Research Article

Bi-Integrable Couplings of a New Soliton Hierarchy Associated with $SO(4)$

Yan Cao,^{1,2} Liangyun Chen,¹ and Baiying He¹

¹School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

²Harbin University of Science and Technology, Rongcheng Campus, Rongcheng 264300, China

Correspondence should be addressed to Liangyun Chen; chenly640@nenu.edu.cn

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Based on the six-dimensional real special orthogonal Lie algebra $SO(4)$, a new Lax integrable hierarchy is obtained by constructing an isospectral problem. Furthermore, we construct bi-integrable couplings for this hierarchy from the enlarged matrix spectral problems and the enlarged zero curvature equations. Hamiltonian structures of the obtained bi-integrable couplings are constructed by the variational identity.

1. Introduction

As is well known, integrable equations are a remarkable class of nonlinear equations. To enrich multicomponent integrable equations, it has been an important task to explore more integrable properties for multi-integrable couplings. For example, one can find work on the integrable couplings [1, 2]. It is always interesting to explore any new procedure for generating integrable couplings for different soliton hierarchies, even from existing nonsemisimple Lie algebras.

The trace identity provides a simple and effective way to search for Hamiltonian structures of the hierarchies, which is proposed by Tu [3, 4]. However the trace identity is based on the semisimple Lie algebras. It is no longer applicable, when the trace identity comes to the nonsemisimple Lie algebras. So the variational identity is proposed, which can be used to obtain the Hamiltonian structures in the case of nonsemisimple Lie algebras [5–7].

Searching for integrable couplings of systems will become more and more meaningful in the area of research in mathematical physics. In [8], Ma et al. propose a new way to generate integrable couplings through a few classes of matrix Lie algebras consisting of block matrices. Recently, bi-integrable couplings and tri-integrable couplings for the KDV hierarchy and the AKNS hierarchy have been studied

considerably [9, 10]. Bi-integrable couplings of a new soliton hierarchy associated with $SO(3)$ have been constructed [11].

An integrable coupling of a given system

$$u_t = K(u) = K(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) \quad (1)$$

is a triangular integrable system of the following form [12]:

$$\begin{aligned} u_t &= K(u), \\ v_t &= S(u, v), \end{aligned} \quad (2)$$

where u is a function of variables t and x , $u_x = \partial u / \partial x$, and $u_t = \partial u / \partial t$. If S is nonlinear with respect to the second dependent variable v , the integrable coupling is called nonlinear.

A bi-integrable coupling of given integrable system (1) is an enlarged triangular integrable system of the following form [8]:

$$\begin{aligned} u_t &= K(u), \\ u_{1,t} &= S_1(u, u_1), \\ u_{2,t} &= S_2(u, u_1, u_2). \end{aligned} \quad (3)$$

Integrable couplings correspond to nonsemisimple Lie algebras $\bar{g}$, and such Lie algebras can be written as semidirect sums [13]:

$$\bar{g} = g \uplus g_c, \quad g\text{—semisimple, } g_c\text{—solvable.} \quad (4)$$

The notion of semidirect sums $\bar{g} = g \uplus g_c$ means that g and g_c satisfy $[g, g_c] \subseteq g_c$, where $[g, g_c] = \{[A, B] \mid A \in g, B \in g_c\}$, with $[\cdot, \cdot]$ denoting the Lie bracket of $\bar{g}$. Obviously, g_c is an ideal of $\bar{g}$. The subscript c indicates a contribution to the construction of coupling systems. We also require the closure property between g and g_c under the matrix multiplication: $gg_c, g_c g \subseteq g_c$, where $gg_c = \{AB \mid A \in g, B \in g_c\}$.

In this paper, we will construct bi-integrable couplings associated with $SO(4)$ for a new hierarchy from the enlarged matrix spectral problems and the enlarged zero curvature equations. Our work is essentially motivated by [11, 14, 15].

2. A New Soliton Hierarchy

Firstly, we will use the six-dimensional real special orthogonal Lie algebra $SO(4)$ [16] to construct a new soliton hierarchy and consider corresponding Hamiltonian structure by means of the trace identity [3, 4]. This Lie algebra G has the following basis:

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ e_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ e_5 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ e_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (5)$$

Set $[a, b] = ab - ba$, $\forall a, b \in G$. Then the circular commutator relations are

$$\begin{aligned} [e_1, e_2] &= 0, \\ [e_1, e_3] &= -e_6, \\ [e_1, e_4] &= -e_5, \\ [e_1, e_5] &= e_4, \\ [e_1, e_6] &= e_3, \\ [e_2, e_3] &= -e_5, \\ [e_2, e_4] &= -e_6, \\ [e_2, e_5] &= e_3, \\ [e_2, e_6] &= e_4, \\ [e_3, e_4] &= 0, \\ [e_3, e_5] &= -e_2, \\ [e_3, e_6] &= -e_1, \\ [e_4, e_5] &= -e_1, \\ [e_4, e_6] &= -e_2, \\ [e_5, e_6] &= 0. \end{aligned} \quad (6)$$

The corresponding loop algebra $\tilde{G}$ is given by

$$\begin{aligned} \tilde{G} &= \{e_1(n), e_2(n), e_3(n), e_4(n), e_5(n), e_6(n)\}, \\ e_i(n) &= e_i \lambda^n, \quad i = 1, \dots, 6, \end{aligned} \quad (7)$$

along with the commutators $[e_i(m), e_j(n)] = [e_i, e_j] \lambda^{m+n}$, $m, n \in \mathbb{Z}$, $1 \leq i, j \leq 6$.

