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Research Article

Constant Mean Curvature Spacelike Surfaces in Lorentzian Warped Products

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We characterize the spacelike slices of a Lorentzian warped product as the only constant mean curvature spacelike surfaces under suitable geometrical and physical assumptions. As a consequence of our study, we derive a Bernstein-type result which widely improves and extends the state-of-the-art results in this setting.

1. Introduction

Spacelike hypersurfaces in ($n \geq 3$)-dimensional Lorentzian spacetimes are geometrical objects of great physical and mathematical interest. Roughly speaking, each of them represents the physical space in a given instant of a time function. In electromagnetism, a spacelike hypersurface is an initial data set that univocally determines the future of the electromagnetic field which satisfies the Maxwell equations [1, Theorem 3.11.1], analogously, for the simple matter equations [1, Theorem 3.11.2]. In Causality Theory, the mere existence of a particular spacelike hypersurface implies that the spacetime obeys a certain causal property. Let us remark that the completeness of a spacelike hypersurface is required whenever we study its global properties, and, also, from a physical viewpoint, completeness implies that the whole physical space is taken into consideration. To know more details on the relevance of constant mean curvature spacelike hypersurfaces in General Relativity see [2].

From a mathematical point of view, the interest of spacelike hypersurfaces is motivated by their nice Bernstein-type properties. In fact, spacelike hypersurfaces of constant mean curvature (CMC) in ($n \geq 3$)-dimensional spacetime are critical points of the area functional under a certain volume constraint [3]. When the ambient spacetime is the Lorentz-Minkowski spacetime $\mathbb{L}^n$, many results have been obtained from different viewpoints. For instance, as an application of the generalized maximum principle due to Omori-Yau [4, 5] and of the Calabi-Bernstein theorem, Aiyama [6] and Xin [7] (see also [8] for a first weaker version given by Palmer) obtained simultaneously and independently a characterization of spacelike hyperplanes as the only complete spacelike hypersurfaces with constant mean curvature (CMC) in the Lorentz-Minkowski spacetime whose hyperbolic image is bounded. As another application of the previous mentioned results, a characterization of the spacelike hyperplanes as the only complete CMC spacelike hypersurfaces in the Lorentz-Minkowski spacetime which lie between two parallel spacelike hyperplanes has been given by Aledo and Alías [9].

In this work, we will deal with spacelike surfaces but in a more general setting (some previous works in this scene, though with a completely different approach, are [10–12]; see Sections 4 and 5). Indeed, we will consider CMC spacelike surfaces in three-dimensional Lorentzian warped products, also called generalized Robertson-Walker spacetimes (see [13]). Note that in our study we make use (see Lemmas 1 and 7) of Lemmas 12 and 17 in [14], where the authors (with a completely different approach) study complete spacelike constant mean curvature hypersurfaces in warped products with parabolic fiber. Nonetheless, unlike our current study, all the principal results in [14] require the Timelike Convergence

Condition and the hypothesis of boundedness of the hyperbolic angle between the normal vector field of the spacelike hypersurface and the timelike coordinate vector field of the warped product.

Finally, let us remark that although three-dimensional spacetimes may be considered unrealistic from a physical point of view, they have been deeply studied from a purely geometrical perspective. In fact, they can be used to light suitable extensions of geometrical properties to usual four-dimensional relativistic models.

The paper is organized as follows. In Section 2 we revise some notions regarding spacelike surfaces in a 3-dimensional Lorentzian space and introduce the notation to be used. We continue, in Section 3, undertaking some technical computations. Section 4 is devoted to present our main results. Thus, in Theorems 2 and 5 we characterize the CMC spacelike surfaces of a Lorentzian warped product as the only spacelike slices under suitable geometrical and physical assumptions. As a consequence of our study, in Section 5 we derive a Bernstein-type result (Theorem 8) which widely improves and extends the state-of-the-art results in this setting.

2. Preliminaries

Let (F, g_F) be a connected Riemannian surface, I an open interval in $\mathbb{R}$, and f a positive smooth function defined on I. Then, the product manifold $I \times F$ endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^* \left(dt^2\right) + f\left(\pi_I\right)^2 \pi_F^* \left(g_F\right), \tag{1}$$

where π_I and π_F denote the projections onto I and F, respectively, is called a *Lorentzian warped product* with *fiber* (F, g_F), *base* $(I, -dt^2)$, and *warping function* f. Along this paper we will represent this 3-dimensional Lorentzian manifold by $\overline{M} = I \times_f F$.

