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Research Article

Optimal Homotopy Asymptotic Solution for Exothermic Reactions Model with Constant Heat Source in a Porous Medium

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The heat flow patterns profiles are required for heat transfer simulation in each type of the thermal insulation. The exothermic reaction models in porous medium can prescribe the problems in the form of nonlinear ordinary differential equations. In this research, the driving force model due to the temperature gradients is considered. A governing equation of the model is restricted into an energy balance equation that provides the temperature profile in conduction state with constant heat source on the steady state. The proposed optimal homotopy asymptotic method (OHAM) is used to compute the solutions of the exothermic reactions equation.

1. Introduction

In physical systems, energy is obtained from chemical bonds. If bonds are broken, energy is needed. If bonds are formed, energy is released. Each type of bond has specific bond energy. It can be predicted whether a chemical reaction would release or need heat by using bond energies. If there is more energy used to form the bonds than to break the bonds, heat is given off. This is well known as an exothermic reaction. On the other hand, if a reaction needs an input of energy, it is said to be an endothermic reaction. The ability to break bonds is activated energy.

Convection has obtained growth uses in many areas such as solar energy conversion, underground coal gasification, geothermal energy extraction, ground water contaminant transport, and oil reservoir simulation. The exothermic reaction model is focused on the system in which the driving force was due to the applied temperature gradients at the boundary of the system. In [1–4], they proposed the investigation of Rayleigh-Bernard-type convection. They also study the convective instabilities that arise due to exothermic reactions model in a porous medium. The exothermic reactions release the heat, create density differences within the fluid, and induce natural convection that turn out the rate of reaction affects [5]. The nonuniform flow of convective motion that is generated by heat sources is investigated by [6–8]. In [9–13], they propose the two- and three-dimensional models of natural convection among different types of porous medium.

In this research, the optimal homotopy asymptotic method for conduction solutions is proposed. The model equation is a steady-state energy balance equation of the temperature profile in conduction state with constant heat source.

The optimal homotopy asymptotic method is an approximate analytical tool that is simple and straightforward and does not require the existence of any small or large parameter as does traditional perturbation method. As observed by Herişanu and Marinca [14], the most significant feature OHAM is the optimal control of the convergence of solutions via a particular convergence-control function H and this ensures a very fast convergence when its components (known as convergence-control parameters) are optimally

determined. In the recent paper of Herişanu et al. [15] where the authors focused on nonlinear dynamical model of a permanent magnet synchronous generator, in their study a different way of construction of homotopy is developed to ensure the fast convergence of the OHAM solutions to the exact one. Optimal Homotopy Asymptotic Method (OHAM) has been successfully been applied to linear and nonlinear problems [16, 17]. This paper is organized as follows. First, in Section 2, exothermic reaction model is presented. In Section 3, we described the basic principles of the optimal homotopy asymptotic method. The optimal homotopy asymptotic method solution of the problem is given in Section 4. Section 5 is devoted for the concluding remarks.

2. Exothermic Reactions Model

In this section, we introduce a pseudohomogeneous model to express convective driven by an exothermic reaction. The case of a porous medium wall thickness ($0 < z' < L$) is focused. The normal assumption in the continuity and momentum equations in the steady-state energy balance presents a nondimensional form of a BVP for the temperature profile [5, 13]:

$$\frac{d^2\theta_0}{dz^2} + B\phi^2\left(1 - \frac{\theta_0}{B}\right)\exp\left(\frac{\gamma\theta_0}{\gamma + \theta_0}\right) = 0. \tag{1}$$

Here, θ_0 is the temperature, the parameter B is the maximum feasible temperature in the absence of natural convection, ϕ^2 is the ratio of the characteristic time for diffusion of heat generator, and γ is the dimensionless activation energy. In the case of the constant heat source, (1) can be written as

$$\frac{d^2\theta_0}{dz^2} + B\phi^2\left(1 - \frac{\theta_0}{B}\right) = 0, \tag{2}$$

subject to boundary condition

$$\begin{aligned} \frac{d\theta_0}{dz} &= 0, \quad \text{at } z = 0, \\ \theta_0 &= 0, \quad \text{at } z = 1. \end{aligned} \tag{3}$$

3. Basic Principles of Optimal Homotopy Asymptotic Method

We review the basic principles of the optimal homotopy asymptotic method as follows.

(i) Consider the following differential equation:

$$A[u(x)] + a(x) = 0, \quad x \in \Omega, \tag{4}$$

where Ω is problem domain, $A(u) = L(u) + N(u)$, where L, N are linear and nonlinear operators, $u(x)$ is an unknown function, and $a(x)$ is a known function.