Consider an isospectral problem

$$\begin{aligned} \varphi_x &= U\varphi, \\ \varphi_t &= V\varphi, \\ \lambda_t &= 0, \end{aligned} \quad (8)$$

where U, V are defined as

$$\begin{aligned} U &= U(u, \lambda) = \begin{pmatrix} 0 & -u_2 & u_4 & -u_1 \\ u_2 & 0 & -\lambda & -u_5 \\ -u_4 & \lambda & 0 & -u_3 \\ u_1 & u_5 & u_3 & 0 \end{pmatrix}, \\ V &= V(u, \lambda) = \begin{pmatrix} 0 & -c & f & -a \\ c & 0 & -b & -g \\ -f & b & 0 & -d \\ a & g & d & 0 \end{pmatrix}, \end{aligned} \quad (9)$$

$$\begin{aligned}
a &= \sum_{i \geq 0} a_i \lambda^{-i}, \\
b &= \sum_{i \geq 0} b_i \lambda^{-i}, \\
c &= \sum_{i \geq 0} c_i \lambda^{-i}, \\
d &= \sum_{i \geq 0} d_i \lambda^{-i}, \\
f &= \sum_{i \geq 0} f_i \lambda^{-i}, \\
g &= \sum_{i \geq 0} g_i \lambda^{-i}.
\end{aligned} \tag{10}$$

The stationary zero curvature representation $V_x = [U, V]$ gives

$$\begin{aligned}
a_x &= -u_2 g - u_3 f + u_4 d + u_5 c, \\
b_x &= -u_2 f - u_3 g + u_4 c + u_5 d, \\
c_x &= \lambda f + u_1 g - u_4 b - u_5 a, \\
d_x &= \lambda g + u_1 f - u_4 a - u_5 b, \\
f_x &= -\lambda c - u_1 d + u_2 b + u_3 a, \\
g_x &= -\lambda d - u_1 c + u_2 a + u_3 b.
\end{aligned} \tag{11}$$

Take the initial values

$$\begin{aligned}
a_0 &= \alpha, \\
b_0 &= \beta, \\
c_0 &= d_0 = f_0 = g_0 = 0.
\end{aligned} \tag{12}$$

From (11), we have

$$\begin{aligned}
a_1 &= b_1 = 0, \\
c_1 &= \alpha u_3 + \beta u_2, \\
d_1 &= \alpha u_2 + \beta u_3, \\
f_1 &= \alpha u_5 + \beta u_4, \\
g_1 &= \alpha u_4 + \beta u_5, \\
a_2 &= -\beta (u_2 u_3 + u_4 u_5) - \frac{\alpha}{2} (u_2^2 + u_3^2 + u_4^2 + u_5^2), \\
c_2 &= -\alpha u_{5x} - \beta u_{4x} - u_1 (\alpha u_2 + \beta u_3), \\
b_2 &= -\alpha (u_2 u_3 + u_4 u_5) - \frac{\beta}{2} (u_2^2 + u_3^2 + u_4^2 + u_5^2), \\
d_2 &= -\alpha u_{4x} - \beta u_{5x} - u_1 (\alpha u_3 + \beta u_2), \\
f_2 &= \alpha u_{3x} + \beta u_{2x} - u_1 (\alpha u_4 + \beta u_5), \\
g_2 &= \alpha u_{2x} + \beta u_{3x} - u_1 (\alpha u_5 + \beta u_4).
\end{aligned} \tag{13}$$