The coordinate vector field $\partial_t := \partial/\partial t$ globally defined on $\overline{M}$ is (unitary) timelike, and so $\overline{M}$ is time-orientable. We will also consider on $\overline{M}$ the conformal closed timelike vector field $\xi := f(\pi_I)\partial_t$. From the relationship between the Levi-Civita connections of $\overline{M}$ and those of the base and the fiber [15, Corollary 7.35], it follows that

$$\overline{\nabla}_X \xi = f'\left(\pi_I\right) X \tag{2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (1). Thus, ξ is conformal with $\mathscr{L}_\xi \langle , \rangle = 2f'(\pi_I)\langle , \rangle$ and its metrically equivalent 1-form is closed.

Recall that a Lorentzian manifold obeys the *Null Convergence Condition (NCC)* if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies $\overline{\mathrm{Ric}}(X, X) \geq 0$, for all null vector $X \in \mathfrak{X}(\overline{M})$. In the case, when $\overline{M} = I \times_f F$ is a Lorentzian warped product with a 2-dimensional fiber, it can be easily checked that $\overline{M}$ obeys the NCC if and only if

$$\frac{K^F\left(\pi_F\right)}{f^2} - \left(\log f\right)'' \geq 0, \tag{3}$$

where K^F stands for the Gaussian curvature of (F, g_F) (see, e.g., [11]).

Throughout this paper we will assume that (F, g_F) is a complete Riemannian surface with *finite total curvature*; that is, its Gaussian curvature K^F satisfies that

$$\int_F \max\left\{0, -K^F\right\} dA_F < +\infty, \tag{4}$$

where dA_F is the area element of (F, g_F) and the integral above is defined with a compact exhaustion procedure.

A smooth immersion $\psi : S \to \overline{M}$ of a (connected) surface S is said to be a *spacelike surface* if the induced metric via ψ is a Riemannian metric g on S.

Since $\overline{M}$ is time-orientable we can take, for each spacelike surface $\psi : S \to \overline{M}$, a unique unitary timelike vector field $N \in \mathfrak{X}^\perp(S)$ globally defined on S with the same time orientation as $-\partial_t$, such that $\overline{g}(N, \partial_t) > 0$. From the wrong-way Cauchy-Schwarz inequality (see, e.g., [15, Proposition 5.30]), we have $\overline{g}(N, \partial_t) \geq 1$, and the equality holds at a point $p \in S$ if and only if $N = -\partial_t$ at p. In fact, $\overline{g}(N, \partial_t) = \cosh\theta$, where θ is the *hyperbolic angle*, at each point, between the unit timelike vectors $-\partial_t$ and N.

We will denote by A and $H := -(1/n)\mathrm{trace}(A)$ the *shape operator* and the *mean curvature function* associated with N. A spacelike surface with constant H is called a constant mean curvature (CMC) spacelike surface.

In any Lorentzian warped product $\overline{M}$ there is a remarkable family of spacelike surfaces, namely, its spacelike *slices* $\{t_0\} \times F$, $t_0 \in I$. Note that a spacelike surface in $\overline{M}$ is a (piece of) spacelike slice if and only if the function $\tau := \pi_I \circ \psi$ is constant. Furthermore, a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the hyperbolic angle θ vanishes identically.

3. Set Up

Let $\psi : S \to \overline{M}$ be a spacelike hypersurface in a Lorentzian warped product $\overline{M} = I \times_f F$. If we put $\partial_t^\top = \partial_t + \overline{g}(\partial_t, N)N$, the tangential part of ∂_t, on S, it is easy to check that the gradient of $\tau = \pi_I \circ \psi$ on S is given by

$$\nabla\tau = -\partial_t^\top. \tag{5}$$

Now, from the Gauss formula and using $\xi^\top = f(\tau)\partial_t^\top$ and (5), the Laplacian of τ gives

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\left\{2 + |\nabla\tau|^2\right\} - 2H\overline{g}\left(N, \partial_t\right), \tag{6}$$

where $f'(\tau) := f' \circ \tau$.

On the other hand, in [11, Formula 10] the Gaussian curvature K of a spacelike surface in $\overline{M}$ is computed and can be written as

$$\begin{aligned} K = {} & \left(\frac{f'(\tau)^2}{f(\tau)^2} - H^2\right) \\ & + \left\{\frac{K^F\left(\pi_F\right)}{f(\tau)^2} - \left(\log f\right)''(\tau)\right\}|\nabla\tau|^2 + \frac{K^F\left(\pi_F\right)}{f(\tau)^2} \\ & + \left(\frac{1}{2}\mathrm{trace}\left(A^2\right) - H^2\right). \end{aligned} \tag{7}$$

We will use the following result [14, Lemma 12].

