Research Article

Hom- $\mathcal{O}$ -Operators and Hom-Yang-Baxter Equations

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In Hom-Lie set, we introduce the concept of Hom- $\mathcal{O}$ -operators and study its relation with classical Hom-Yang-Baxter equation, as well as left-symmetric Hom-algebras. We construct the corresponding relation between left-symmetric Hom-algebras and Hom-1-cocycles, which are both related to classical Hom-Yang-Baxter equation. Moreover, in Hom-algebra setting, we establish the equivalent relationship between AHYBE (associative Hom-Yang-Baxter equations) and $\mathcal{O}$ -operators on Frobenius monoidal Hom-algebras.

1. Introduction

Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized, or deformed derivatives and corresponding generalizations, discretizations, and deformations of vector fields and differential calculus. Hom-Lie algebras have been first introduced within context of more general quasi-Lie algebras and quasi-Hom-Lie algebras. Discretization of vector fields via twisted derivations leads to quasi-Hom-Lie structures in which the Jacobi identity is twisted by linear maps in [1–4]. The first examples and constructions of quasi-Hom-Lie algebras and Hom-Lie algebras have been concerned with the q -deformations of Witt and Virasoro algebras obtained when the derivations are replaced by σ -derivations [1]. In Hom-Lie algebras, the Jacobi identity is replaced by the so-called Hom-Jacobi identity,

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad (1)$$

where α is an endomorphism of Lie algebras. Recently, various Hom-Lie structures have been studied further by many scholars in [5–12]. Among them are such structures as Hom-Lie bialgebras, quasi-Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras, Hom-Lie admissible Hom-algebras, and Hom-Nambu-Lie algebras. In [13], the construction of Hom-Lie bialgebras both from Hom-Lie algebras and Hom-Lie coalgebras was investigated, respectively.

Quasi-triangular Hom-Lie bialgebras were considered further in [14].

The ideal of tailoring associativity-like conditions by endomorphisms was migrated to other algebraic structures. Hom-associative algebras, as an analogue and generalization of associative algebras for Hom-Lie algebras, have been introduced in [15]. Then the concepts of Hom-algebras, Hom-coalgebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, and quasi-triangular Hom-bialgebras were introduced and further developed in [8, 15–22]. The integral and smash products of monoidal Hom-Hopf algebras were introduced in [23]. Further, some actions and coactions on these Hom-algebras structures such as Hom-modules, Hom-comodules, Hom-Hopf modules, and Yetter-Drinfeld Hom-modules were considered in [24], and the fundamental structure theorem of Hom-Hopf modules was investigated in [16].

Hom-associative algebras and Hom-coassociative coalgebras were introduced in [8, 19]. Here the associativity of algebras and the coassociativity of coalgebras were twisted by endomorphisms. Hom-bialgebras are both Hom-associative algebras and Hom-coassociative coalgebras such that the comultiplication and counit are morphisms of algebras. These objects are slightly different from the ones (called monoidal Hom-structures) studied in this paper (see Section 5).

Yang-Baxter system plays a crucial role in many fields like integrable systems, quantum groups, quantum field, and so on and has become an important topic in both mathematics and mathematical physics since 1980s. Classical Yang-Baxter equation (CYBE) first arose in the study of inverse scattering theory. The different operator forms of classical Yang-Baxter equation are given in the tensor expression through a unified method in [25]. It is closely related to left-symmetric algebras which play an important role in mathematics and mathematical physics. A direct relation between the $\mathcal{O}$ -operators and CYBE was obtained in [25].

The Rota-Baxter operator has appeared in a wide range of areas both in mathematics and physics. As a generalization of Rota-Baxter operators, $\mathcal{O}$ -operators also have a close relationship with the associative Yang-Baxter equation (AYBE). Bai introduced the extended $\mathcal{O}$ -operator which generalizes the concept of $\mathcal{O}$ -operators and studied the relation between the extended $\mathcal{O}$ -operator and AYBE in [26].

The main purpose of this paper is to investigate these objects above, equations, and their relations above in Hom-setting. Explicitly, we study Hom- $\mathcal{O}$ -operators and their relation with classical Hom-Yang-Baxter equation, as well as left-symmetric Hom-algebras. On the other hand, the equivalent relationship between AHYBE and $\mathcal{O}$ -operators (on Frobenius monoidal Hom-algebras) is obtained.

This paper is organized as follows. In Section 3, we introduce the concept of Hom- $\mathcal{O}$ -operators, which generalize $\mathcal{O}$ -operators. And the relationship between Hom- $\mathcal{O}$ -operators and classical Hom-Yang-Baxter equation (CHYBE) is studied. In Section 4, we study the relations between left-symmetric Hom-algebras and CHYBE through Hom-1-cocycles. Left-symmetric Hom-algebras can be constructed from Hom-1-cocycles. Conversely, there is a Hom-1-cocycle associated with left-symmetric Hom-algebras. Further, the equivalent relations among Hom-1-cocycles, Hom- $\mathcal{O}$ -operators, and CHYBE are constructed. In Sections 5 and 6, we study similar operators related to monoidal Hom-algebras in Hom-category $\mathcal{H}(\mathcal{M}_k)$ and establish the equivalent relation between AHYBE and $\mathcal{O}$ -operators as well as the relationship between $\mathcal{O}$ -operators (Rota-Baxter operators) and the AHYBE on Frobenius monoidal Hom-algebras.

Throughout, k will be a fixed field and all vector spaces, tensor products, and homomorphisms are over k . We use Sweedler's notation for terminologies on coalgebras and comodules. For a coalgebra C , we write its comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$; for a right C -comodule M , we denote its coaction by $\rho(m) = m_{(0)} \otimes m_{(1)}$, for any $m \in M$, in which we often omit the summation symbols for convenience. τ denotes the transposition map.

2. Preliminaries

For a Lie coalgebra Γ , we write its Lie-cobracket $\Delta : \Gamma \rightarrow \Gamma^{\otimes 2}$, $x \mapsto x_1 \otimes x_2$, for any $x \in \Gamma$; for a left Γ -comodule M , we denote its location by $\rho(m) = m_{(-1)} \otimes m_{(0)}$ for any $m \in M$. Let ξ be the cyclic permutation $(1 \ 2 \ 3)$; we denote the symbol ς by the sum over id, ξ, ξ^2 . Namely, we denote the Hom-Jacobi identity by $\varsigma [[x, y], \alpha(z)] = 0$ in place of $[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] = 0$, for any $x, y, z \in \Gamma$.

In what follows, we recall some concepts used in this paper.

Definition 1. A Hom-Lie algebra [14] is a triple $(L, [-, -], \alpha)$ consisting of vector space L , bilinear map $[-, -] : L^{\otimes 2} \rightarrow L$ (called the “bracket”), and a linear endomorphism $\alpha : L \rightarrow L$ satisfying

$$\begin{aligned} \alpha[x, y] &= [\alpha(x), \alpha(y)], \quad (\text{multiplicativity}), \\ [x, y] + [y, x] &= 0, \quad (\text{skew-symmetric}), \\ \varsigma [[x, y], \alpha(z)] &= 0, \end{aligned} \tag{2}$$

(Hom-Jacobi identity)

for any $x, y, z \in L$.

Let $(L, [-, -])$ be a Lie algebra, and let $\alpha : L \rightarrow L$ be a Lie algebra endomorphism. Define a new bracket $[-, -]_\alpha$ on L by setting $[x, y]_\alpha = \alpha[x, y]$; then a direct calculation shows that $L_\alpha = (L, [-, -]_\alpha, \alpha)$ is a Hom-Lie algebra.

Definition 2. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra. An (L, α) -Hom-Lie module (V, β) consists of a vector space V and a linear endomorphism $\beta : V \rightarrow V$ together with a bilinear function $\rho : L \otimes V \rightarrow V$, $x \otimes v \mapsto x \cdot v$ satisfying

$$[x, y] \cdot \beta(v) = \alpha(x) \cdot (y \cdot v) - \alpha(y) \cdot (x \cdot v), \tag{3}$$

$$\beta(x \cdot v) = \alpha(x) \cdot \beta(v), \tag{4}$$

for all $x, y \in L, v \in V$.

It is straightforward that Hom-Lie algebra $(L, [-, -], \alpha)$ is itself an (L, α) -Hom-Lie module via its Lie-bracket $[-, -]$.

Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra. For any $x \in L$ and integer number $n \geq 1$, we define the *adjoint action* $\text{ad}_x : L^{\otimes n} \rightarrow L^{\otimes n}$ by

$$\text{ad}_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^n \alpha(y_1) \otimes \cdots \otimes \alpha(y_{i-1}) \tag{5}$$

$$\otimes [x, y_i] \otimes \alpha(y_{i+1}) \cdots \otimes \alpha(y_n).$$

In particular, for $n = 1$ and $n = 2$, we have

$$\text{ad}_x(y) = [x, y], \tag{6}$$

$$\text{ad}_x(y_1 \otimes y_2) = [x, y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [x, y_2].$$

Definition 3. A Hom-Lie coalgebra introduced in [7] is a triple (Γ, Δ, α) with a vector space Γ , linear map $\Delta : \Gamma \rightarrow \Gamma^{\otimes 2}$ (called the “cobracket”), and a linear endomorphism $\alpha : \Gamma \rightarrow \Gamma$, such that

$$\begin{aligned} \Delta\alpha &= \alpha^{\otimes 2}\Delta, \quad (\text{comultiplicativity}), \\ \Delta + \tau\Delta &= 0, \quad (\text{antisymmetric}), \\ \varsigma (\alpha \otimes \Delta)\Delta &= 0, \quad (\text{Hom-co-Jacobi identity}). \end{aligned} \tag{7}$$

Definition 4. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra and $r = \sum_i a_i \otimes b_i \in L^{\otimes 2}$. One calls it the coboundary triangular Hom-Lie algebra, if it satisfies the classical Hom-Yang-Baxter equation (CHYBE for short)

$$\text{CH}(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \quad (8)$$

where

$$\begin{aligned} [r_{12}, r_{13}] &= [a_i, a_j] \otimes \alpha(b_i) \otimes \alpha(b_j), \\ [r_{12}, r_{23}] &= \alpha(a_i) \otimes [b_i, a_k] \otimes \alpha(b_k), \\ [r_{13}, r_{23}] &= \alpha(a_j) \otimes \alpha(a_k) \otimes [b_j, b_k], \end{aligned} \quad (9)$$

where $r_{12} = r \otimes 1 = a_i \otimes b_i \otimes 1$, $r_{13} = (\tau \otimes \text{id})(1 \otimes r) = a_j \otimes 1 \otimes b_j$, and $r_{23} = 1 \otimes r = 1 \otimes a_k \otimes b_k$.

3. Hom- $\mathcal{O}$ -Operator and Classical Hom-Yang-Baxter Equation

In this section, we introduce the concept of Hom- $\mathcal{O}$ -operators, which generalized that of $\mathcal{O}$ -operators. And we also study the relationship between Hom- $\mathcal{O}$ -operators and classical Hom-Yang-Baxter equation.

Definition 5. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra and let $(V, \cdot, \beta)$ be an (L, α) -Hom-Lie module. A linear map $T : (V, \beta) \rightarrow (L, \alpha)$ is called a Hom- $\mathcal{O}$ -operator associated with $(L, [-, -], \alpha)$ if it satisfies the following two conditions:

$$\begin{aligned} T\beta &= \alpha T, \\ [T(u), T(v)] &= T(T(\beta(u)) \cdot v) - T(T(\beta(v)) \cdot u). \end{aligned} \quad (10)$$

Proposition 6. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra and let (V, ρ, β) be an (L, α) -Hom-Lie module with $\rho : L \otimes V \rightarrow V$, $x \otimes v \mapsto x \cdot v$. Then there is a natural Hom-Lie algebra structure on $(L \oplus V, \gamma_{\alpha, \beta})$ (denoted by $(L \ltimes_{\rho} V, \gamma_{\alpha, \beta})$) given by

$$\begin{aligned} \gamma_{\alpha, \beta} : L \oplus V &\longrightarrow L \oplus V, \\ x + v &\longmapsto \alpha(x) + \beta(v), \end{aligned} \quad (11)$$

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + x_1 \cdot v_2 - x_2 \cdot v_1, \quad (12)$$

for any $x, x_1, x_2 \in L$, $v, v_1, v_2 \in V$.

Proof. For any $x_1, x_2 \in L$, $v_1, v_2 \in V$, $[x_1 + v_1, x_2 + v_2] + [x_2 + v_2, x_1 + v_1] = 0$ by the bracket of $(L \oplus V, \gamma_{\alpha, \beta})$ and the antisymmetry of (L, α) . Next, $\gamma_{\alpha, \beta}[x_1 + v_1, x_2 + v_2] = [\alpha(x_1) + \beta(v_1), \alpha(x_2) + \beta(v_2)]$, since $[-, -]$ is multiplicative and ρ satisfies equality (4). The Hom-Jacobi identity is also true: for any $x_1, x_2, x_3 \in L$, $v_1, v_2, v_3 \in V$,

$$\begin{aligned} \mathcal{O} [[x_1 + v_1, x_2 + v_2], \alpha(x_3) + \beta(v_3)] &= \mathcal{O} [[x_1, x_2] \\ &+ x_1 \cdot v_2 - x_2 \cdot v_1, \alpha(x_3) + \beta(v_3)] \\ &= \mathcal{O} [[x_1, x_2], \alpha(x_3)] + \mathcal{O} (\alpha(x_1) \cdot (x_2 \cdot v_3) \\ &- \alpha(x_2) \cdot (x_1 \cdot v_3) - \alpha(x_3) \cdot (x_1 \cdot v_2) + \alpha(x_3) \\ &\cdot (x_2 \cdot v_1)) = 0. \end{aligned} \quad (13)$$

□

Lemma 7. Let $(L, \alpha, [-, -])$ be a Hom-Lie algebra; let (V, ρ, β) be an (L, α) -Hom-Lie module of m -dimension. And let (V^*, β^*) be the dual of (V, β) with $\beta^*(v^*) = v^* \beta$, for $v^* \in V^*$. Then (V^*, β^*) is also a Hom-Lie module with the structure

$$\begin{aligned} \rho^* : L \otimes V^* &\longrightarrow V^*, \\ \rho^*(x \otimes v^*) &:= x \cdot v^* = - \sum_{i=1}^m v^*(x \cdot v_i) v_i^*, \end{aligned} \quad (14)$$

if and only if $\alpha^2 = \text{id}$, where $\{v_i\}$ is a basis of V and $\{v_i^*\}$ is the corresponding dual basis.

Proof. By the definition of ρ^* , for any $x \in L$, $v \in V$, and $v^* \in V^*$,

$$\begin{aligned} \rho^*(\alpha(x) \otimes \beta^*(v^*)) (v) &= -v^*(\alpha^2(x) \cdot \beta(v)), \\ \beta^* \rho^*(x \otimes v^*) (v) &= -v^*(x \cdot \beta(v)), \end{aligned} \quad (15)$$

so $\rho^*(\alpha(x) \otimes \beta^*(v^*)) = \beta^* \rho^*(x \otimes v^*)$ if and only if $\alpha^2 = \text{id}$.

For any $x, y \in L$,

$$\begin{aligned} ([x, y] \cdot \beta^*(v^*)) (v) &= -v^* \beta([x, y] \cdot v), \\ (\alpha(x) \cdot (y \cdot v^*) - \alpha(y) \cdot (x \cdot v^*)) (v) \\ &= \sum_{j=0}^m \sum_{i=0}^m (v_i^*(\alpha(x) \cdot v_j) v^*(y \cdot v_i) v_j^* \\ &- v_i^*(\alpha(y) \cdot v_j) v^*(x \cdot v_i) v_j^*) = v^*(y \cdot (\alpha(x) \cdot v) \\ &- x \cdot (\alpha(y) \cdot v)). \end{aligned} \quad (16)$$

Then $[x, y] \cdot \beta^*(v^*) = \alpha(x) \cdot (y \cdot v^*) - \alpha(y) \cdot (x \cdot v^*)$ if and only if $\alpha^2 = \text{id}$ implied by the compatibility condition (3). Thus (V^*, β^*) is also a Hom-Lie module with the structure ρ^* given as above if and only if $\alpha^2 = \text{id}$. □

If we assume that $\alpha^2 = \text{id}$, then there is a Hom-Lie algebra $(L \ltimes_{\rho^*} V^*, \gamma_{\alpha, \beta^*})$.