Then the zero curvature equation $U_{t_m} - V_x^{[m]} + [U, V^{[m]}] = 0$ leads to the following Lax integrable hierarchy:

$$\begin{aligned}
u_{t_m} &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}_{t_m} = \begin{pmatrix} -a_{mx} \\ c_{mx} - u_1 g_m + u_4 b_m \\ d_{mx} - u_1 f_m + u_5 b_m \\ f_{mx} + u_1 d_m - u_2 b_m \\ g_{mx} + u_1 c_m - u_3 b_m \end{pmatrix} \\
&= J_1 \begin{pmatrix} -2a_m \\ -2c_m \\ -2d_m \\ -2f_m \\ -2g_m \end{pmatrix} = J_1 P_{1,m},
\end{aligned} \tag{14}$$

where

$$J_1 = \frac{1}{2} \begin{pmatrix} \partial & 0 & 0 & 0 & 0 \\ 0 & -(\partial + u_4 \partial^{-1} u_4) & -u_4 \partial^{-1} u_5 & u_4 \partial^{-1} u_2 & E \\ 0 & -u_5 \partial^{-1} u_4 & -(\partial + u_5 \partial^{-1} u_5) & u_5 \partial^{-1} u_2 + u_1 & F \\ 0 & u_2 \partial^{-1} u_4 & -(u_1 - u_2 \partial^{-1} u_5) & -(\partial + u_2 \partial^{-1} u_2) & G \\ 0 & -(u_1 - u_3 \partial^{-1} u_4) & u_3 \partial^{-1} u_5 & -u_3 \partial^{-1} u_2 & H \end{pmatrix}, \tag{15}$$

$$E = u_4 \partial^{-1} u_3 + u_1,$$

$$F = u_5 \partial^{-1} u_3,$$

$$\begin{aligned}
G &= -u_2 \partial^{-1} u_3, \\
H &= -(\partial + u_3 \partial^{-1} u_3),
\end{aligned} \tag{16}$$

where $\partial = d/dx$ and $\partial^{-1} = \int (d/dx) dx$.

From recurrence relations (11), we have

$$\begin{aligned}
P_{1,m} &= \begin{pmatrix} 0 & l_{12} & l_{13} & l_{14} & l_{15} \\ u_3 & u_2 \partial^{-1} u_4 & l_{23} & l_{24} & -u_2 \partial^{-1} u_3 \\ u_2 & l_{32} & u_3 \partial^{-1} u_5 & -u_3 \partial^{-1} u_2 & l_{35} \\ u_5 & l_{42} & u_4 \partial^{-1} u_5 & -u_4 \partial^{-1} u_2 & l_{45} \\ u_4 & u_5 \partial^{-1} u_4 & l_{53} & l_{54} & -u_5 \partial^{-1} u_3 \end{pmatrix} P_{1,m-1} \tag{17} \\
&= L_1 P_{1,m-1},
\end{aligned}$$

where L_1 is a recurrence operator and

$$\begin{aligned}
l_{12} &= \partial^{-1} (-u_3 \partial - u_4 u_1), \\
l_{13} &= \partial^{-1} (-u_2 \partial - u_5 u_1), \\
l_{14} &= \partial^{-1} (u_2 u_1 - u_5 \partial), \\
l_{15} &= \partial^{-1} (u_3 u_1 - u_4 \partial), \\
l_{23} &= -u_1 + u_2 \partial^{-1} u_5, \\
l_{24} &= -\partial - u_2 \partial^{-1} u_2, \\
l_{32} &= -u_1 + u_3 \partial^{-1} u_4, \\
l_{35} &= -\partial - u_3 \partial^{-1} u_3, \\
l_{42} &= \partial + u_4 \partial^{-1} u_4, \\
l_{45} &= -u_1 - u_4 \partial^{-1} u_3, \\
l_{53} &= \partial + u_5 \partial^{-1} u_5, \\
l_{54} &= -u_1 - u_5 \partial^{-1} u_2.
\end{aligned} \tag{18}$$

In what follows, we will construct the Hamiltonian structure for soliton hierarchy (14) by the trace identity [3, 4]:

$$\begin{aligned}
\frac{\delta}{\delta u} \int \text{tr} \left(\frac{\partial U}{\partial \lambda} V \right) dx &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(\frac{\partial U}{\partial u} V \right), \\
\gamma &= \frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(V^2)|.
\end{aligned} \tag{19}$$

In this $SO(4)$ case, we can easily obtain

$$\begin{aligned}
\text{tr} \left(V \frac{\partial U}{\partial \lambda} \right) &= -2b, \\
\text{tr} \left(V \frac{\partial U}{\partial u_1} \right) &= -2a, \\
\text{tr} \left(V \frac{\partial U}{\partial u_2} \right) &= -2c, \\
\text{tr} \left(V \frac{\partial U}{\partial u_3} \right) &= -2d, \\
\text{tr} \left(V \frac{\partial U}{\partial u_4} \right) &= -2f, \\
\text{tr} \left(V \frac{\partial U}{\partial u_5} \right) &= -2g.
\end{aligned} \tag{20}$$

Substituting the above equations into trace identity (19) yields

$$\frac{\delta}{\delta u} \int (-2b) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{pmatrix} -2a \\ -2c \\ -2d \\ -2f \\ -2g \end{pmatrix}. \tag{21}$$

Balancing coefficients of each power of λ in the above equality gives rise to

$$\frac{\delta}{\delta u} \int (-2b_{m+1}) dx = (\gamma - m) \begin{pmatrix} -2a_m \\ -2c_m \\ -2d_m \\ -2f_m \\ -2g_m \end{pmatrix}. \tag{22}$$

