A^{-1}w$ can be also expressed as [14, 30]

$$v(x) = A^{-1}w = g * w = \int_0^1 g(x-y)w(y) dy, \quad (11)$$

where $g(x)$ is the periodic extension of $\tilde{g}(x)$ with $\tilde{g}(x) = (1/2)x(x-1) + 13/12$, $x \in [0, 1]$. Besides, as A^{-1} and ∂_x commute, then

$$\begin{aligned} A^{-1}\partial_x w(x) &= \left(x - \frac{1}{2} \right) \int_0^1 w(x) dx - \int_0^x w(y) dy \\ &\quad + \int_0^1 \int_0^x w(y) dy dx, \\ A^{-1}\partial_x^2 w(x) &= -w(x) + \int_0^1 w(x) dx. \end{aligned} \quad (12)$$

We will show that the nonlocal terms $F(u)$ map $B_{2,1}^{3/2}$ to $B_{2,1}^{3/2}$ (see Lemma 10) and therefore we can expect the local well-posedness of (1) in $B_{2,1}^{3/2}$.

Notations. We use $\lesssim$, $\gtrsim$, and $\approx$ to denote estimates that hold up to some universal constant. C is also a generic constant that may assume different values in different lines. $\mathcal{D}(\mathbb{S})$ is the space of all infinitely differentiable functions on $\mathbb{S}$ and $\mathcal{D}'(\mathbb{S})$ is its dual space. All function spaces are over $\mathbb{S}$ and we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity.

The main results of this paper are as follows.

Theorem 1. *When $u_0 \in B_{2,1}^{3/2}$, we have the following results:*

- (1) *There exists a $T = T(u_0) > 0$ such that (1), (2) have a unique solution $u \in C([0, T]; B_{2,1}^{3/2}) \cap C^1([0, T]; B_{2,1}^{1/2})$ which depends continuously on the initial data u_0 . Furthermore, for some $C > 0$, we have*

$$\|u(t)\|_{B_{2,1}^{3/2}} \leq 2\|u_0\|_{B_{2,1}^{3/2}}, \quad \text{for } 0 \leq t \leq T = \frac{1}{4C\|u_0\|_{B_{2,1}^{3/2}}}. \quad (13)$$

- (2) *If u, v are the solutions with initial data $u_0, v_0 \in B(0, R) \subset B_{2,1}^{3/2}$, respectively, then*

$$\|u(t) - v(t)\|_{B_{2,1}^q} \leq \|u_0 - v_0\|_{B_{2,\infty}^{1/2}}^\alpha, \quad 0 \leq t \leq \tilde{T} = \frac{1}{(4CR + 4C\lambda)}, \quad (14)$$

where $\alpha = (3/2 - q)\exp\{-C_R\}$ with the constant $C_R > 0$ depending on the radius R and $1/2 \leq q < 3/2$. That is to say, the solution map $u_0 \mapsto u(t, \cdot)$ defined by (1), (2) is Hölder continuous as a map from $B(0, R) \subset B_{2,1}^{3/2}$, with the $B_{2,\infty}^{1/2}$ topology, to $C([0, \tilde{T}], B_{2,1}^q)$.

Now we outline the proof of Theorem 1. We construct a sequence of the approximate solutions $\{u_n\}$. We need to prove that $\{u_n\}$ is a Cauchy sequence. However, in the critical case, the following estimate

$$\|F(u) - F(v)\|_{B_{2,1}^{2/1}} \lesssim \|u - v\|_{B_{2,1}^{2/1}} \quad (15)$$

cannot be obtained due to the lack of appropriate product estimate in $B_{2,1}^{-1/2}$ (this is one of the significant differences between the critical case $s = 3/2$ and the supercritical case $s > 3/2$). Therefore we will use a product estimate in $B_{2,\infty}^{-1/2}$ obtained in Danchin's celebrated paper [31] (see Lemma 7) and the Log interpolation inequality (see Lemma 8) to find

$$\begin{aligned} \|F(u) - F(v)\|_{B_{2,\infty}^{1/2}} &\lesssim \|u - v\|_{B_{2,1}^{1/2}} \\ &\leq C \|u - v\|_{B_{2,\infty}^{1/2}} \ln \left(e + \frac{\|u - v\|_{B_{2,\infty}^{3/2}}}{\|u - v\|_{B_{2,\infty}^{1/2}}} \right). \end{aligned} \quad (16)$$

Then we prove the convergence of $\{u_n\}$ in $B_{2,\infty}^{1/2}$ firstly and then to obtain the convergence in $B_{2,1}^{1/2}$ by interpolation. To this end, we will use the Osgood lemma (Lemma 9) to estimate $\|u - v\|_{B_{2,\infty}^{1/2}}$. Our idea is motivated by [32, 33], where the authors used the Osgood lemma to studied the b-family equation and the modified Camassa-Holm equation, respectively.