Let $\{e_1, \dots, e_n\}$ be a basis of L , let $\{v_1, \dots, v_m\}$ be a basis of V , and let $\{v_1^*, \dots, v_m^*\}$ be its dual basis; that is, $v_i^*(v_j) = \delta_{ij}$. Set $T : (V, \beta) \rightarrow (L, \alpha)$ by

$$T(v_i) = \sum_{j=1}^n a_{ij} e_j \quad (17)$$

with $T\beta = \alpha T$, where $a_{ij} \in k$ ($i = 1, \dots, m$). Since $\text{Hom}(V, L) \cong L \otimes V^*$ as spaces, we have

$$\begin{aligned} T &= \sum_{i=1}^m T(v_i) \otimes v_i^* = \sum_{i=1}^m \sum_{j=1}^n a_{ij} e_j \otimes v_i^* \in L \otimes V^* \\ &\subseteq (L \ltimes_{\rho^*} V^*) \otimes (L \ltimes_{\rho^*} V^*). \end{aligned} \quad (18)$$

Lemma 8. With the above assumption, the skew-symmetric part $\gamma = T - T^{21}$ is a solution of CHYBE in $(L \ltimes_{\rho^*} V^*, \gamma_{\alpha, \beta^*})$ if and only if T is a Hom- $\mathcal{O}$ -operator, where $T^{21} = \tau(T)$.

Proof. In fact, the Lie-bracket on $L \ltimes_{\rho^*} V^*$ given in (12) and equality (18) imply that

$$\begin{aligned}
 [r_{12}, r_{13}] &= \sum_{i,k=1}^m \{ [T(v_i), T(v_k)] \otimes \beta^*(v_i^*) \otimes \beta^*(v_k^*) \\
 &\quad - T(v_i) \cdot v_k^* \otimes \beta^*(v_i^*) \otimes \alpha T(v_k) - T(v_k) \cdot v_i^* \\
 &\quad \otimes \alpha T(v_i) \otimes \beta^*(v_k^*) \}; \\
 [r_{12}, r_{23}] &= \sum_{i,k=1}^m \{ -\beta^*(v_i^*) \otimes [T(v_i), T(v_k)] \otimes \beta^*(v_k^*) \\
 &\quad - \alpha T(v_i) \otimes T(v_k) \cdot v_i^* \otimes \beta^*(v_k^*) + \beta^*(v_i^*) \\
 &\quad \otimes T(v_i) \cdot v_k^* \otimes \alpha T(v_k) \}; \\
 [r_{13}, r_{23}] &= \sum_{i,k=1}^m \{ \beta^*(v_i^*) \otimes \beta^*(v_k^*) \otimes [T(v_i), T(v_k)] \\
 &\quad + \alpha T(v_i) \otimes \beta^*(v_k^*) \otimes T(v_k) \cdot v_i^* - \beta^*(v_i^*) \\
 &\quad \otimes \alpha T(v_k) \otimes T(v_i) \cdot v_k^* \}.
 \end{aligned} \tag{19}$$

By the definition of ρ^* , we have

$$\begin{aligned}
 &\sum_{i,k=1}^m \alpha T(v_i) \otimes T(v_k) \cdot v_i^* \otimes \beta^*(v_k^*) \\
 &= - \sum_{i,j,k=1}^m \alpha T(v_i) \otimes v_i^* (T(v_k) \cdot v_j) v_j^* \otimes \beta^*(v_k^*) \\
 &= - \sum_{i,j,k=1}^m \alpha T(v_j) \otimes v_j^* (T(v_k) \cdot v_i) v_i^* \otimes \beta^*(v_k^*) \\
 &= - \sum_{i,j,k=1}^m \alpha T(T(v_k) \cdot v_i) \otimes v_i^* \otimes \beta^*(v_k^*) \\
 &= - \sum_{i,j,k=1}^m T(T\beta(v_k) \cdot \beta(v_i)) \otimes v_i^* \otimes \beta^*(v_k^*) \\
 &= - \sum_{i,j,k=1}^m T(T\beta(v_k) \cdot v_i) \otimes \beta^*(v_i^*) \otimes \beta^*(v_k^*),
 \end{aligned} \tag{20}$$

where the last step is from the fact that pairs $(\beta(v_i), v_i^*)$ and $(v_i, \beta^*(v_i^*))$ are equivalent to each other in the sense of dual basis.

Thus

$$\begin{aligned}
 &[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \\
 &= \sum_{i,k=1}^m ([T(v_i), T(v_k)] + T(T\beta(v_k) \cdot v_i) \\
 &\quad - T(T\beta(v_i) \cdot v_k)) \otimes \beta^*(v_i^*) \otimes \beta^*(v_k^*) + \beta^*(v_i^*) \\
 &\quad \otimes \beta^*(v_k^*) \otimes ([T(v_i), T(v_k)] + T(T\beta(v_k) \cdot v_i)
 \end{aligned}$$

$$\begin{aligned}
 &- T(T\beta(v_i) \cdot v_k)) - \beta^*(v_i^*) \otimes ([T(v_i), T(v_k)] \\
 &\quad + T(T\beta(v_k) \cdot v_i) - T(T\beta(v_i) \cdot v_k)) \otimes \beta^*(v_k^*).
 \end{aligned} \tag{21}$$

So γ is a solution of CHYBE in $(L \ltimes_{\rho^*} V^*, \alpha + \beta^*)$ if and only if T is a Hom- $\mathcal{O}$ -operator. $\square$

In particular, set the Hom-Lie module action $\rho = \text{ad}$ on Hom-Lie algebra (L, α) . In this case, we suppose that the Hom-Lie algebra is equipped with a *nondegenerate symmetric Hom-associative bilinear form* $B : (L, \alpha) \otimes (L, \alpha) \rightarrow (L, \alpha)$. That is,

$$\begin{aligned}
 B(x, y) &= B(y, x), B([x, y], \alpha(z)) \\
 &= B(\alpha(x), [y, z]), \quad \forall x, y, z \in L.
 \end{aligned} \tag{22}$$

Hence L^* can be identified with L . Let $\gamma \in L \otimes L$ be skew-symmetric; that is, $\gamma = -\gamma^{21} = -\tau(\gamma)$. Then γ is a solution of CHYBE in (L, α) if and only if it satisfies the equation

$$[r(x), r(y)] = r([r\beta(x), y] + [x, \gamma\beta(y)]). \tag{23}$$

Thus, in this case, the CHYBE is equivalent to (23).

4. Left-Symmetric Hom-Algebras and CHYBE

In this section, we study the relations between left-symmetric Hom-algebras and CHYBE through Hom-1-cocycles. Left-symmetric Hom-algebras can be constructed from Hom-1-cocycle and, conversely, there are Hom-1-cocycles associated with left-symmetric Hom-algebras. As the main result of this section, we construct the equivalent relation among Hom-1-cocycles, Hom- $\mathcal{O}$ -operators, and CHYBE.

Definition 9. A left-symmetric Hom-algebra (A, α) is a vector space A over a field k equipped with a bilinear product $m : x \otimes y \mapsto xy$ and an endomorphism $\alpha : A \rightarrow A$ such that α is multiplicative; that is, $\alpha(xy) = \alpha(x)\alpha(y)$, and the associator

$$(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz) \tag{24}$$

is symmetric in x, y , for any $x, y, z \in A$; that is, $(x, y, z) = (y, x, z)$, or equivalently

$$(xy)\alpha(z) - \alpha(x)(yz) = (yx)\alpha(z) - \alpha(y)(xz). \tag{25}$$

Similarly, we can define *right-symmetric Hom-algebras*. A Hom-associative algebra in [8] is obviously a left-symmetric Hom-algebra since the associator is zero.

A left-symmetric Hom-algebra (A, α) can define a Hom-Lie algebra via the commutator

$$[x, y] = xy - yx, \tag{26}$$

for any $x, y \in A$. Indeed, the skew-symmetry is obvious, and the Hom-Jacobi identity is also true by

$$\begin{aligned}
 &\cup [\alpha(x), [y, z]] = \cup [\alpha(x), yz - zy] \\
 &= \alpha(x)(yz) - \alpha(x)(zy) - (yz)\alpha(x) + (zy)\alpha(x)
 \end{aligned}$$

$$\begin{aligned}
& + \alpha(y)(zx) - \alpha(y)(xz) - (zx)\alpha(y) \\
& + (xz)\alpha(y) + \alpha(z)(xy) - \alpha(z)(yx) \\
& - (xy)\alpha(z) + (yx)\alpha(z) = 0.
\end{aligned} \tag{27}$$

We denote this above induced Hom-Lie algebra by $(L(A), \alpha)$.