Lemma 1. *Let $\psi : S \to \overline{M}$ be a CMC complete spacelike surface in $\overline{M} = I \times_f F$ whose warping function f is such that $(\log f)'' \leq 0$. If the Gaussian curvature of S is bounded from below and S is contained between two slices, then*

$$H = -\frac{f'(\tau)}{f(\tau)}. \tag{8}$$

Finally, note that, given a complete spacelike surface $\psi : S \to \overline{M}$, the projection $\phi := \pi_F \circ \psi : S \to F$ is a covering map provided that $\sup f(\tau) < +\infty$ [13]. In particular, if the fiber F is simply connected then ϕ is a diffeomorphism. Then, the area elements dA_S and dA_F of (S, g) and (F, g_F), respectively, satisfy the relationship,

$$\phi^*(dA_F) = \frac{\cosh\theta}{f(\tau)^2} dA_S; \tag{9}$$

see [16, equation 8].

4. Main Results

Next, we will characterize the spacelike slices under suitable geometrical and physical assumptions.

Theorem 2. *Let (F, g_F) be a simply connected Riemannian surface with finite total curvature and whose Gaussian curvature is bounded from below. Assume that the Lorentzian warped product $\overline{M} = I \times_f F$ satisfies the NCC and that the warping function f is such that $(\log f)'' \leq 0$.*

Let $\psi : S \to \overline{M}$ be a CMC complete spacelike surface in $\overline{M} = I \times_f F$. If S is contained between two slices, then S is a spacelike slice.

Proof. Since $\overline{M}$ satisfies the NCC, from (7) we get by using (3) that the Gaussian curvature K of S is bounded from below when K^F is bounded from below. To see that, note that $f'(\tau)^2/f(\tau)^2$ is bounded because S is contained between two slices and that, from the Cauchy-Schwarz inequality, it is $((1/2)\operatorname{trace}(A^2) - H^2) \geq 0$. Then, from Lemma 1 we get that the (constant) mean curvature of S is given by (8).

With all of this, the Gaussian curvature of S (7) satisfies that

$$K \geq \frac{K^F(\pi_F)}{f(\tau)^2}. \tag{10}$$

Now, since F has finite total curvature we have, using (9), that

$$\begin{aligned}&\int_S \max\{0, -K\}\, dA_S \\ &\quad \leq \int_F \frac{1}{\cosh(\theta \circ \phi^{-1})} \max\left\{0, -K^F\left(\pi_F \circ \phi^{-1}\right)\right\} dA_F \\ &\quad < +\infty.\end{aligned} \tag{11}$$

Therefore, (S, g) has finite total curvature and, in particular, (S, g) is parabolic.

Let us consider the function F on I given by

$$F(t) := \int_{t_0}^{t} f(s)\, ds, \quad t_0 \in I. \tag{12}$$

Using (5), (6), and (8), the Laplacian of $F(\tau)$ can be computed to obtain

$$\Delta F(\tau) = 2Hf(\tau)(1 + \cosh\theta), \tag{13}$$

which is signed because H is constant. Since $F(\tau)$ is bounded (because S is contained between two slices), it follows from the parabolicity of (S, g) that $F(\tau)$ is constant. Then

$$0 = \nabla F(\tau) = f(\tau)\nabla\tau \tag{14}$$

and consequently $\nabla\tau = -\partial_t^\top = 0$.

If we put $N^F = N + \overline{g}(N, \partial_t)\partial_t$, we get from $\overline{g}(N, N) = -1 = \overline{g}(\partial_t, \partial_t)$ that

$$0 = \left|\partial_t^\top\right|^2 = \left|N^F\right|^2 = \sinh^2\theta; \tag{15}$$

that is, the hyperbolic angle vanishes identically on S and therefore S is a spacelike slice. □

Remark 3. As we commented in Section 2, we need $\sup f(\tau) < +\infty$ to assure that ϕ is a diffeomorphism. In Theorem 2 this inequality holds since τ is bounded.

Note that the previous theorem is a wide extension of [9, Theorem 1]. To see this, it is enough to take a suitable splitting of $\mathbb{L}^3$.