The remainder of this paper is organized as follows. In the next section, we state some preliminaries. We prove Theorem 1 in Section 3.

2. Preliminaries

Now we state some preliminaries on the Littlewood-Paley theory. We refer to [34–36] for the elementary properties of them.

Let χ, ϕ be two smooth radial functions satisfying $0 \leq \chi, \phi \leq 1$, $\text{supp } \chi \subset \{\xi \in \mathbb{R} : |\xi| \leq 4/3\}$, $\text{supp } \phi \subset \{\xi \in \mathbb{R} : 3/4 \leq |\xi| \leq 8/3\}$, and

$$\begin{aligned} \chi(\xi) + \sum_{j \in \mathbb{N}} \varphi(2^{-j}\xi) &= 1, \quad \forall \xi \in \mathbb{R}, \\ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) &= 1, \quad \forall \xi \in \mathbb{R} \setminus \{0\}, \end{aligned} \quad (17)$$

$$\text{supp } \varphi(2^{-j}\cdot) \cap \text{supp } \varphi(2^{-j'}\cdot) = \emptyset \quad \text{if } |j - j'| \geq 2,$$

$$\text{supp } \varphi(\cdot) \cap \text{supp } \varphi(2^{-j}\cdot) = \emptyset \quad \text{if } j \geq 1.$$

We decompose $u \in \mathcal{D}'(\mathbb{S})$ into Fourier series; that is, $u = (1/2\pi) \sum_{\xi \in \mathbb{Z}} \mathcal{F}_x u(\xi) e^{ix\xi}$, where $\mathcal{F}_x u(\xi) = \int_{\mathbb{S}} e^{-ix\xi} u(x) dx$ is the Fourier transform and the inverse transform is given by $\mathcal{F}_x^{-1} u(x) = (1/2\pi) \sum_{\xi \in \mathbb{Z}} u(\xi) e^{ix\xi}$. Now we define the periodic dyadic blocks as

$$\Delta_{-1}u = \mathcal{F}_x^{-1} \chi \mathcal{F}_x u = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \chi(\xi) (\mathcal{F}_x u)(\xi) e^{ix\xi},$$

$$\Delta_j u = \mathcal{F}_x^{-1} \varphi(2^{-j}\xi) \mathcal{F}_x u \quad (18)$$

$$= \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \varphi(2^{-j}\xi) (\mathcal{F}_x u)(\xi) e^{ix\xi}, \quad \text{if } j \geq 0.$$

Then we define the low frequency cut-off S_j as $S_j u = \sum_{i=-1}^{j-1} \Delta_i u$. Direct computation implies that, for any $1 \leq p \leq \infty$, we have the quasi-orthogonality properties:

$$\Delta_i \Delta_j u \equiv 0, \quad \text{if } |i - j| \geq 2, \quad (19)$$

$$\Delta_j (S_{i-1} u \Delta_i v) \equiv 0, \quad \text{if } |i - j| \geq 5.$$

Furthermore, for all $u \in L^p$, $\|\Delta_i u\|_{L^p} \leq \|u\|_{L^p}$ and $\|S_j u\|_{L^p} \leq C \|u\|_{L^p}$.

Definition 2 (Besov spaces). Let $s \in \mathbb{R}$, $1 \leq p$, and $r \leq +\infty$. The nonhomogeneous Besov space $B_{p,r}^s(\mathbb{S})$ is defined by

$$B_{p,r}^s(\mathbb{S}) = \left\{ f \in \mathcal{D}'(\mathbb{S}) : \|f\|_{B_{p,r}^s(\mathbb{S})} < \infty \right\}, \quad (20)$$

where $\|f\|_{B_{p,r}^s(\mathbb{S})} = \|2^{js} \Delta_j f\|_{l^r(L^p)} = \|(2^{js} \|\Delta_j f\|_{L^p})_{j \geq -1}\|_{l^r}$.

The following lemma summarizes some useful properties of $B_{p,r}^s$.

Lemma 3 (see [34–36]). Let $s \in \mathbb{R}$ and $1 \leq p, r, p_j, r_j \leq \infty$, $j = 1, 2$. Then the following properties hold:

- (1) Consider $B_{p_1,r_1}^{s_1} \hookrightarrow B_{p_2,r_2}^{s_2}$ if $p_1 \leq p_2$; consider $r_1 \leq r_2$ and $s_2 = s_1 - (1/p_1 - 1/p_2)$.
 $B_{p_1,r_1}^{s_1} \hookrightarrow B_{p_2,r_2}^{s_2}$ is locally compact if $s_2 < s_1$; $r_1 \leq r_2$.
- (2) $\forall s > 0$, $B_{p,r}^s \cap L^\infty$ is a Banach algebra. $B_{p,r}^s$ is a Banach algebra $\Leftrightarrow B_{p,r}^s \hookrightarrow L^\infty \Leftrightarrow s > 1/p$ (or $s \geq 1/p$ and $r = 1$).
- (3) Consider $\forall \theta \in [0, 1]$, $s = \theta s_1 + (1 - \theta)s_2$, and

$$\|f\|_{B_{p,r}^s} \leq C \|f\|_{B_{p,r}^{s_1}}^\theta \|f\|_{B_{p,r}^{s_2}}^{1-\theta}, \quad \forall f \in B_{p,r}^{s_1} \cap B_{p,r}^{s_2}. \quad (21)$$