Checking the case with $m = 1$ gives $\gamma = 0$. Thus, we get the Hamiltonian structure for soliton hierarchy (14):

$$\begin{aligned}
u_{t_m} &= K_{1,m}(u) = \begin{pmatrix} -a_{mx} \\ c_{mx} - u_1 g_m + u_4 b_m \\ d_{mx} - u_1 f_m + u_5 b_m \\ f_{mx} + u_1 d_m - u_2 b_m \\ g_{mx} + u_1 c_m - u_3 b_m \end{pmatrix} \\
&= J_1 \frac{\delta H_{1,m}}{\delta u}, \quad H_{1,m} = 2 \int \frac{b_{m+1}}{m} dx, \quad m \geq 1.
\end{aligned} \tag{23}$$

It is said that hierarchy (14) has the Hamiltonian structure, and it is easy to verify that $J_1 L_1 = L_1^* J_1$. Therefore, hierarchy (14) is Liouville integrable.

3. Bi-Integrable Couplings and Hamiltonian Structures

3.1. *Bi-Integrable Couplings Associated with $SO(4)$.* We take the class of 3×3 block matrices in the form of

$$M_2(A_1, A_2, A_3) = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2 \\ 0 & 0 & A_1 \end{pmatrix}, \quad (24)$$

where α is an arbitrary nonzero constant and A_1, A_2 , and A_3 are square matrices of the same order. In the following, we define the corresponding nonsemisimple Lie algebra $\bar{g}$ by a semidirect sum

$$\bar{g}(\lambda) = g \ltimes g_c, \quad (25)$$

with

$$\begin{aligned} g &= \{M_2(A_1, 0, 0) \mid A \in \widetilde{SO(4)}\}, \\ g_c &= \{M_2(0, A_2, A_3) \mid A_2, A_3 \in \widetilde{SO(4)}\}, \end{aligned} \quad (26)$$

where loop algebra $\widetilde{SO(4)}$ is defined by

$$\begin{aligned} \widetilde{SO(4)} &= \{A(\lambda) \in SO(4) \mid \text{entries of } A(\lambda) \\ &\quad \text{—Laurent series in } \lambda\}. \end{aligned} \quad (27)$$

The corresponding matrix product reads

$$\begin{aligned} [M_2(A_1, A_2, A_3), M_2(B_1, B_2, B_3)] \\ = M_2(C_1, C_2, C_3), \end{aligned} \quad (28)$$

with C_1, C_2 , and C_3 being defined by

$$\begin{aligned} C_1 &= [A_1, B_1], \\ C_2 &= [A_1, B_2] + [A_2, B_1], \\ C_3 &= [A_1, B_3] + [A_3, B_1] + \alpha [A_2, B_2]. \end{aligned} \quad (29)$$

Based on this special nonsemisimple Lie algebra $\bar{g}(\lambda)$, we choose the corresponding enlarged spectral matrix

$\bar{U}_1 = M_2(U, U_1, U_2)$, where U is defined as in (9) U_1 and U_2 are defined as follows:

$$\begin{aligned} U_1 = U_1(u_1, \lambda) &= \begin{pmatrix} 0 & -u'_2 & u'_4 & -u'_1 \\ u'_2 & 0 & 0 & -u'_5 \\ -u'_4 & 0 & 0 & -u'_3 \\ u'_1 & u'_5 & u'_3 & 0 \end{pmatrix}, \\ u_1 &= \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \end{pmatrix}, \end{aligned} \quad (30)$$

$$\begin{aligned} U_2 = U_2(u_2, \lambda) &= \begin{pmatrix} 0 & -u''_2 & u''_4 & -u''_1 \\ u''_2 & 0 & 0 & -u''_5 \\ -u''_4 & 0 & 0 & -u''_3 \\ u''_1 & u''_5 & u''_3 & 0 \end{pmatrix}, \\ u_2 &= \begin{pmatrix} u''_1 \\ u''_2 \\ u''_3 \\ u''_4 \\ u''_5 \end{pmatrix}. \end{aligned}$$

In order to solve the enlarged stationary zero curvature equations $\bar{V}_{1x} = [\bar{U}_1, \bar{V}_1]$, we take $\bar{V}_1 = M_2(V, V_1, V_2)$, where V is defined by (9):

$$\begin{aligned} V = V(u, \lambda) &= \begin{pmatrix} 0 & -c & f & -a \\ c & 0 & -b & -g \\ -f & b & 0 & -d \\ a & g & d & 0 \end{pmatrix} \\ &= \sum_{i \geq 0} \begin{pmatrix} 0 & -c_i & f_i & -a_i \\ c_i & 0 & -b_i & -g_i \\ -f_i & b_i & 0 & -d_i \\ a_i & g_i & d_i & 0 \end{pmatrix} \lambda^{-i}. \end{aligned} \quad (31)$$