A Lorentzian warped product $\mathbb{R} \times_{e^t} F$, where F is an n-dimensional Riemannian manifold, is called a *steady state spacetime* (see [17] for more details). The following result constitutes a partial extension of [17, Theorem 8], when $n = 2$.

Corollary 4. *Let $\mathbb{R} \times_{e^t} F$ be a 3-dimensional type steady state spacetime whose fiber is a simply connected Riemannian surface with finite total curvature and whose Gaussian curvature is bounded from below. Then, the only complete CMC spacelike surfaces contained between two slices are the spacelike slices.*

Alternatively, the assumption of NCC can be changed for the one of bounded hyperbolic angle as follows.

Theorem 5. *Let (F, g_F) be a simply connected Riemannian surface with finite total curvature and whose Gaussian curvature is bounded from below. Assume that the warping function f of the Lorentzian warped product $\overline{M} = I \times_f F$ is such that $(\log f)'' \leq 0$.*

Let $\psi : S \to \overline{M}$ be a CMC complete spacelike surface in $\overline{M} = I \times_f F$. If the hyperbolic angle of ψ is bounded and S is contained between two slices, then S is a spacelike slice.

Proof. Reasoning as in the proof of Theorem 2, we get now that

$$K \geq \frac{K^F(\pi_F)}{f(\tau)^2}\left(|\nabla\tau|^2 + 1\right) = \frac{K^F(\pi_F)}{f(\tau)^2}\cosh^2\theta, \tag{16}$$

where we have also used (15). Then

$$\begin{aligned}\int_S \max\{0,-K\}\,dA_S &\\ \leq \int_F \cosh\left(\theta\circ\phi^{-1}\right)\max\left\{0,-K^F\left(\pi_F\circ\phi^{-1}\right)\right\}dA_F &\\ < +\infty &\end{aligned} \tag{17}$$

and the proof finishes as in Theorem 2. □

Notice that the comment in Remark 3 is also valid for Theorem 5. On the other hand, it is worth pointing out that the boundedness of the surface and the one of the hyperbolic angle have no relation at all [12, Remark 5.3]. Hence, in Theorem 5, from the assumption of boundedness of the surface the boundedness of the hyperbolic angle cannot be derived and so this last condition must be required in order to conclude that S is a spacelike surface.

The assumption of boundedness of the hyperbolic angle admits the following physical interpretation. Along S there exist two families of instantaneous observers, $\mathcal{T}_p = -\partial_t(p)$, $p \in S$ (the sign minus depends on the time orientation chosen here), and the normal observers N_p, $p \in S$. The quantities $\cosh\theta(p)$ and $v(p) := (1/\cosh\theta(p))N_p^F$ are, respectively, the energy and the velocity that $\mathcal{T}_p$ measures for N_p, and we have $\|v\| = \tanh\theta$ on S. Therefore, the relative speed function satisfies $\|v\| < \lambda$ and so it does not approach the speed of light in vacuum on S.

In order to apply the previous results to more general cases we need an extra topological hypothesis.

Let us consider a GRW spacetime $M = I\times_f F$, whose fiber is a 2-dimensional complete Riemannian manifold. Recall that if the warping function is bounded on a complete spacelike surface $x : S \to M$, then

$$\widetilde{\pi} := \pi_F\circ x : S \longrightarrow F \tag{18}$$

is a covering map [13].

Now, let us take $p_0 \in F$ and $\widetilde{p}_0 \in S$ such that $\phi(\widetilde{p}_0) = p_0$. Denote by

$$A = \frac{\pi_1(F,p_0)}{\widetilde{\pi}_*(\pi_1(S,\widetilde{p}_0))} \tag{19}$$

the set of all left cosets of $\widetilde{\pi}_*(\pi_1(S,\widetilde{p}_0))$ in $\pi_1(F,p_0)$. It is well-known that

$$\sharp\left(\widetilde{\pi}^{-1}(p_0)\right) = \sharp(A). \tag{20}$$

Now, let us assume $\sharp(A) < \infty$. Thus, S covers $\sharp(A)$-times the fiber. Moreover, taking into account the reasoning in Theorems 2 and 5, it is not difficult to see that S also has finite total curvature under the same assumptions.