- (4) Consider $\forall \theta \in (0, 1)$, $s_1 > s_2$, $s = \theta s_1 + (1 - \theta)s_2$, and

$$\|u\|_{B_{p,1}^s} \leq \frac{C(\theta)}{s_1 - s_2} \|u\|_{B_{p,\infty}^{s_1}}^\theta \|u\|_{B_{p,\infty}^{s_2}}^{1-\theta}, \quad \forall u \in B_{p,\infty}^{s_1}. \quad (22)$$

- (5) If $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $B_{p,r}^s$ and u_n converges to u in $\mathcal{D}'(\mathbb{S})$, then $u \in B_{p,r}^s$ and

$$\|u\|_{B_{p,r}^s} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{B_{p,r}^s}. \quad (23)$$

For $A = (\mu - \partial_x^2)$, we see that $\mathcal{F}_x(A^{-1}f) = (1/(\delta(\xi) + \xi^2)) \mathcal{F}_x f$, where $\delta(\xi) = 1$ if $\xi = 0$ and $\delta(\xi) = 0$ if $\xi \neq 0$. Therefore, we see that if $s \in \mathbb{R}$, $f \in B_{p,r}^s$ with $1 \leq p, r \leq \infty$,

then $\|A^{-1}f\|_{B_{p,r}^{s+2}} \leq \|f\|_{B_{p,r}^s}$, which means that A^{-1} is bounded from $B_{p,r}^s$ into $B_{p,r}^{s+2}$.

Now we recall some results of the transport equation (see [34] for the details).

Lemma 4 (a priori estimates). *Let $1 \leq p, r \leq \infty$, and $s > -\min\{1/p, 1 - 1/p\}$. Assume that $f_0 \in B_{p,r}^s$, $F \in L^1(0, T; B_{p,r}^s)$, and $\partial_x v \in L^1(0, T; B_{p,r}^{s-1})$ if $s > 1 + 1/p$ or $\partial_x v \in L^1(0, T; B_{p,r}^{1/p} \cap L^\infty)$ otherwise. If $f \in L^\infty(0, T; B_{p,r}^s) \cap C([0, T]; \mathcal{D}'(\mathbb{S}))$ solves the 1D linear transport equation*

$$\begin{aligned} f_t + v f_x &= F, \quad t > 0, \quad x \in \mathbb{S}, \\ f(0, x) &= f_0, \quad x \in \mathbb{S}, \end{aligned} \quad (24)$$

then there exists a constant C depending only on s, p, r such that the following statements hold:

$$V(t) = \begin{cases} \int_0^t \|v_x(\tau)\|_{B_{p,r}^{1/p} \cap L^\infty} d\tau, & \text{if } s < 1 + \frac{1}{p}, \\ \int_0^t \|v_x(\tau)\|_{B_{p,r}^{s-1}} d\tau, & \text{if } s > 1 + \frac{1}{p} \text{ or } \left(s = 1 + \frac{1}{p}, r = 1\right). \end{cases} \quad (27)$$

(2) If $f = v$, then (1) holds true for all $s > 0$ with $V(t) = \int_0^t \|v_x(\tau)\|_{L^\infty} d\tau$.

Lemma 5 (existence and uniqueness). *Let p, r, s, f_0 , and F be as in the statement of Lemma 4. Assume that $v \in L^p(0, T; B_{\infty,\infty}^{-M})$ for some $\rho > 1, M > 0$, and $v_x \in L^1(0, T; B_{p,r}^{s-1})$ if $s > 1 + 1/p$ or $s = 1 + 1/p, r = 1$, and $v_x \in L^1(0, T; B_{p,\infty}^{1/p} \cap L^\infty)$ if $s < 1 + 1/p$. Then (24) have a unique solution $f \in L^\infty(0, T; B_{p,r}^s) \cap C([0, T]; B_{p,1}^{s'})$ for any $s' < s$ and the inequalities of Lemma 4 hold true. Moreover, if $r < \infty$, then $f \in C([0, T]; B_{p,r}^s)$.*

We need the following lemmas.