Let $L_x : (A, \alpha) \rightarrow (A, \alpha)$ be the left multiplication operator; that is, $L_x(y) = xy$. Then

$$\begin{aligned}
P : (L(A), \alpha) \otimes (L(A), \alpha) & \longrightarrow (L(A), \alpha); \\
x \otimes y & \longmapsto L_x(y) = xy
\end{aligned} \tag{28}$$

which gives an $(L(A), \alpha)$ -Hom-Lie module on itself by the Hom-Jacobi identity and the symmetry of the associator of left-symmetric Hom-algebras.

Definition 10. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra, and let (V, ρ, β) be an (L, α) -Hom-Lie module. A Hom-1-cocycle q of (L, α) associated with ρ (denoted by (q, ρ)) is defined as a linear map $q : (L, \alpha) \rightarrow (V, \beta)$ such that $q\alpha = \beta q$ and

$$q[x, y] = \alpha(x) \cdot q(y) - \alpha(y) \cdot q(x), \tag{29}$$

for any $x, y \in L$.

Proposition 11. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra, and let (V, ρ, β) be an (L, α) -Hom-Lie module. If α is bijective, then there is a compatible left-symmetric Hom-algebra structure on (L, α) if and only if there exists a bijective Hom-1-cocycle of (L, α) .

Proof. Let q be a bijective Hom-1-cocycle of (L, α) associated with ρ . Then, for any $x, y \in L$,

$$x * y = q^{-1}(\alpha(x) \cdot q(y)) \tag{30}$$

defines a compatible left-symmetric Hom-algebra structure on (L, α) . In fact,

$$\begin{aligned}
& (x * y) * \alpha(z) - \alpha(x) * (y * z) - (y * x) * \alpha(z) \\
& + \alpha(y) * (x * z) = q^{-1}(q^{-1}(\alpha^2(x) \cdot q\alpha(y)) \\
& \cdot q\alpha(z)) - q^{-1}(\alpha^2(x) \cdot (\alpha(y) \cdot q(z))) \\
& - q^{-1}(q^{-1}(\alpha^2(y) \cdot q\alpha(x)) \cdot q\alpha(z)) + q^{-1}(\alpha^2(y) \\
& \cdot (\alpha(x) \cdot q(z))) = q^{-1}\{q^{-1}(\alpha^2(x) \cdot q\alpha(y)) \\
& \cdot q\alpha(z) - q^{-1}(\alpha^2(y) \cdot q\alpha(x)) \cdot q\alpha(z) + (\alpha^2(y) \\
& \cdot (\alpha(x) \cdot q(z)) - \alpha^2(x) \cdot (\alpha(y) \cdot q(z)))\} \\
& \stackrel{(3)}{=} q^{-1}\{q^{-1}(\alpha^2(x) \cdot q\alpha(y) - \alpha^2(y) \cdot q\alpha(x)) \\
& \cdot q\alpha(z) + [\alpha(y), \alpha(x)] \cdot \beta q(z)\} \\
& \stackrel{(29)}{=} q^{-1}\{[\alpha(x), \alpha(y)] \cdot q\alpha(z) + [\alpha(y), \alpha(x)] \\
& \cdot q\alpha(z)\} = 0.
\end{aligned} \tag{31}$$

Conversely, for a left-symmetric Hom-algebra (A, α) with bijective α , (α, P) given in (28) is a bijective Hom-1-cocycle of $(L(A), \alpha)$. And the Hom-1-cocycle condition is $\alpha[x, y] = \alpha(x)\alpha(y) - \alpha(y)\alpha(x)$ followed by (26) and the multiplicativity of α . $\square$

Note that, for any Hom-Lie algebra (L, α) and an (L, α) -Hom-Lie module (V, ρ) , a linear isomorphism $T : (V, \beta) \rightarrow (L, \alpha)$ (hence $\dim L = \dim V$) is a Hom- $\mathcal{O}$ -operator associated with ρ if and only if T^{-1} is a bijective Hom-1-cocycle of (L, α) to ρ . Indeed, T is a Hom- $\mathcal{O}$ -operator if and only if, for all $u, v \in V$, $[T(u), T(v)] = T(T\beta(u) \cdot v - T\beta(v) \cdot u)$ if and only if $T^{-1}[T(u), T(v)] = T\beta(u) \cdot v - T\beta(v) \cdot u$ if and only if $T^{-1}[x, y] = \alpha(x) \cdot T^{-1}(y) - \alpha(y) \cdot T^{-1}(x)$ for all $x, y \in L$, which means that T^{-1} is a Hom-1-cocycle of (L, α) .

From the above discussion together with Lemma 8, we obtain the main result of this section.

Theorem 12. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra with $\alpha^2 = \text{id}$, and let (V, ρ, β) be an (L, α) -Hom-Lie module. Then $q : (L, \alpha) \rightarrow (V, \beta)$ is a bijective Hom-1-cocycle of (L, α) associated with ρ if and only if $q^{-1} : (V, \beta) \rightarrow (L, \alpha)$ is a Hom- $\mathcal{O}$ -operator associated with ρ if and only if $\gamma = q^{-1} - (q^{-1})^{21}$ is a solution of CHYBE in $(L \ltimes_{\rho^*} V^*, \gamma_{\alpha, \beta^*})$.

In particular, if a Hom-Lie algebra (L, α) is also a left-symmetric Hom-algebra with the invertible endomorphism α , then $T = \alpha^{-1}$ is a Hom- $\mathcal{O}$ -operator associated with the action P defined in (28) since $[\alpha^{-1}(x), \alpha^{-1}(y)] = \alpha^{-1}(\alpha^{-1}(\alpha(x))y - \alpha^{-1}(\alpha(y))x)$, for any $x, y \in L$. So we have that

$$\begin{aligned}
\gamma & = T - T^{21} = \sum_{i=1}^m T(v_i) \otimes v_i^* - v_i^* \otimes T(v_i) \\
& = \sum_{i=1}^m \{\alpha^{-1}(e_i) \otimes e_i^* - e_i^* \otimes \alpha^{-1}(e_i)\}
\end{aligned} \tag{32}$$

is a solution of CHYBE in $(L \ltimes_{\rho^*} L^*, \alpha + \alpha^*)$, where $\{e_i\}$ is a basis of L and $\{e_i^*\}$ is the corresponding dual basis.

Theorem 13. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra with $\alpha^2 = \text{id}$, and let $(V, \cdot, \beta)$ be an (L, α) -Hom-Lie module. If $T : (V, \beta) \rightarrow (L, \alpha)$ is an invertible Hom- $\mathcal{O}$ -operator associated with $\cdot$, then T is an isomorphism of both left-symmetric Hom-algebras and Hom-Lie algebras.

Proof. If Hom- $\mathcal{O}$ -operator T associated with $\cdot$ is invertible, then T^{-1} is a bijective Hom-1-cocycle of (L, α) associated with $\cdot$. Hence, for any $x, y \in L$,

$$x * y = T(\alpha(x) \cdot T^{-1}(y)) \tag{33}$$

defines a compatible left-symmetric Hom-algebra structure on (L, α) by Proposition 11. Moreover, for any $u, v \in V$, let $x = T(u)$, $y = T(v)$; then we have

$$T(u) * T(v) = T(\alpha(T(u)) \cdot v). \tag{34}$$

Since T is invertible, there is a left-symmetric Hom-algebra structure on (V, β) induced from the left-symmetric Hom-algebra structure on (L, α) by

$$u * v = T^{-1}(T(u) * T(v)) = \alpha(T(u)) \cdot v. \quad (35)$$

Then we obtain $T(u * v) = T(u) * T(v)$; thus T is an isomorphism of left-symmetric Hom-algebras between them.