V_1 and V_2 are defined similarly:

$$\begin{aligned} V_1 = V_1(u, u_1, \lambda) &= \begin{pmatrix} 0 & -c' & f' & -a' \\ c' & 0 & -b' & -g' \\ -f' & b' & 0 & -d' \\ a' & g' & d' & 0 \end{pmatrix} \\ &= \sum_{i \geq 0} \begin{pmatrix} 0 & -c'_i & f'_i & -a'_i \\ c'_i & 0 & -b'_i & -g'_i \\ -f'_i & b'_i & 0 & -d'_i \\ a'_i & g'_i & d'_i & 0 \end{pmatrix} \lambda^{-i}, \end{aligned}$$

$$\begin{aligned}
V_2 = V_2(u, u_1, u_2, \lambda) &= \begin{pmatrix} 0 & -c'' & f'' & -a'' \\ c'' & 0 & -b'' & -g'' \\ -f'' & b'' & 0 & -d'' \\ a'' & g'' & d'' & 0 \end{pmatrix} \\
&= \sum_{i \geq 0} \begin{pmatrix} 0 & -c_i'' & f_i'' & -a_i'' \\ c_i'' & 0 & -b_i'' & -g_i'' \\ -f_i'' & b_i'' & 0 & -d_i'' \\ a_i'' & g_i'' & d_i'' & 0 \end{pmatrix} \lambda^{-i}.
\end{aligned} \tag{32}$$

It now follows from the enlarged stationary zero curvature equations $\bar{V}_{1x} = [\bar{U}_1, \bar{V}_1]$ that

$$\begin{aligned}
V_x &= [U, V], \\
V_{1x} &= [U, V_1] + [U_1, V], \\
V_{2x} &= [U, V_2] + [U_2, V] + \alpha [U_1, V_1].
\end{aligned} \tag{33}$$

The above equation system equivalently leads to (11) and the following equations:

$$\begin{aligned}
a'_x &= u_5 c' - u_3 f' - u_2 g' + u_4 d' + u_5' c - u_3' f - u_2' g \\
&\quad + u_4' d, \\
b'_x &= u_4 c' - u_3 g' - u_2 f' + u_5 d' + u_4' c - u_3' g - u_2' f \\
&\quad + u_5' d, \\
c'_x &= \lambda f' - u_5 a' - u_4 b' + u_1 g' - u_5' a - u_4' b + u_1' g, \\
d'_x &= \lambda g' - u_4 a' - u_5 b' + u_1 f' - u_4' a - u_5' b + u_1' f, \\
f'_x &= -\lambda c' + u_2 b' + u_3 a' - u_1 d' + u_2' b + u_3' a - u_1' d, \\
g'_x &= -\lambda d' + u_2 a' + u_3 b' - u_1 c' + u_2' a + u_3' b - u_1' c, \\
a''_x &= u_5 c'' - u_3 f'' - u_2 g'' + u_4 d'' + u_5'' c - u_3'' f \\
&\quad - u_2'' g + u_4'' d \\
&\quad + \alpha (u_5' c' - u_3' f' - u_2' g' + u_4' d'),
\end{aligned}$$

$$\begin{aligned}
b''_x &= u_4 c'' - u_3 g'' - u_2 f'' + u_5 d'' + u_4'' c - u_3'' g \\
&\quad - u_2'' f + u_5'' d \\
&\quad + \alpha (u_4' c' - u_3' g' - u_2' f' + u_5' d'), \\
c''_x &= \lambda f'' - u_5 a'' - u_4 b'' + u_1 g'' - u_5'' a - u_4'' b + u_1'' g \\
&\quad - \alpha (u_5' a' + u_4' b' - u_1' g'), \\
d''_x &= \lambda g'' - u_4 a'' - u_5 b'' + u_1 f'' - u_4'' a - u_5'' b \\
&\quad + u_1'' f - \alpha (u_4' a' + u_5' b' - u_1' f'), \\
f''_x &= -\lambda c'' + u_2 b'' + u_3 a'' - u_1 d'' + u_2'' b + u_3'' a \\
&\quad - u_1'' d + \alpha (u_2' b' + u_3' a' - u_1' d'), \\
g''_x &= -\lambda d'' + u_2 a'' + u_3 b'' - u_1 c'' + u_2'' a + u_3'' b \\
&\quad - u_1'' c + \alpha (u_2' a' + u_3' b' - u_1' c').
\end{aligned} \tag{34}$$

Now, let us define the enlarged Lax matrices $\bar{V}_1^{[m]} = (\lambda^m \bar{V}_1)_+ = M_2(V^{[m]}, V_1^{[m]}, V_2^{[m]})$, $m \geq 0$, where $V^{[m]}$ is defined as $V^{[m]} = (\lambda^m V)_+$, $m \geq 0$, and $V_i^{[m]} = (\lambda^m V_i)_+$, $i = 1, 2$.