Lemma 6 (Moser-type estimates; see [34, 37]). *Let $1 \leq p, r \leq +\infty$, and $s > 0$; then*

$$\begin{aligned} \|fg\|_{B_{p,r}^s} &\leq C \left(\|f\|_{B_{p,r}^s} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{B_{p,r}^s} \right) \\ &\quad \forall f, g \in B_{p,r}^s \cap L^\infty. \end{aligned} \quad (28)$$

Lemma 7 (see [34]). *For any $f \in B_{2,1}^{-1/2}$, $g \in B_{2,1}^{1/2}$, there holds the product estimate*

$$\|fg\|_{B_{2,\infty}^{-1/2}} \leq \|f\|_{B_{2,1}^{-1/2}} \|g\|_{B_{2,1}^{1/2}}. \quad (29)$$

(1) For all $t \in [0, T]$,

$$\begin{aligned} \|f\|_{B_{p,r}^s} &\leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau \\ &\quad + C \int_0^t V'(\tau) \|f(\tau)\|_{B_{p,r}^s} d\tau, \end{aligned} \quad (25)$$

and hence

$$\|f\|_{B_{p,r}^s} \leq e^{CV(t)} \left(\|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{p,r}^s} d\tau \right), \quad (26)$$

where

Lemma 8 (see [34]). *There is a constant $C > 0$ such that, for $s \in \mathbb{R}, \varepsilon > 0$, and $1 \leq p \leq \infty$,*

$$\begin{aligned} \|f\|_{B_{p,1}^s} &\leq C \frac{1+\varepsilon}{\varepsilon} \|f\|_{B_{p,\infty}^s} \ln \left(e + \frac{\|f\|_{B_{p,\infty}^{s+\varepsilon}}}{\|f\|_{B_{p,\infty}^s}} \right), \\ &\quad \forall f \in B_{p,\infty}^{s+\varepsilon}. \end{aligned} \quad (30)$$

Lemma 9 (Osgood lemma; see [34, 38]). *Let $\rho \geq 0$ be a measurable function, $\gamma > 0$ a locally integrable function, and μ a continuous and increasing function. Assume that, for some nonnegative real number c , the function ρ satisfies*

$$\rho(t) \leq c + \int_{t_0}^t \gamma(t') \mu(\rho(t')) dt'. \quad (31)$$

If $c > 0$, then $-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \leq \int_{t_0}^t \gamma(t') dt'$ with $\mathcal{M} = \int_x^1 (dr/\mu(r))$.

If $c = 0$ and μ verifies the condition $\int_0^1 (dr/\mu(r)) = +\infty$, then the function $\rho = 0$.

Now we establish some estimates on $F(u)$ and the difference of two solutions.

Lemma 10. *Let F be given in (9); then there holds the estimate:*

$$\begin{aligned} \|F(u)\|_{B_{2,1}^{3/2}} &\leq \|u\|_{B_{2,1}^{3/2}} + \|u_x\|_{L^\infty} \|u\|_{B_{2,1}^{3/2}} + \lambda \|u\|_{B_{2,1}^{3/2}} \\ &\leq \left(\|u\|_{B_{2,1}^{3/2}} + \lambda \right)^2, \quad \forall u \in B_{p,r}^s. \end{aligned} \quad (32)$$

Proof. Since $B_{2,1}^{1/2} \hookrightarrow L^\infty$, (32) follows from Lemma 6 easily and we omit the proof here. $\square$

Lemma 11. *For any u, v in $B_{2,1}^{3/2}$, the following estimates hold:*

$$\begin{aligned} & \|F(u) - F(v)\|_{B_{2,\infty}^{1/2}} \\ & \leq \left(\|u\|_{B_{2,1}^{3/2}} + \|v\|_{B_{2,1}^{3/2}} + \lambda \right) \|u - v\|_{B_{2,1}^{1/2}}. \end{aligned} \quad (33)$$

Proof. Equation (33) follows from (9) and Lemma 7. We omit the details here. $\square$

Lemma 12. *If $u, v \in C([0, T]; \mathcal{D}'(\mathbb{S})) \cap L^\infty(0, T; B_{2,1}^{3/2})$ are two solutions to (1) for some $T > 0$ with initial data $u_0, v_0 \in B_{2,1}^{3/2}$, respectively, then, for any $s' \in (1/2, 3/2)$ and $t \in [0, T]$,*

$$\begin{aligned} \|u - v\|_{B_{2,1}^{s'}} & \leq \|u_0 - v_0\|_{B_{2,\infty}^{1/2}}^{\theta \exp\{-CT\}}, \\ \theta & = \frac{3}{2} - s' \in (0, 1), \end{aligned} \quad (34)$$

where $C > 0$ is a constant depending on $\|u\|_{L^\infty(0,T;B_{2,1}^{3/2})}$ and $\|v\|_{L^\infty(0,T;B_{2,1}^{3/2})}$.