On the other hand, let $T : (V, \beta) \rightarrow (L, \alpha)$ be a linear map with $T\beta = \alpha T$. Then for any $u, v \in V$, the product

$$u * v = \alpha(T(u)) \cdot v \quad (36)$$

satisfies

$$\begin{aligned} & (u, v, w) - (v, u, w) \\ &= (u * v) * \beta(w) - \beta(u) * (v * w) - (v * u) \\ & \quad * \beta(w) + \beta(v) * (u * w) \\ &= \alpha T(\alpha T(u) \cdot v) \cdot \beta(w) - \alpha T(\alpha T(v) \cdot u) \cdot \beta(w) \\ & \quad - \alpha T\beta(u) \cdot (\alpha T(v) \cdot w) + \alpha T\beta(v) \cdot (\alpha T(u) \cdot w) \quad (37) \\ &= \alpha T(\alpha T(u) \cdot v - \alpha T(v) \cdot u) \cdot \beta(w) \\ & \quad + [T\beta(v), T\beta(u)] \cdot \beta(w) \\ &= \alpha([T(v), T(u)] + T(\alpha T(u) \cdot v - \alpha T(v) \cdot u)) \\ & \quad \cdot \beta(w). \end{aligned}$$

Hence (36) defines a left-symmetric Hom-algebra if and only if, for any $u, v, w \in V$, $\alpha([T(v), T(u)] + T(\alpha T(u) \cdot v - \alpha T(v) \cdot u)) \cdot \beta(w) = 0$. In particular, for any Hom- $\mathcal{O}$ -operator $T : (V, \beta) \rightarrow (L, \alpha)$ associated with $\cdot$, (36) defines a left-symmetric Hom-algebra on (V, β) . Further we can define a Hom-Lie algebra structure on (V, β) by (26). And $T[u, v] = [T(u), T(v)]$; hence T is also a Hom-Lie algebraic homomorphism. $\square$

5. Monoidal Hom-Algebras and Associative Hom-Yang-Baxter Equations

In this section, we give the equivalent relation between AHYBE and $\mathcal{O}$ -operators over monoidal Hom-algebras.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the monoidal category of k -modules. There is a new monoidal category $\mathcal{H}(\mathcal{M}_k)$. The objects of $\mathcal{H}(\mathcal{M}_k)$ are couples (M, μ) , where $M \in \mathcal{M}_k$ and $\mu \in \text{Aut}_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are morphisms $f : (M, \mu) \rightarrow (N, \nu)$ in $\mathcal{M}_k$ such that $\nu f = f\mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu), \quad (k, \text{id}). \quad (38)$$

Briefly speaking, all Hom-structures are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \text{id}), \tilde{a}, \tilde{l}, \tilde{r})$ introduced in [16], where the associator $\tilde{a}$ is given by the formula

$$\begin{aligned} \tilde{a}_{M,N,L} &= a_{M,N,L}((\mu \otimes \text{id}) \otimes \zeta^{-1}) \\ &= (\mu \otimes (\text{id} \otimes \zeta^{-1})) a_{M,N,L}, \end{aligned} \quad (39)$$

for any objects $(M, \mu), (N, \nu), (L, \zeta) \in \mathcal{H}(\mathcal{M}_k)$, and the unitors $\tilde{l}$ and $\tilde{r}$ are

$$\begin{aligned} \tilde{l}_M &= \mu l_M = l_M(\text{id} \otimes \mu); \\ \tilde{r}_M &= \mu r_M = r_M(\mu \otimes \text{id}). \end{aligned} \quad (40)$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the Hom-category associated with monoidal category $\mathcal{M}_k$.

In the following, we recall from [16] some definitions about Hom-structures.

Definition 14. A *unital monoidal Hom-associative algebra* (which is called monoidal Hom-algebra in [16, Proposition 2.1]) is a vector space A together with an element $1_A \in A$ and linear maps

$$\begin{aligned} m : A \otimes A &\longrightarrow A; \\ a \otimes b &\longmapsto ab, \\ \alpha &\in \text{Aut}_k(A) \end{aligned} \quad (41)$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \quad (42)$$

$$\alpha(ab) = \alpha(a)\alpha(b), \quad (43)$$

$$a1_A = 1_A a = \alpha(a), \quad (44)$$

$$\alpha(1_A) = 1_A, \quad (45)$$

for all $a, b, c \in A$.

Throughout, we use the conception of [16] for convenience. The definition of monoidal Hom-associative algebras is different from the unital Hom-associative algebras in [8, 19] in the following sense. The same twisted associativity condition (42) holds in both cases. However, the unitality condition on unital Hom-associative algebras is the usual untwisted one: $a1_A = 1_A a = a$, for any $a \in A$, and the twisting map α does not need to be monoidal (i.e., (43) and (45) are not required).

In the language of Hopf algebras, m is called the *Hom-multiplication*, α is the *twisting automorphism*, and 1_A is the *unit*. Let (A, α) and (A', α') be two monoidal Hom-associative algebras. A *monoidal Hom-algebra map* $f : (A, \alpha) \rightarrow (A', \alpha')$ is a linear map such that $f\alpha = \alpha'f$, $f(ab) = f(a)f(b)$, and $f(1_A) = 1_{A'}$.

Definition 15. A *counital monoidal Hom-coassociative coalgebra* (which is called monoidal Hom-coalgebra in [16, Proposition 2.4]) is a vector space C together with linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \rightarrow k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad (46)$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \quad (47)$$

$$c_1 \varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1) c_2, \quad (48)$$

$$\varepsilon(\gamma(c)) = \varepsilon(c), \quad (49)$$

for all $c \in C$.

Note that (46) is equivalent to $c_1 \otimes (c_{21} \otimes \gamma(c_{22})) = (\gamma(c_{11}) \otimes c_{12}) \otimes c_2$, which is used often in the rest of the paper. Let (C, γ) and (C', γ') be two monoidal Hom-coassociative coalgebras. A *monoidal Hom-coalgebra map* $f : (C, \gamma) \rightarrow (C', \gamma')$ is a linear map such that $f\gamma = \gamma'f$, $\Delta f = (f \otimes f)\Delta$, and $\varepsilon'f = \varepsilon$.

The definition of monoidal Hom-coassociative coalgebra here is somewhat different from the counital Hom-coassociative coalgebra in [8, 19]. Their coassociativity condition is twisted by some endomorphism, not necessarily by the inverse of the automorphism γ . The counitality condition is the usual untwisted one. Counital Hom-coassociative coalgebras are not monoidal; that is, (47) and (49) are not required.

Definition 16. Let (A, α) be a monoidal Hom-algebra. A *right (A, α) -Hom-module* consists of (M, μ) in $\mathcal{H}(\mathcal{M}_k)$ together with a morphism $\psi : M \otimes A \rightarrow M$, $\psi(m \otimes a) = m \cdot a$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab), \quad m \cdot 1_A = \mu(m), \quad (50)$$

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a), \quad (51)$$

for all $a, b \in A$ and $m \in M$.

Similarly, we can define left (A, α) -Hom-modules. Monoidal Hom-algebra (A, α) can be considered as a Hom-module on itself by the Hom-multiplication. Let (M, μ) , (N, ν) be two left (A, α) -Hom-modules. A morphism $f : M \rightarrow N$ is called *left A -linear* if $f(a \cdot m) = a \cdot f(m)$, for any $a \in A$, $m \in M$, and $fu = \nu f$. We denote the category of left (A, α) -Hom-modules by $\mathcal{H}_{(A)}(\mathcal{M})$.

(M, μ) is called an (A, α) -Hom-bimodule if (M, μ) is both a left (A, α) -Hom-module and right (A, α) -Hom-module satisfying the following compatibility condition:

$$\alpha(a) \cdot (m \cdot b) = (a \cdot m) \cdot \alpha(b), \quad (52)$$

for any $a, b \in A$, $m \in M$.

Lemma 17. Let (A, α) be a monoidal Hom-algebra, and let $(M, \cdot, \mu)$ be a finite-dimensional (A, α) -Hom-bimodule. Then there is an (A, α) -Hom-bimodule structure on the dual $(M^*, (\mu^*)^{-1})$ of (M, μ) , where the actions are given by

$$\begin{aligned} \langle ax, m \rangle &= \langle x, \mu^{-2}(m) \cdot \alpha^n(a) \rangle, \\ \langle xa, m \rangle &= \langle x, \alpha^n(a) \cdot \mu^{-2}(m) \rangle, \end{aligned} \quad (53)$$

for any $a \in A$, $m \in M$, $x \in M^*$, and $n \in \mathbb{Z}$ the set of integer numbers.