Solving the enlarged zero curvature equations $\bar{U}_{1t_m} - \bar{V}_{1x}^{[m]} + [\bar{U}_1, \bar{V}_1^{[m]}] = 0$, $m \geq 0$, we get bi-integrable couplings for soliton hierarchy (14):

$$\begin{aligned}
\bar{u}_{t_m} &= \begin{pmatrix} u_{t_m} \\ u_{1t_m} \\ u_{2t_m} \end{pmatrix} = \begin{pmatrix} J_1 & 0 & 0 \\ J_2 & J_1 & 0 \\ J_3 & \alpha J_2 & J_1 \end{pmatrix} \begin{pmatrix} P_{1m} \\ P_{2m} \\ P_{3m} \end{pmatrix} \\
&= \begin{pmatrix} J_1 & 0 & 0 \\ J_2 & J_1 & 0 \\ J_3 & \alpha J_2 & J_1 \end{pmatrix} \bar{P}_{1,m},
\end{aligned} \tag{35}$$

with u_{t_m} , P_{1m} , and J_1 being defined by (14) and (15), with u_{1t_m} , u_{2t_m} , P_{2m} , P_{3m} , J_2 , and J_3 being defined as follows:

$$u_{1t_m} = \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \\ u'_5 \end{pmatrix}_{t_m} = \begin{pmatrix} -a'_{mx} \\ c'_{mx} - u_1 g'_m + u_4 b'_m - u'_1 g_m + u'_4 b_m \\ d'_{mx} - u_1 f'_m + u_5 b'_m - u'_1 f_m + u'_5 b_m \\ f'_{mx} + u_1 d'_m - u_2 b'_m + u'_1 d_m - u'_2 b_m \\ g'_{mx} + u_1 c'_m - u_3 b'_m + u'_1 c_m - u'_3 b_m \end{pmatrix},$$

$$\begin{aligned}
u_{2t_m} &= \begin{pmatrix} u_1'' \\ u_2'' \\ u_3'' \\ u_4'' \\ u_5'' \end{pmatrix}_{t_m} = \begin{pmatrix} -a_{mx}'' \\ c_{mx}'' + u_4 b_m'' - u_1 g_m'' + u_4'' b_m - u_1'' g_m + \alpha (u_4' b_m' - u_1' g_m') \\ d_{mx}'' + u_5 b_m'' - u_1 f_m'' + u_5'' b_m - u_1'' f_m + \alpha (u_5' b_m' - u_1' f_m') \\ f_{mx}'' - u_2 b_m'' + u_1 d_m'' - u_2'' b_m + u_1'' d_m + \alpha (u_1' d_m' - u_2' b_m') \\ g_{mx}'' - u_3 b_m'' + u_1 c_m'' - u_3'' b_m + u_1'' c_m + \alpha (u_1' c_m' - u_3' b_m') \end{pmatrix}, \\
P_{2m} &= \begin{pmatrix} -2a_m' \\ -2c_m' \\ -2d_m' \\ -2f_m' \\ -2g_m' \end{pmatrix}, \\
P_{3m} &= \begin{pmatrix} -2a_m'' \\ -2c_m'' \\ -2d_m'' \\ -2f_m'' \\ -2g_m'' \end{pmatrix}, \\
J_2 &= \frac{1}{2} \begin{pmatrix} \partial & 0 & 0 & 0 & 0 \\ 0 & -(u_4 \partial^{-1} u_4' + u_4' \partial^{-1} u_4) & -(u_4' \partial^{-1} u_5 + u_4 \partial^{-1} u_5') & B_1 & B_2 \\ 0 & -(u_5 \partial^{-1} u_4' + u_5' \partial^{-1} u_4) & -(u_5 \partial^{-1} u_5' + u_5' \partial^{-1} u_5) & B_3 & B_4 \\ 0 & u_2 \partial^{-1} u_4' + u_2' \partial^{-1} u_4 & (u_2 \partial^{-1} u_5' + u_2' \partial^{-1} u_5 - u_1') & B_5 & B_6 \\ 0 & u_3 \partial^{-1} u_4' + u_3' \partial^{-1} u_4 & u_3 \partial^{-1} u_5' + u_3' \partial^{-1} u_5 & B_7 & B_8 \end{pmatrix},
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
B_1 &= u_4 \partial^{-1} u_2' + u_4' \partial^{-1} u_2, \\
B_2 &= u_4 \partial^{-1} u_3' + u_4' \partial^{-1} u_3 + u_1', \\
B_3 &= u_5 \partial^{-1} u_2' + u_5' \partial^{-1} u_2 + u_1', \\
B_4 &= u_5 \partial^{-1} u_3' + u_5' \partial^{-1} u_3, \\
B_5 &= -(u_2 \partial^{-1} u_2' + u_2' \partial^{-1} u_2), \\
B_6 &= -(u_2 \partial^{-1} u_3' + u_2' \partial^{-1} u_3), \\
B_7 &= -(u_3 \partial^{-1} u_2' + u_3' \partial^{-1} u_2), \\
B_8 &= -(u_3 \partial^{-1} u_3' + u_3' \partial^{-1} u_3),
\end{aligned} \tag{37}$$