Proof. Since u, v are two solutions to (1) with initial data $u_0, v_0 \in B_{2,1}^{3/2}$, respectively, we know that $w = u - v$ satisfies

$$\begin{aligned} w_t + v \partial_x w & = -w \partial_x u - (F(u) - F(v)), \\ w(0, x) & = u_0 - v_0, \end{aligned} \quad (35)$$

where $F(\cdot)$ is given in (9). Let $U(t) = \int_0^t \|v_x(\tau)\|_{B_{2,\infty}^{1/2} \cap L^\infty} d\tau$, by Lemma 4; we have

$$\begin{aligned} \|w(t)\|_{B_{2,\infty}^{1/2}} & \leq \|w_0\|_{B_{2,\infty}^{1/2}} e^{CU(t)} \\ & + \int_0^t e^{CU(t)-CU(\tau)} \|\widehat{F}\|_{B_{2,\infty}^{1/2}} d\tau, \end{aligned} \quad (36)$$

where $\widehat{F} = -(u - v) \partial_x u - [F(u) - F(v)]$. Since $u, v \in L^\infty(0, T; B_{2,1}^{3/2})$, we see that

$$\begin{aligned} \|v(t)\|_{B_{2,1}^{3/2}} + \|u(t)\|_{B_{2,1}^{3/2}} & \leq M, \\ 0 \leq t \leq T, \text{ for some } M > 0, \end{aligned} \quad (37)$$

and therefore we have $U(t) \leq 1$. Since $B_{2,1}^s \hookrightarrow B_{2,\infty}^s$, by Lemma 11 and (37), we arrive at

$$\begin{aligned} & \|F(u) - F(v)\|_{B_{2,\infty}^{1/2}} \\ & \leq \left(\|u\|_{B_{2,1}^{1/2}} + \|v\|_{B_{2,1}^{1/2}} + \lambda \right) \|u - v\|_{B_{2,1}^{1/2}} \leq \|u - v\|_{B_{2,1}^{1/2}}. \end{aligned} \quad (38)$$

$$\|b(u - v) \partial_x u\|_{B_{2,\infty}^{1/2}} \leq \|u - v\|_{B_{2,1}^{1/2}} \|u\|_{B_{2,1}^{3/2}} \leq \|u - v\|_{B_{2,1}^{1/2}}.$$

Hence $\|\widehat{F}\|_{B_{2,\infty}^{1/2}} \leq \|u - v\|_{B_{2,1}^{1/2}}$. From Lemma 8 ($\varepsilon = 1, s = 1/2$) and (37), we obtain that

$$\begin{aligned} \|\widehat{F}\|_{B_{2,\infty}^{1/2}} & \leq \|u - v\|_{B_{2,1}^{1/2}} \\ & \leq \|u - v\|_{B_{2,\infty}^{1/2}} \ln \left(e + \frac{M}{\|u - v\|_{B_{2,\infty}^{1/2}}} \right), \end{aligned} \quad (39)$$

$$0 \leq t \leq T.$$

Consequently, for all $t \in [0, T]$,

$$\begin{aligned} & \|w(t)\|_{B_{2,\infty}^{1/2}} \\ & \leq \|w(0)\|_{B_{2,\infty}^{1/2}} \\ & + C \int_0^t \|w(\tau)\|_{B_{2,\infty}^{1/2}} \ln \left(e + \frac{M}{\|w(\tau)\|_{B_{2,\infty}^{1/2}}} \right) d\tau. \end{aligned} \quad (40)$$

Using the fact that $\ln(e + M/x) \leq \ln(e + 1)(1 - \ln(x/M))$ for $x \in (0, M]$, we can infer from (40) that

$$\begin{aligned} \frac{\|w(t)\|_{B_{2,\infty}^{1/2}}}{M} & \leq \frac{\|w(0)\|_{B_{2,\infty}^{1/2}}}{M} \\ & + C \int_0^t \frac{\|w(\tau)\|_{B_{2,\infty}^{1/2}}}{M} \left(1 - \ln \frac{\|w(\tau)\|_{B_{2,\infty}^{1/2}}}{M} \right) d\tau, \end{aligned} \quad (41)$$

$$\forall t \in [0, T].$$

Thanks to Lemma 9, we obtain

$$\frac{\|w(t)\|_{B_{2,\infty}^{1/2}}}{eM} \leq \left(\frac{\|w(0)\|_{B_{2,\infty}^{1/2}}}{eM} \right)^{\exp\{-CT\}}. \quad (42)$$

Therefore, we have

$$\|w(t)\|_{B_{2,\infty}^{1/2}} \leq \|w(0)\|_{B_{2,\infty}^{1/2}}^{\exp\{-CT\}}. \quad (43)$$