Proof. The rationality, unity, associativity, and the compatibility of left and right actions on $(M^*, (\mu^*)^{-1})$ are all derived from the same properties of (A, α) -Hom-bimodule $(M, \cdot, \mu)$. This proof is left to the readers. $\square$

Due to the selection of n in Lemma 17, there is a family of Hom-bimodule structures on $(M^*, (\mu^*)^{-1})$ of (M, μ) . In

particular, the dual $(A^*, (\alpha^*)^{-1})$ of (A, α) can be considered as an (A, α) -Hom-bimodule by selection $n = 0$ or -1 :

$$\begin{aligned} \langle a \rightarrow x, b \rangle &= \langle x, \alpha^{-2}(b) \cdot a \rangle, \\ \langle x \leftarrow a, b \rangle &= \langle x, a\alpha^{-2}(b) \rangle; \\ \langle a \rightarrow x, b \rangle &= \langle x, \alpha^{-2}(b) \alpha^{-1}(a) \rangle, \\ \langle x \leftarrow a, b \rangle &= \langle x, \alpha^{-1}(a) \alpha^{-2}(b) \rangle, \end{aligned} \quad (54)$$

for any $a, b \in A$, $x \in A^*$.

Definition 18. Let (A, α) be a monoidal Hom-algebra and $r = r_1 \otimes r_2 \in A^{\otimes 2}$. r is called a *solution of the associative Hom-Yang-Baxter equation (AHYBE for short)*, if r is $\alpha^{\otimes 2}$ -invariant, that is, $(\alpha \otimes \alpha)(r) = r$, and satisfies the AHYBE

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0, \quad (55)$$

where $r_{12} = r_1 \otimes r_2 \otimes 1$, $r_{13} = r_1 \otimes 1 \otimes r_2$, and $r_{23} = 1 \otimes r_1 \otimes r_2$.

Explicitly, the AHYBE in (55) can be rewritten as

$$\begin{aligned} r_1 R_2 \otimes \alpha(r_2) \otimes \alpha(R_2) + \alpha(r_1) \otimes \alpha(R_1) \otimes r_2 R_2 \\ - \alpha(r_1) \otimes R_1 r_2 \otimes \alpha(R_2) = 0, \end{aligned} \quad (56)$$

where $R = R_1 \otimes R_2$ is another copy of r .

For a finite-dimensional monoidal Hom-algebra (A, α) , any $\alpha^{\otimes 2}$ -invariant element $r \in A^{\otimes 2}$ can be considered as a morphism from $(A^*, (\alpha^*)^{-1})$ to (A, α) in $\mathcal{H}(\mathcal{M}_k)$ by

$$\begin{aligned} r : x &\mapsto \langle x, r_1 \rangle r_2 \\ \text{or } r^t : x &\mapsto \langle x, r_2 \rangle r_1. \end{aligned} \quad (57)$$

And the $\alpha^{\otimes 2}$ -invariance of r implies $r, r^t \in \mathcal{H}(\mathcal{M}_k)$; that is, $r(\alpha^*)^{-1} = \alpha r$ and $r^t(\alpha^*)^{-1} = \alpha r^t$. Further, if r is symmetric (resp., skew-symmetric), then $r = r^t$ (resp., $r = -r^t$).

Definition 19. Let (A, α) be a monoidal Hom-algebra and $\lambda \in k$. And let (M, μ) be an (A, α) -Hom-bimodule. A morphism $q : (M, \mu) \rightarrow (A, \alpha)$ in $\mathcal{H}(\mathcal{M}_k)$ is called an $\mathcal{O}$ -operator of weight 0 associated with (M, μ) if it satisfies

$$q(m)q(n) = q(m \cdot q(n) + q(m) \cdot n), \quad (58)$$

for any $m, n \in M$.

Particularly, (A, α) is an (A, α) -Hom-bimodule on itself. Then $q : (A, \alpha) \rightarrow (A, \alpha)$ is called a *Rota-Baxter operator* of weight 0 if

$$q(a)q(b) = q(aq(b) + q(a)b), \quad (59)$$

for any $a, b \in A$.

A Rota-Baxter operator is associated with Rota-Baxter algebra [27] which was studied further by Guo.

Proposition 20. Let (A, α) be a finite-dimensional monoidal Hom-algebra and $r \in A^{\otimes 2}$. If r is skew-symmetric, then r is a solution of AHYBE if and only if it is an $\mathcal{O}$ -operator of weight 0 associated with $(A^*, \rightarrow, \leftarrow, (\alpha^*)^{-1})$.

Proof. The $\alpha^{\otimes 2}$ -invariance of r is equivalent to $r \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ by the definitions.

For any $x, y, z \in A^*$,

$$\begin{aligned}
 & \langle r_{12}r_{13}, x \otimes y \otimes z \rangle \\
 &= \langle r_1 R_1, x \rangle \langle \alpha(r_2), y \rangle \langle \alpha(R_2), z \rangle \\
 &= \langle r^t \alpha^*(y) R_1, x \rangle \langle \alpha(R_2), z \rangle \\
 &= \langle \alpha^{-1} r^t(y) R_1, x \rangle \langle \alpha(R_2), z \rangle \\
 &= \langle \alpha^2(R_1), x \leftarrow r^t(y) \rangle \langle \alpha(R_2), z \rangle \\
 &= \langle \alpha(R_1), x \leftarrow r^t(y) \rangle \langle R_2, z \rangle \\
 &= \langle r\alpha^*(x \leftarrow r^t(y)), z \rangle \\
 &= -\langle r\alpha^*(x \leftarrow r(y)), z \rangle.
 \end{aligned} \tag{60}$$

Similarly, we have

$$\begin{aligned}
 & \langle r_{13}r_{23}, x \otimes y \otimes z \rangle = \langle r\alpha^*(x) r\alpha^*(y), z \rangle, \\
 & \langle r_{23}r_{12}, x \otimes y \otimes z \rangle = \langle r\alpha^*(r(x) \rightarrow y), z \rangle.
 \end{aligned} \tag{61}$$

So, $\langle r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}, x \otimes y \otimes z \rangle = \langle r\alpha^*(x)r\alpha^*(y) - r\alpha^*(x \leftarrow r(y)) - r\alpha^*(r(x) \rightarrow y), z \rangle$. Thus $r \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ and the multiplicativity of α imply that r is a solution of AHYBE if and only if it is an $\mathcal{O}$ -operator of weight 0 associated with $(A^*, \rightarrow, \leftarrow, (\alpha^*)^{-1})$. $\square$

6. Frobenius Monoidal Hom-Algebras and Hom- $\mathcal{O}$ -Operators

In this section, we consider the relationship between $\mathcal{O}$ -operators (Rota-Baxter operators) and the AHYBE on Frobenius monoidal Hom-algebras.

Definition 21. A finite-dimensional monoidal Hom-algebra (A, α) is called *Frobenius* if

$$f : (A, \alpha) \longrightarrow (A^*, (\alpha^*)^{-1}) \tag{62}$$

is an isomorphism as right (A, α) -Hom-modules.

Lemma 22 (see [23, Proposition 5.2]). For a finite-dimensional monoidal Hom-algebra (A, α) , the following assertions are equivalent:

- (1) (A, α) is Frobenius.
- (2) $(A, \alpha) \cong (A^*, (\alpha^*)^{-1})$ as left (A, α) -Hom-modules.
- (3) There exist elements $\varphi \in A^*$, and $e^1 \otimes e^2 \in A \otimes A$ such that $\varphi \circ \alpha = \varphi$, $\alpha(e^1) \otimes \alpha(e^2) = e^1 \otimes e^2$, and $\varphi(ae^1)e^2 = e^1\varphi(e^2a) = a$.

(4) There exists a monoidal Hom-coalgebra $(A, \Delta, \varepsilon, \alpha)$, such that Δ is an (A, α) -Hom-bimodule morphism.

(5) There exists a Hom-associative, nondegenerate bilinear form for (A, α) . That is, there exists a bilinear map $B : A \times A \rightarrow k$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, such that $B(\alpha(a), bc) = B(ab, \alpha(c))$, and if $B(x, a) = 0$ or $B(a, x) = 0$ for any $x \in A$, then $a = 0$.