(ii) Construct an optimal homotopy equation as

$$\begin{aligned} &(1-p)\left[L(\phi(x;p)) + a(x)\right] \\ &\quad - H(p)\left[A(\phi(x;p)) + a(x)\right] = 0, \end{aligned} \tag{5}$$

where $0 \le p \le 1$ is an embedding parameter and $H(p) = \sum_{i=1}^{m} p^i K_i$ is auxiliary function on which the convergence of the solution greatly dependent. Here, K_j are the convergence-control parameters. The auxiliary function $H(p)$ also adjusts the convergence domain and controls the convergence region.

(iii) Expand $\phi(x; p, K_j)$ in Taylor's series about p, one has an approximate solution:

$$\phi(x; p, K_j) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, K_j)\, p^k, \qquad j = 1, 2, 3, \ldots. \tag{6}$$

Many researchers have observed that the convergence of the series equation (6) depends upon K_j, $(j = 1, 2, \ldots, m)$; if it is convergent then, we obtain

$$\tilde{v} = v_0(x) + \sum_{k=1}^{m} v_k(x; K_j). \tag{7}$$

(iv) Substituting (7) in (4), we have the following residual:

$$R(x; K_j) = L(\tilde{u}(x; K_j)) + a(x) + N(\tilde{u}(x; K_j)). \tag{8}$$

If $R(x; K_j) = 0$, then $\tilde{v}$ will be the exact solution. For nonlinear problems, generally, this will not be the case. For determining K_j, $(j = 1, 2, \ldots, m)$, collocation method, Ritz method, or the method of least squares can be used.

(v) Finally, substituting the optimal values of the convergence-control parameters K_j in (7) one can get the approximate solution.

4. Application of OHAM to an Exothermic Reaction Model

Applying OHAM on (2), the zeroth, first, and second order problems are

$$(1-p)(\theta_0'') - H(p)\left(\theta'' + B\phi^2\left(1 - \frac{\theta_0}{B}\right)\right) = 0. \tag{9}$$

We consider θ_0, $H(p)$ in the following manner:

$$\begin{aligned} \theta &= \theta_{0,0} + p\theta_{0,1} + p^2\theta_{0,2}. \\ H_1(p) &= pK_1 + p^2K_2, \end{aligned} \tag{10}$$

4.1. Zeroth Order Problem

$$\theta_{0,0}'' = 0, \tag{11}$$

with boundary conditions

$$\begin{aligned} \theta_{0,0}(1) &= 0, \\ \theta_{0,0}'(0) &= 0. \end{aligned} \tag{12}$$

The solution of (11) with boundary condition (12) is

$$\theta_{0,0}(z) = 0. \tag{13}$$

4.2. First Order Problem

$$\theta''_{0,1} - K_1\phi^2 B = 0, \tag{14}$$

with boundary conditions

$$\begin{aligned} \theta_{0,1}(1) &= 0, \\ \theta'_{0,1}(0) &= 0. \end{aligned} \tag{15}$$

The solution of (14) with boundary condition (15) is

$$\theta_{0,1}(z, K_1) = \frac{K_1\phi^2 B}{2}\left(z^2 - 1\right). \tag{16}$$

4.3. Second Order Problem

$$\begin{aligned} \theta''_{0,2}(z, K_1, K_2) &= K_1\phi^2 B + K_1^2\phi^2 B - \frac{1}{2}K_1^2\phi^4 B z^2 \\ &\quad + \frac{1}{2}K_1^2\phi^4 B + \frac{1}{2}K_2\phi^2 B, \end{aligned} \tag{17}$$

with boundary conditions

$$\begin{aligned} \theta_{0,2}(1) &= 0, \\ \theta'_{0,2}(0) &= 0. \end{aligned} \tag{18}$$

The solution of (17) with boundary condition (18) is

$$\begin{aligned} \theta_{0,2}(z, K_1, K_2) &= \frac{-1}{24}\phi^4 K_1^2 B z^4 + \frac{1}{2}\phi^2 K_1 B z^2 \\ &\quad + \frac{1}{2}\phi^2 K_1^2 B z^2 + \frac{1}{4}\phi^4 K_1^2 B z^2 \\ &\quad + \frac{1}{2}\phi^2 K_2 B z^2 - \frac{5}{24}\phi^4 K_1^2 B \\ &\quad - \frac{1}{2}\phi^2 K_1 B - \frac{1}{2}\phi^2 K_1^2 B - \frac{1}{2}\phi^2 K_2 B. \end{aligned} \tag{19}$$

The final three terms solution via OHAM for $p = 1$ is

$$\begin{aligned} \tilde{\theta}_0(z, K_1, K_2) &= \theta_{0,0}(z) + \theta_{0,1}(z, K_1) \\ &\quad + \theta_{0,2}(z, K_1, K_2). \end{aligned} \tag{20}$$

The method of least squares is used to determine the convergence control parameters K_1 and K_2 in (20). In particular case for $\phi = 1$, $B = 10$, the values of the convergence control parameters are $K_1 = -0.8337205022$ and $K_2 = -0.02092667470$.