If $1/2 < s' < 3/2$, by interpolation, the embedding $B_{2,1}^{1/2} \hookrightarrow B_{2,\infty}^{1/2}$, and (37), we obtain

$$\begin{aligned} \|w\|_{B_{2,1}^{s'}} & \leq \|w\|_{B_{2,\infty}^{1/2}}^\theta \|w\|_{B_{2,\infty}^{3/2}}^{1-\theta} \\ & \leq \|w\|_{B_{2,\infty}^{1/2}}^\theta \left(\|u\|_{B_{2,1}^{3/2}} + \|v\|_{B_{2,1}^{3/2}} \right)^{1-\theta} \leq \|w\|_{B_{2,\infty}^{1/2}}^\theta, \end{aligned} \quad (44)$$

where $\theta = 3/2 - s'$. Using (43), we obtain that

$$\|w\|_{B_{2,1}^{s'}} \leq \|w\|_{B_{2,\infty}^{1/2}}^\theta \leq \|w(0)\|_{B_{2,\infty}^{1/2}}^{\theta \exp\{-CT\}}. \quad (45)$$

We finish the proof by combining (43) and (45). $\square$

Lemma 13 (see [31]). *Denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $\{a_k\}_{k \in \overline{\mathbb{N}}}$ be a sequence of continuous bounded functions on $[0, T] \times \mathbb{S}$ with*

$\sup_{k \in \mathbb{N}} \|a_k(t)\|_{B_{2,1}^{3/2}} \leq \alpha(t)$ for some $\alpha(t) \in L^1(0, T)$ and $T > 0$. Assume that v_k solves

$$\begin{aligned} \partial_t v_k + a_k \partial_x v_k &= g, \quad t \geq 0, \quad x \in \mathbb{S}, \\ v_k(0) &= v_0, \quad x \in \mathbb{S}, \end{aligned} \quad (46)$$

with $v_0 \in B_{2,1}^{3/2}$, $g \in L^1(0, T; B_{2,1}^{1/2})$. If $a_n \rightarrow a_\infty$ in $L^1(0, T; B_{2,1}^{1/2})$, then $v_n \rightarrow v_\infty$ in $C([0, T]; B_{2,1}^{1/2})$.

3. Proof of Theorem 1

In this section, we prove Theorem 1 by the following several steps.

3.1. Approximate Solutions and Their Uniform Bounds. Starting from $u_1 = 0$ and by induction, we define a sequence of smooth functions $\{u_n\}$, $n \in \mathbb{N}$ by solving the following transport equation iteratively:

$$\begin{aligned} \partial_t u_{n+1} + u_n \partial_x u_{n+1} &= -F(u_n), \\ u_{n+1}(0, x) &= S_{n+1} u_0, \end{aligned} \quad (47)$$

where $F(u) = \sum_{1 \leq i \leq 3} F_i(u)$ and $F_i(u)$ given in (9). Since all the data belong to $B_{2,1}^\infty$, from Lemma 5 and by induction, we can show that, for all $n \geq 1$, the above equation has a global solution u_{n+1} belonging to $C(\mathbb{R}^+, B_{2,1}^\infty)$. For $n \in \mathbb{N}$, set $U_n(t) = \int_0^t \|u_n\|_{B_{2,1}^{3/2}} d\tau$. Since $\|u_{n+1}(0, x)\|_{B_{2,1}^{3/2}} \leq \|u_0\|_{B_{2,1}^{3/2}}$, from the estimate (26) of Lemmas 4 and 10, we have

$$\begin{aligned} \|u_{n+1}(t)\|_{B_{2,1}^{3/2}} &\leq e^{CU_n(t)} \left(\|u_0\|_{B_{2,1}^{3/2}} \right. \\ &\quad \left. + C \int_0^t e^{-CU_n(\tau)} \left(\|u_n\|_{B_{2,1}^{3/2}} + \lambda \right)^2 d\tau \right). \end{aligned} \quad (48)$$

Reset the constant C on the above estimate such that

$$\begin{aligned} \|u_{n+1}(t)\|_{B_{2,1}^{3/2}} + \lambda &\leq e^{CU_n(t)} \left(\|u_0\|_{B_{2,1}^{3/2}} + \lambda \right. \\ &\quad \left. + C \int_0^t e^{-CU_n(\tau)} \left(\|u_n\|_{B_{2,1}^{3/2}} + \lambda \right)^2 d\tau \right). \end{aligned} \quad (49)$$

Let $T > 0$ such that $2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)T < 1$; we claim that

$$\begin{aligned} \|u_n(t)\|_{B_{2,1}^{3/2}} + \lambda &\leq \frac{\|u_0\|_{B_{2,1}^{3/2}} + \lambda}{1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)t} \\ &\quad \forall t \in [0, T]. \end{aligned} \quad (50)$$

Assume (50) is true for n . We now prove that it also holds true for $n + 1$. Actually, by (50), we have

$$\begin{aligned} e^{CU_n(t) - CU_n(\tau)} &\leq \exp \left\{ C \int_\tau^t \frac{(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)}{(1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)t') dt'} \right\} \\ &= \left(\frac{1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)\tau}{1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)t} \right)^{1/2}. \end{aligned} \quad (51)$$