$$J_3 = \frac{1}{2} \begin{pmatrix} \partial & 0 & 0 & 0 & 0 \\ 0 & -(u_4 \partial^{-1} u_4'' + u_4'' \partial^{-1} u_4 + \alpha u_4' \partial^{-1} u_4') & C_1 & C_2 & C_3 \\ 0 & -(u_5 \partial^{-1} u_4'' + u_5'' \partial^{-1} u_4 + \alpha u_5' \partial^{-1} u_4') & C_4 & C_5 & C_6 \\ 0 & u_2 \partial^{-1} u_4'' + u_2'' \partial^{-1} u_4 + \alpha u_2' \partial^{-1} u_4' & C_7 & C_8 & C_9 \\ 0 & u_3 \partial^{-1} u_4'' + u_3'' \partial^{-1} u_4 + \alpha u_3' \partial^{-1} u_4' - u_1'' & C_{10} & C_{11} & C_{12} \end{pmatrix}, \tag{38}$$

where

$$\begin{aligned}
C_1 &= -\left(u_4 \partial^{-1} u_5'' + u_4'' \partial^{-1} u_5 + \alpha u_4' \partial^{-1} u_5'\right), \\
C_2 &= u_4 \partial^{-1} u_2'' + u_4'' \partial^{-1} u_2 + \alpha u_4' \partial^{-1} u_2', \\
C_3 &= u_4 \partial^{-1} u_3'' + u_4'' \partial^{-1} u_3 + \alpha u_4' \partial^{-1} u_3' + u_1'', \\
C_4 &= -\left(u_5 \partial^{-1} u_5'' + u_5'' \partial^{-1} u_5 + \alpha u_5' \partial^{-1} u_5'\right), \\
C_5 &= u_5 \partial^{-1} u_2'' + u_5'' \partial^{-1} u_2 + \alpha u_5' \partial^{-1} u_2' + u_1'', \\
C_6 &= u_5 \partial^{-1} u_3'' + u_5'' \partial^{-1} u_3 + \alpha u_5' \partial^{-1} u_3', \\
C_7 &= u_2 \partial^{-1} u_5'' + u_2'' \partial^{-1} u_5 + \alpha u_2' \partial^{-1} u_5' - u_1'', \\
C_8 &= -\left(u_2 \partial^{-1} u_2'' + u_2'' \partial^{-1} u_2 + \alpha u_2' \partial^{-1} u_2'\right), \\
C_9 &= -\left(u_2 \partial^{-1} u_3'' + u_2'' \partial^{-1} u_3 + \alpha u_2' \partial^{-1} u_3'\right), \\
C_{10} &= u_3 \partial^{-1} u_5'' + u_3'' \partial^{-1} u_5 + \alpha u_3' \partial^{-1} u_5', \\
C_{11} &= -\left(u_3 \partial^{-1} u_2'' + u_3'' \partial^{-1} u_2 + \alpha u_3' \partial^{-1} u_2'\right), \\
C_{12} &= -\left(u_3 \partial^{-1} u_3'' + u_3'' \partial^{-1} u_3 + \alpha u_3' \partial^{-1} u_3'\right).
\end{aligned} \tag{39}$$

3.2. Hamiltonian Structures. In this section, in order to generate the Hamiltonian structures of hierarchy (35), we use the corresponding variational identity [17]

$$\frac{\delta}{\delta u} \int \{\bar{V}, \bar{U}_\lambda\} dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \{\bar{V}, \bar{U}_u\}, \tag{40}$$

where $\{\cdot, \cdot\}$ is a required bilinear form, which is symmetric, nondegenerate, and invariant under the loop algebra $\bar{g}(\lambda)$.

First, we need to define a mapping [8]

$$\begin{aligned}
\sigma : \bar{g}(\lambda) &\longrightarrow R^{18}, \\
A &\longmapsto (a_1, a_2, \dots, a_{18})^T,
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
A &= M(A_1, A_2, A_3) \in \bar{g}(\lambda), \\
A_i &= \begin{pmatrix} 0 & -a_{6i-3} & a_{6i-1} & -a_{6i-5} \\ a_{6i-3} & 0 & -a_{6i-4} & -a_{6i} \\ -a_{6i-1} & a_{6i-4} & 0 & -a_{6i-2} \\ a_{6i-5} & a_{6i} & a_{6i-2} & 0 \end{pmatrix}, \quad i = 1, 2, 3.
\end{aligned} \tag{42}$$