Definition 23. A Frobenius monoidal Hom-algebra (A, α) is called *symmetric* if $B(x, y) = B(y, x)$, for any $x, y \in A$, where B is the nondegenerate Hom-associative bilinear form of the Frobenius monoidal Hom-algebra (A, α) .

A linear map $\phi : (A, \alpha) \rightarrow (A, \alpha)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called *self-adjoint* (resp., *skew-adjoint*) with respect to a bilinear form B if for any $x, y \in A$, $B(\phi(x), y) = B(x, \phi(y))$ (resp., $B(\phi(x), y) = -B(x, \phi(y))$).

Then there is an induced injective linear map $\varphi : (A, \alpha) \rightarrow (A^*, (\alpha^*)^{-1})$ defined by the bilinear form for (A, α) :

$$B(a, b) = \langle \varphi(a), b \rangle, \tag{63}$$

for any $a, b \in A$. B being a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ implies that φ is too; that is, $\varphi\alpha = (\alpha^*)^{-1}\varphi$. For a given Frobenius monoidal Hom-algebra (A, α) , we always assume that φ is the induced morphism given by (63).

Lemma 24. Let (A, α) be a finite-dimensional monoidal Hom-algebra and let $e = e^1 \otimes e^2 \in A \otimes A$ be symmetric and $\alpha^{\otimes 2}$ -invariance. Then the following assertions are equivalent.

(1) e is invariant; that is, for any $a \in A$,

$$\alpha(e^1) \otimes ae^2 = e^1 a \otimes \alpha(e^2). \tag{64}$$

(2) e regarded as a linear map from $(A^*, (\alpha^*)^{-1})$ to (A, α) is balanced; that is, for any $x, y \in A^*$,

$$e(x) \longrightarrow (\alpha^*)^{-1}(y) = (\alpha^*)^{-1}(x) \longleftarrow e(y). \tag{65}$$

(3) e regarded as a linear map from $(A^*, \rightarrow, \leftarrow, (\alpha^*)^{-1})$ to (A, α) is an (A, α) -Hom-bimodule homomorphism; that is, for any $x \in A^*$, $a \in A$,

$$\begin{aligned}
 e(x)a &= e(x \leftarrow a), \\
 ae(x) &= e(a \rightarrow x).
 \end{aligned} \tag{66}$$

Proof. “(1) $\Leftrightarrow$ (2).” For any $a \in A$, $x, y \in A^*$, we have

$$\begin{aligned}
 & \langle \alpha(e^1) \otimes ae^2 - e^1 a \otimes \alpha(e^2), x \otimes y \rangle = \langle \alpha(e^1), x \rangle \\
 & \cdot \langle ae^2, y \rangle - \langle e^1 a, x \rangle \langle \alpha(e^2), y \rangle = \langle \alpha(e^1), x \rangle \\
 & \cdot \langle \alpha^2(a), e^2 \rightarrow y \rangle - \langle \alpha^2(a), x \leftarrow e^2 \rangle \\
 & \cdot \langle \alpha(e^1), y \rangle = \langle e^1, \alpha^*(x) \rangle \langle \alpha^2(a), e^2 \rightarrow y \rangle
 \end{aligned}$$

$$\begin{aligned}
& -\langle \alpha^2(a), x \leftarrow e^2 \rangle \langle e^1, \alpha^*(y) \rangle = \langle e\alpha^*(x) \\
& \rightarrow y - x \leftarrow e\alpha^*(y), \alpha^2(a) \rangle = \langle e(x) \\
& \rightarrow (\alpha^*)^{-1}(y) - (\alpha^*)^{-1}(x) \leftarrow e(y), \alpha^3(a) \rangle,
\end{aligned} \tag{67}$$

where the second step is followed by the actions $\rightarrow$ and $\leftarrow$ of $(A \cdot \alpha)$ on $(A^*, (\alpha^*)^{-1})$ and the symmetry of e , and the last step is true by the $\alpha^{\otimes 2}$ -invariance of e . So e is invariant if and only if it is balanced.

“(1) $\Leftrightarrow$ (3).” For any $a \in A$, $x, y \in A^*$, we have

$$\begin{aligned}
& \langle \alpha(e^1) \otimes ae^2 - e^1a \otimes \alpha(e^2), x \otimes y \rangle \\
& = \langle \alpha(e^1), x \rangle \langle ae^2, y \rangle - \langle e^1a, x \rangle \langle \alpha(e^2), y \rangle \\
& = \langle ae\alpha^*(x), y \rangle \\
& \quad - \langle \alpha^2(e^1), \alpha(a) \rightarrow x \rangle \langle \alpha(e^2), y \rangle \\
& = \langle ae\alpha^*(x), y \rangle - \langle e^1, a \rightarrow \alpha^*(x) \rangle \langle e^2, y \rangle \\
& = \langle ae\alpha^*(x), y \rangle - \langle e(a \rightarrow \alpha^*(x)), y \rangle \\
& = \langle ae\alpha^*(x) - e(a \rightarrow \alpha^*(x)), y \rangle,
\end{aligned} \tag{68}$$

where the third step is followed by the $\alpha^{\otimes 2}$ -invariance of e and axiom (51) of action $\rightarrow$. So $\alpha(e^1) \otimes ae^2 = e^1a \otimes \alpha(e^2)$ if and only if $e(a \rightarrow x) = ae(x)$. Similarly, $\alpha(e^1) \otimes ae^2 = e^1a \otimes \alpha(e^2)$ if and only if $e(x \leftarrow a) = e(x)a$, for any $a \in A$, $x \in A^*$. Thus, e is invariant if and only if e regarded as a linear map from $(A^*, \rightarrow, \leftarrow, (\alpha^*)^{-1})$ to (A, α) is an (A, α) -Hom-bimodule homomorphism. $\square$

Proposition 25. Let (A, α) be a finite-dimensional monoidal Hom-algebra, let $e = A \otimes A$ be symmetric and invariant. Suppose that e regarded as a linear map from $(A^*, (\alpha^*)^{-1})$ to (A, α) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ by $x \mapsto \langle x, e^1 \rangle e^2$ is invertible. Then $e^{-1} : (A, \alpha) \rightarrow (A^*, (\alpha^*)^{-1})$ regarded as a bilinear form $B(-, -) : A \otimes A \rightarrow k$ through equality (63) for $\varphi = e^{-1}$ is symmetric, nondegenerate, and Hom-associative. Thus (A, B, α) is a symmetric Frobenius monoidal Hom-algebra.

Proof. Since e is symmetric and e regarded as a linear map from $(A^*, (\alpha^*)^{-1})$ to (A, α) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is invertible, $B(-, -)$ is symmetric and nondegenerate. In addition, the invariance of e implies that e is balanced by Lemma 24. Then for any $a, b, c \in A$, $x = e^{-1}(a)$, $y = e^{-1}(b)$, and $z = e^{-1}(c)$, we have

$$\begin{aligned}
B(ab, \alpha(c)) &= \langle e^{-1}(ab), \alpha(c) \rangle \\
&= \langle e^{-1}(e(x)e(y)), \alpha e(z) \rangle \\
&= \langle e(x)e(y), (\alpha^*)^{-1}(z) \rangle \\
&= \langle \alpha^2 e(y), (\alpha^*)^{-1}(z) \leftarrow e(x) \rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle \alpha^2 e(y), e(z) \rightarrow (\alpha^*)^{-1}(x) \rangle \\
&= \langle e(y)e(z), (\alpha^*)^{-1}(x) \rangle \\
&= \langle e^{-1}(e(y)e(z)), \alpha e(x) \rangle \\
&= \langle e^{-1}(bc), \alpha(a) \rangle = B(bc, \alpha(a)) \\
&= B(\alpha(a), bc),
\end{aligned} \tag{69}$$

where the third step is from the equality $\langle e(x), y \rangle = \langle x, e(y) \rangle$ followed by the symmetry of e , and the last step is true by the fact that B is symmetric. Thus $B(-, -)$ is Hom-associative, and (A, α, B) is a symmetric Frobenius monoidal Hom-algebra. $\square$

Lemma 26. Let (A, α, B) be a symmetric Frobenius monoidal Hom-algebra of finite dimension. Suppose that $\beta : (A, \alpha) \rightarrow (A, \alpha)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is self-adjoint with respect to B . Then $\tilde{\beta} := \beta\varphi^{-1} : (A^*, (\alpha^*)^{-1}) \rightarrow (A, \alpha)$ regarded as an element of $A \otimes A$ is symmetric, where $\varphi : (A, \alpha) \rightarrow (A, \alpha)$ is defined by equality (63). Moreover, β is a balanced (A, α) -Hom-bimodule homomorphism if and only if $\tilde{\beta}$ is a balanced (A, α) -Hom-bimodule homomorphism.