From the above equation, we see clearly that when $\tau = 0$, $U_n(\tau) = 0$, we have

$$e^{CU_n(t)} \leq \frac{1}{(1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)t)^{1/2}}. \quad (52)$$

Combining (49), (50), (51), and (52) gives rise to

$$\begin{aligned} \|u_{n+1}(t)\|_{B_{2,1}^{3/2}} + \lambda &\leq \frac{1}{(1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)t)^{1/2}} \left[\|u_0\|_{B_{2,1}^{3/2}} + \lambda \right. \\ &\quad \left. - \frac{(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)}{2} \int_0^t \frac{d(1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)\tau)}{(1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)\tau)^{3/2}} \right] \\ &\leq \frac{\|u_0\|_{B_{2,1}^{3/2}} + \lambda}{1 - 2C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)t}. \end{aligned} \quad (53)$$

Hence (50) is true for $n + 1$. Setting $T = 1/4C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)$, for $n \in \mathbb{N}$, we can conclude that u_n exists for $0 \leq t \leq T$ and satisfies the bound

$$\|u_n(t)\|_{B_{2,1}^{3/2}} \leq 2\|u_0\|_{B_{2,1}^{3/2}} + \lambda \leq 1, \quad 0 \leq t \leq T. \quad (54)$$

Furthermore, for $t \in [0, T]$, using (54) yields

$$\|\partial_t u_n\|_{B_{2,1}^{1/2}} \leq \|u_n\|_{B_{2,1}^{1/2}} \|\partial_x u_{n+1}\|_{B_{2,1}^{1/2}} + \|F(u_n)\|_{B_{2,1}^{3/2}} \leq 1. \quad (55)$$

Hence we conclude that $\{u_n\} \subset C([0, T]; B_{2,1}^{3/2}) \cap C^1([0, T]; B_{2,1}^{1/2})$ is uniformly bounded.

3.2. Convergence of the Approximate Solutions. We now prove that $\{u_n\}$ is a Cauchy sequence in $C([0, T]; B_{2,1}^{1/2})$. Firstly, we see that the estimate of $u_{n+m+1} - u_{n+1}$ is essential in

the derivation of Lemma 12. Similar to Lemma 12, we obtain that, for $m, n \in \mathbb{N}$,

$$\begin{aligned} & \| (u_{n+m+1} - u_{n+1}) (t) \|_{B_{2,\infty}^{1/2}} \leq 2^{-n} \\ & + \int_0^t \| (u_{n+m} - u_n) (\tau) \|_{B_{2,\infty}^{1/2}} \\ & \cdot \ln \left(e + \frac{\| (u_{n+m} - u_n) (\tau) \|_{B_{2,\infty}^{3/2}}}{\| (u_{n+m} - u_n) (\tau) \|_{B_{2,\infty}^{1/2}}} \right) d\tau, \end{aligned} \quad (56)$$

where we used the following estimate and we have (see, e.g., in [39], page 2142)

$$\begin{aligned} & \| (u_{n+m+1} - u_{n+1}) (0) \|_{B_{2,\infty}^{1/2}} \leq \| (u_{n+m+1} - u_{n+1}) (0) \|_{B_{2,1}^{1/2}} \\ & \leq C 2^{-n} \| u_0 \|_{B_{2,1}^{3/2}}. \end{aligned} \quad (57)$$

Let $H_{n,m}(t) = \| (u_{n+m} - u_n)(t) \|_{B_{2,\infty}^{1/2}}$. Since $\{u_n\} \subset C([0, T]; B_{2,1}^{3/2}) \cap C^1([0, T]; B_{2,1}^{1/2})$ is uniformly bounded, from (56), we obtain that

$$\begin{aligned} & H_{n+1,m}(t) \\ & \leq \left(2^{-n} + \int_0^t H_{n,m}(\tau) \ln \left(e + \frac{1}{H_{n,m}(\tau)} \right) d\tau \right), \end{aligned} \quad (58)$$

$t \in [0, T]$.

As the function $x \ln(e + 1/x)$ is nondecreasing, we see that $H_n(t) \triangleq \sup_{m \in \mathbb{N}} H_{n,m}(t)$ satisfies

$$\begin{aligned} & H_{n+1}(t) \leq \left(2^{-n} + \int_0^t H_n(\tau) \ln \left(e + \frac{1}{H_n(\tau)} \right) d\tau \right), \end{aligned} \quad (59)$$

$t \in [0, T]$.