By substituting the values of K_1 and K_2 in (20) and after simplification, we can obtain the second order approximate solution via OHAM. To check the accuracy of the OHAM solution, a comparison between the solutions determined by OHAM and numerical methods was made and is presented in Table 1. Graphical representation of the solution using *finite difference technique* [5], OHAM, and *Runge-Kutta Fehlberg fourth fifth* order method is shown in Figure 1; an excellent agreement can be observed. We can see that the OHAM gives a better accurate solution than the traditional finite difference technique of [5]. On the other hand, the OHAM gives a continuity solution, but the traditional finite difference technique gives a discrete solution. It follows that the solutions of the OHAM is easier to implement than the finite difference solutions.

In Figure 2, we exhibit the effect of different values of ϕ with fixed value of B on temperature profile.

Table 1: Comparison of $\theta_0(z)$ via OHAM and RKF45 for $\phi = 1$, $B = 10$.

Z	FDM [5]	RKF45	OHAM	Percentage error
0.0	3.114344	3.518277	3.518285	0.000227
0.1	3.046176	3.485927	3.485969	0.001204
0.2	2.911251	3.388613	3.388675	0.001829
0.3	2.711819	3.225339	3.225359	0.000620
0.4	2.451166	2.994264	2.994284	0.000667
0.5	2.133897	2.693071	2.693037	0.001262
0.6	1.766284	2.318441	2.318432	0.000388
0.7	1.356680	1.866723	1.866701	0.001178
0.8	0.915960	1.333395	1.333311	0.006299
0.9	0.457980	0.713042	0.713046	0.000560
1.0	0.000000	0.000000	0.000000	—

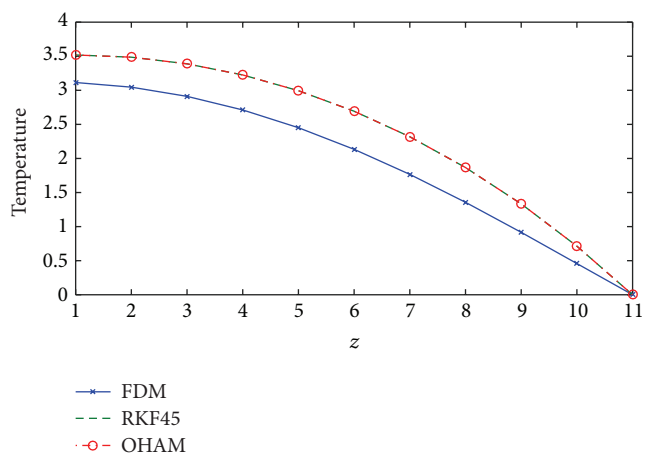

Figure 1: Comparison of analytical and numerical solution.

5. Concluding Remarks

In this paper, one has described an optimal homotopy asymptotic technique for obtaining the temperature profiles in porous medium. We can see that the temperature reduces to the end. The OHAM scheme for obtaining the model is convenient to implement. The OHAM gives fourth order accurate solutions. It follows that the method has no instability problem. The model should be considered in the case of nonconstant heat source.

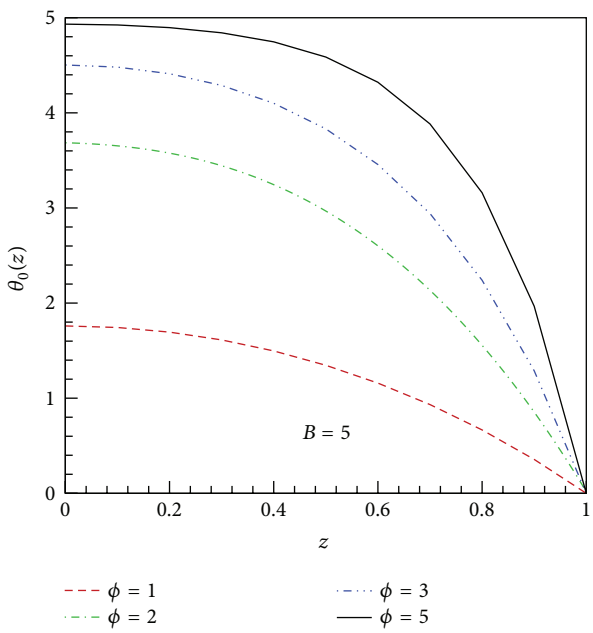

Figure 2

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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