Proof. For any $x, y, z \in A^*$, set $a = \varphi^{-1}(x)$, $b = \varphi^{-1}(y)$, and $c = \varphi^{-1}(z)$. Since β is self-adjoint with respect to B , we obtain $\langle \tilde{\beta}(x), y \rangle = \langle x, \tilde{\beta}(y) \rangle$. Hence, $\tilde{\beta}$ is symmetric as an element of $A \otimes A$. Further, since B is symmetric and Hom-associative, and β is self-adjoint with respect to B , for any $x, y \in A^*$, $c \in A$, and $a = \varphi^{-1}(x)$, $b = \varphi^{-1}(y)$, we have

$$\begin{aligned}
& \langle \tilde{\beta}(x) \rightarrow (\alpha^*)^{-1}(y), c \rangle \\
&= \langle \beta(a) \rightarrow (\alpha^*)^{-1}\varphi(b), c \rangle \\
&= \langle (\alpha^*)^{-1}\varphi(b), \alpha^{-2}(c)\beta(a) \rangle \\
&= \langle \varphi(b), \alpha^{-3}(c)\beta\alpha^{-1}(a) \rangle \\
&= B(b, \alpha^{-3}(c)\beta\alpha^{-1}(a)), \\
& \langle (\alpha^*)^{-1}(x) \leftarrow \tilde{\beta}(y), c \rangle \\
&= \langle (\alpha^*)^{-1}\varphi(a) \leftarrow \beta(b), c \rangle \\
&= \langle (\alpha^*)^{-1}\varphi(a), \beta(b)\alpha^{-2}(c) \rangle \\
&= \langle \varphi(a), \alpha^{-1}\beta(b)\alpha^{-3}(c) \rangle \\
&= B(a, \alpha^{-1}\beta(b)\alpha^{-3}(c)) = B(\alpha^{-1}\beta(b)\alpha^{-3}(c), a) \\
&= B(\beta(b), \alpha^{-3}(c)\alpha^{-1}(a)) \\
&= B(b, \beta(\alpha^{-3}(c)\alpha^{-1}(a))).
\end{aligned} \tag{70}$$

So $\tilde{\beta}(x) \rightarrow (\alpha^*)^{-1}(y) = (\alpha^*)^{-1}(x) \leftarrow \tilde{\beta}(y)$ if and only if $\beta(\alpha^{-3}(c)\alpha^{-1}(a)) = \alpha^{-3}(c)\beta\alpha^{-1}(a)$. Since $\alpha \in \text{Aut}_k(A)$, they are all equivalent to $\beta(ab) = a\beta(b)$.

On the other hand, according to the above proof, we have

$$\begin{aligned} \langle \tilde{\beta}(x) \rightarrow (\alpha^*)^{-1}(y), c \rangle &= B(b, \alpha^{-3}(c) \beta \alpha^{-1}(a)) \\ &= B(\alpha^{-1}(b) \alpha^{-3}(c), \beta(a)) \\ &= B(\beta(a), \alpha^{-1}(b) \alpha^{-3}(c)) \\ &= B(a, \beta(\alpha^{-1}(b) \alpha^{-3}(c))), \end{aligned} \quad (71)$$

and $\langle (\alpha^*)^{-1}(x) \leftarrow \tilde{\beta}(y), c \rangle = B(a, \beta \alpha^{-1}(b) \alpha^{-3}(c))$. So $\tilde{\beta}(x) \rightarrow (\alpha^*)^{-1}(y) = (\alpha^*)^{-1}(x) \leftarrow \tilde{\beta}(y)$ if and only if $\beta(\alpha^{-1}(b) \alpha^{-3}(c)) = \beta \alpha^{-1}(b) \alpha^{-3}(c)$ if and only if $\beta(ab) = \beta(a)b$.

In summary, $\tilde{\beta}$ is balanced if and only if β is an (A, α) -Hom-bimodule homomorphism. Then Lemma 24 implies that β is a balanced (A, α) -Hom-bimodule homomorphism if and only if $\tilde{\beta}$ is too. $\square$

In particular, set $\beta = \text{id}$ in Lemma 26; then $\tilde{\beta} = \varphi^{-1}$ is a balanced (A, α) -Hom-bimodule homomorphism.

Corollary 27. *Let (A, α, B) be a symmetric Frobenius monoidal Hom-algebra of finite-dimension, and let $\varphi : (A, \alpha) \rightarrow (A^*, (\alpha^*)^{-1})$ be the linear map defined in equality (63). Suppose $\beta \in A \otimes A$ is symmetric. Then β regarded as a linear map from $(A^*, (\alpha^*)^{-1})$ to (A, α) is a balanced (A, α) -Hom-bimodule homomorphism if and only if $\tilde{\beta} := \beta\varphi : (A, \alpha) \rightarrow (A, \alpha)$ is a balanced (A, α) -Hom-bimodule homomorphism too.*

Proof. Since $\beta \in A \otimes A$ is symmetric, for any $a, b \in A$,

$$\begin{aligned} \langle \beta, \varphi(a) \otimes \varphi(b) \rangle &= \langle \beta, \varphi(b) \otimes \varphi(a) \rangle \\ &\iff \langle \beta\varphi(a), \varphi(b) \rangle \\ &= \langle \beta\varphi(b), \varphi(a) \rangle \\ &\iff B(\hat{\beta}(a), b) = B(\hat{\beta}(b), a). \end{aligned} \quad (72)$$

Thus $\hat{\beta}$ is self-adjoint with respect to B , and the assertion follows from Lemma 26. $\square$

Theorem 28. *Let (A, α, B) be a symmetric Frobenius monoidal Hom-algebra of finite-dimension and let β be an endomorphism of (A, α) in $\mathcal{H}(\mathcal{M}_k)$. Suppose that β is skew-adjoint with respect to B . Then $\tilde{\beta} := \beta\varphi^{-1}$ regarded as an element of $(A \otimes A, \alpha \otimes \alpha)$ is skew-symmetric and $\tilde{\beta}$ is a solution of AHYBE if and only if β is a Hom- $\mathcal{O}$ -operator of weight zero.*

Proof. Since β is skew-adjoint with respect to B , we have

$$\langle \beta(a), \varphi(b) \rangle + \langle \varphi(a), \beta(b) \rangle = 0, \quad (73)$$

for any $a, b \in A$. Hence,

$$\langle \tilde{\beta}(x), y \rangle + \langle x, \tilde{\beta}(y) \rangle = 0, \quad (74)$$

for any $x, y \in A^*$. So $\tilde{\beta}$ regarded as an element of $(A \otimes A, \alpha \otimes \alpha)$ is skew-symmetric. Then Proposition 20 implies that $\tilde{\beta}$ is a solution of AHYBE if and only if $\tilde{\beta}$ is an $\mathcal{O}$ -operator of weight zero, since for any $x, y \in A^*$, and $a = \varphi^{-1}(x)$, $b = \varphi^{-1}(y)$,

$$\begin{aligned} \tilde{\beta}(x) \tilde{\beta}(y) &= \tilde{\beta}(x \leftarrow \tilde{\beta}(y)) + \tilde{\beta}(\tilde{\beta}(x) \rightarrow y) \\ &\iff \beta\varphi^{-1}(x) \beta\varphi^{-1}(y) \\ &= \beta\varphi^{-1}(x \leftarrow \beta\varphi^{-1}(y)) \\ &\quad + \beta\varphi^{-1}(\beta\varphi^{-1}(x) \rightarrow y) \\ &\iff \beta(a) \beta(b) \\ &= \beta\varphi^{-1}(\varphi(a) \leftarrow \beta(b)) \\ &\quad + \beta\varphi^{-1}(\beta(a) \rightarrow \varphi(b)) \\ &\iff \beta(a) \beta(b) \\ &= \beta(a \leftarrow \beta(b)) + \beta(\beta(a) \rightarrow b), \end{aligned} \quad (75)$$

where the last equivalence follows by the fact that φ^{-1} is a homomorphism of Hom-bimodules. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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