Let $\tilde{H}(t) = \limsup_{n \rightarrow \infty} H_n(t)$. For any given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n > N$, we have $H_n(t) \leq \tilde{H}(t) + \varepsilon$, from which it follows that

$$\begin{aligned} & H_{n+1}(t) \\ & \leq \left(2^{-n} + \int_0^t (\tilde{H}(\tau) + \varepsilon) \ln \left(e + \frac{1}{\tilde{H}(\tau) + \varepsilon} \right) d\tau \right), \end{aligned} \quad (60)$$

$t \in [0, T], n > N$.

Hence we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} H_n(t) \leq \limsup_{n \rightarrow \infty} \left(2^{-n} \right. \\ & \left. + \int_0^t (\tilde{H}(\tau) + \varepsilon) \ln \left(e + \frac{1}{\tilde{H}(\tau) + \varepsilon} \right) d\tau \right) \\ & = \int_0^t (\tilde{H}(\tau) + \varepsilon) \ln \left(e + \frac{1}{\tilde{H}(\tau) + \varepsilon} \right) d\tau. \end{aligned} \quad (61)$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\tilde{H}(t) \leq \int_0^t \tilde{H}(\tau) \ln \left(e + \frac{1}{\tilde{H}(\tau)} \right) d\tau. \quad (62)$$

Since $x \ln(e + 1/x)$ is nondecreasing and $\int_0^1 (1/(x \ln(e + 1/x))) dx = +\infty$, we can infer from Lemma 9 that $\tilde{H}(t) = 0$; in other words, $\{u_n\}$ is a Cauchy sequence in $C([0, T]; B_{2,\infty}^{1/2})$. By (5) in Lemma 3, we see

$$\begin{aligned} & \| u_{n+m} - u_n \|_{B_{2,1}^1} \leq \| u_{n+m} - u_n \|_{B_{2,\infty}^{1/2}}^{1/2} \| u_{n+m} - u_n \|_{B_{2,\infty}^{3/2}}^{1/2} \\ & \leq \| u_{n+m} - u_n \|_{B_{2,\infty}^{1/2}}^{1/2}. \end{aligned} \quad (63)$$

Hence $\{u_n\}_{n \in \mathbb{N}}$ is actually Cauchy sequence in $C([0, T]; B_{2,1}^1)$ and therefore $\{u_n\}$ converges to some function $u \in C([0, T]; B_{2,1}^1)$.

3.3. Existence, Regularity, and Uniqueness of the Solution. Since $\|u_n\|_{B_{2,1}^{3/2}}$ is uniformly bounded by $2\|u_0\|_{B_{2,1}^{3/2}}$, property (5) in Lemma 3 guarantees that $\|u\|_{B_{2,1}^{3/2}} \leq 2\|u_0\|_{B_{2,1}^{3/2}}$, which means that $u \in L^\infty(0, T; B_{2,1}^{3/2})$. If $s' < 1$, we have

$$\|u_n - u\|_{B_{2,1}^{s'}} \leq \|u_n - u\|_{B_{2,1}^1}. \quad (64)$$

If $1 < s' < 3/2$, by interpolation again, we have

$$\begin{aligned} & \|u_n - u\|_{B_{2,1}^{s'}} \leq \|u_n - u\|_{B_{2,1}^1}^\theta \|u_n - u\|_{B_{2,1}^{3/2}}^{1-\theta} \\ & \leq \|u_n - u\|_{B_{2,1}^1}^\theta \|u_0\|_{B_{2,1}^{3/2}}^{1-\theta}, \end{aligned} \quad (65)$$

where $\theta = 3 - 2s'$. From (64) and (65), we see that $u_k \rightarrow u$ in $C([0, T]; B_{2,1}^{s'})$ for all $s' < 3/2$. Taking limits to (47), we can deduce that u indeed solves (8). Since $r = 1 < \infty$, from Lemmas 5 and 10, we know that $u \in C([0, T]; B_{2,1}^{3/2})$. By (8), we obtain that $u_t \in C([0, T]; B_{2,1}^{1/2})$. The uniqueness is a corollary of Lemma 12.

3.4. Hölder Continuity of the Solution Map. The continuity of the solution map can be obtained by following the steps in [31] and using Lemma 13 and we omit the details in this paper. Now we consider the Hölder continuity of the solution map. For the initial data $u_0 \in B(0, R) \subset B_{2,1}^{3/2}$, by (13), we see that the lifespan T_u of the corresponding solution u to (8) satisfies

$$T_u > \frac{1}{4C(\|u_0\|_{B_{2,1}^{3/2}} + \lambda)} > \frac{1}{4CR + 4C\lambda} \triangleq \tilde{T}, \quad (66)$$

where $\tilde{T}$ does not depend on u . Therefore, we can find $\tilde{T} > 0$ such that, for all $u_0 \in B(0, R) \subset B_{2,1}^{3/2}$, the corresponding solution $u \in C([0, \tilde{T}]; B_{2,1}^{3/2})$. Directly from (13) and Lemma 12, (14) is proved.

We complete the proof of Theorem 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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