Remark 6. (a) Consider $F = \mathbb{S}^1\times\mathbb{R}$, endowed with its canonical product metric, and f, an arbitrary positive smooth function. Set $S = F$. For each positive integer m, let $x : S \to I\times_f F$ be the spacelike immersion given by $x(e^{i\theta}, s) = (t_0, e^{im\theta}, s)$. This example shows that there exist surfaces with arbitrary $\sharp(A)$. (b) However, we cannot force the fact that the fundamental group of the fiber is finite unless it is trivial. This is due to the fact that the fundamental group of any noncompact surface must be free (see, e.g., [18]).

We end this section pointing out that Theorems 2 and 5 extend and improve widely the conclusion given in [11, Corollary 6.11].

5. Calabi-Bernstein-Type Problems

Let (F, g_F) be a (noncompact) complete Riemannian surface, $I \subset \mathbb{R}$ an open interval in $\mathbb{R}$, and f a positive smooth function on I. For each $u \in C^\infty(F)$ such that $u(F) \subset I$ we can consider its graph $\Sigma_u := \{(u(p), p) : p \in F\}$ in the Lorentzian warped product $\overline{M} = I\times_f F$. The graph of u inherits a metric, represented on F by $g_u := -du^2 + f(u)^2 g_F$, which is Riemannian if and only if u satisfies $g_F(Du, Du) < f(u)^2$ everywhere on F, where Du denotes the gradient of u in (F, g_F). In this case, the graph is a spacelike surface in $\overline{M}$. Note that $\tau(u(p), p) = u(p)$ for any $p \in F$, and so, on the spacelike graph, τ and u can be naturally identified.

When Σ_u is spacelike, the unitary normal vector field on Σ_u satisfying $\overline{g}(N, \partial_t) > 0$ is

$$N = -\frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}}\left(f(u)^2\partial_t + Du\right), \tag{21}$$

and the corresponding mean curvature function is

$$\begin{aligned}H(u) = &-\operatorname{div}\left(\frac{Du}{2f(u)\sqrt{f(u)^2-|Du|^2}}\right)\\ &- \frac{f'(u)}{2\sqrt{f(u)^2-|Du|^2}}\left(2+\frac{|Du|^2}{f(u)^2}\right).\end{aligned} \tag{22}$$

Our aim in this section is to study the *entire* solutions of the CMC spacelike surface equation

$$\begin{aligned}&\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2-|Du|^2}}\right)\\ &\quad= -2H - \frac{f'(u)}{\sqrt{f(u)^2-|Du|^2}}\left(2+\frac{|Du|^2}{f(u)^2}\right),\end{aligned} \tag{E.1}$$

$$|Du| < \lambda f(u), \quad 0<\lambda<1, \tag{E.2}$$

under suitable geometrical and analytical assumptions.

Note that the constraint (E.2) can be written as

$$\cosh\theta < \frac{1}{\sqrt{1-\lambda^2}}, \tag{23}$$

where θ is the hyperbolic angle of Σ_u. Therefore, (E.2) implies that Σ_u has bounded hyperbolic angle. Moreover, this constraint means that the differential equation (E.1) is, in fact, uniformly elliptic.

An important fact when we deal with entire graphs in this context is that, in general, the induced metric g_u is not complete. In this sense, we will make use of the following result [14, Lemma 17].

Lemma 7. *Let* $\overline{M} = I \times_f F$ *be a Lorentzian warped product whose fiber is a (noncompact) complete Riemannian surface. Consider a function* $u \in C^\infty(F)$, *with* $\mathrm{Im}(u) \subseteq I$, *such that the entire graph* $\Sigma_u = \{(u(p), p) : p \in F\} \subset \overline{M}$ *endowed with the metric* $g_u = -du^2 + f(u)^2 g_F$ *is spacelike. If the hyperbolic angle of* Σ_u *is bounded and* $\inf f(u) > 0$, *then the graph* (Σ_u, g_{Σ_u}) *is complete or, equivalently, the Riemannian surface* (F, g_u) *is complete.*

Now, from Theorem 5 and taking into account that every graph in a Lorentzian product is diffeomorphic to its fiber, we can state the following.

Theorem 8. *Let* (F, g) *be a simply connected Riemannian surface with finite total curvature and whose Gaussian curvature is bounded from below, and let* $f : I \to \mathbb{R}$ *be a smooth positive function such that* $(\log f)'' \leq 0$. *Then, the only bounded solutions to the uniformly elliptic equation* (E.1) + (E.2) *are the constant functions.*

Note that Theorem 8 thoroughly extends and improves [11, Theorem 7.1], as well as [12, Theorem 5.2].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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