This mapping σ induces a Lie algebraic structure on R^{18} . The Lie bracket $[\cdot, \cdot]$ on R^{18} can be computed as follows:

$$\begin{aligned}
[a, b] &= a^T \bar{R}_1(b), \\
\forall a &= (a_1, \dots, a_{18})^T, \quad b = (b_1, \dots, b_{18})^T \in R^{18},
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
\bar{R}_1(b) &= M_2(R_1, R_2, R_3), \\
R_i &= \begin{pmatrix} 0 & 0 & b_{6i} & b_{6i-1} & -b_{6i-2} & -b_{6i-3} \\ 0 & 0 & b_{6i-1} & b_{6i} & -b_{6i-3} & -b_{6i-2} \\ -b_{6i} & -b_{6i-1} & 0 & 0 & b_{6i-4} & b_{6i-5} \\ -b_{6i-1} & -b_{6i} & 0 & 0 & b_{6i-5} & b_{6i-4} \\ b_{6i-2} & b_{6i-3} & -b_{6i-4} & -b_{6i-5} & 0 & 0 \\ b_{6i-3} & b_{6i-2} & -b_{6i-5} & -b_{6i-4} & 0 & 0 \end{pmatrix}, \tag{44} \\
i &= 1, 2, 3.
\end{aligned}$$

Mapping (41) is a Lie algebra isomorphism between the two Lie algebras.

Then we define a bilinear form on R^{18} as follows:

$$\begin{aligned}
\{a, b\} &= a^T F_1 b, \\
\forall a &= (a_1, \dots, a_{18})^T, \quad b = (b_1, \dots, b_{18})^T \in R^{18}.
\end{aligned} \tag{45}$$

Following the properties of the matrix F_1 : $F_1(\bar{R}_1(b))^T = -\bar{R}_1(b)F_1$ and $F_1 = F_1^T$, we have

$$F_1 = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} r_1 & r_2 & 0 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_1 & r_2 & 0 & 0 \\ 0 & 0 & r_2 & r_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_1 & r_2 \\ 0 & 0 & 0 & 0 & r_2 & r_1 \end{pmatrix}, \tag{46}$$

where $\eta_1, \eta_2, \eta_3, r_1$, and r_2 are arbitrary constants. We choose $r_1 \neq \pm r_2$; then the nondegenerate condition requires

$$\det(F_1) = \alpha^6 \eta_3^{18} (r_1^2 - r_2^2)^9 \neq 0. \tag{47}$$

So, the bilinear form on the semidirect sum $\bar{g}(\lambda)$ is given by

$$\begin{aligned}
\{A, B\} &= (a_1, \dots, a_{18}) F_1 (b_1, \dots, b_{18}) = (r_1 a_1 b_1 \\
&+ r_2 a_2 b_1 + r_2 a_1 b_2 + r_1 a_2 b_2 + r_1 a_3 b_3 + r_2 a_4 b_3 \\
&+ r_2 a_3 b_4 + r_1 a_4 b_4 + r_1 a_5 b_5 + r_2 a_6 b_5 + r_2 a_5 b_6 \\
&+ r_1 a_6 b_6) \eta_1 + (r_1 a_7 b_1 + r_2 a_8 b_1 + r_2 a_7 b_2 + r_1 a_8 b_2 \\
&+ r_1 a_9 b_3 + r_2 a_{10} b_3 + r_2 a_9 b_4 + r_1 a_{10} b_4 + r_1 a_{11} b_5 \\
&+ r_2 a_{12} b_5 + r_2 a_{11} b_6 + r_1 a_{12} b_6 + r_1 a_1 b_7 + r_2 a_2 b_7 \\
&+ r_2 a_1 b_8 + r_1 a_2 b_8 + r_1 a_3 b_9 + r_2 a_4 b_9 + r_2 a_3 b_{10}
\end{aligned}$$

$$\begin{aligned}
& + r_1 a_4 b_{10} + r_1 a_5 b_{11} + r_2 a_6 b_{11} + r_2 a_5 b_{12} + r_1 a_6 b_{12}) \\
& \cdot \eta_2 + (r_1 a_{13} b_1 + r_2 a_{14} b_1 + r_2 a_{13} b_2 + r_1 a_{14} b_2 \\
& + r_1 a_{15} b_3 + r_2 a_{16} b_3 + r_2 a_{15} b_4 + r_1 a_{16} b_4 + r_1 a_{17} b_5 \\
& + r_2 a_{18} b_5 + r_1 a_{18} b_6 + r_2 a_{17} b_6 + \alpha r_1 a_7 b_7 + \alpha r_2 a_8 b_7 \\
& + \alpha r_2 a_7 b_8 + \alpha r_1 a_8 b_8 + \alpha r_1 a_9 b_9 + \alpha r_2 a_{10} b_9 \\
& + \alpha r_2 a_9 b_{10} + \alpha r_1 a_{10} b_{10} + \alpha r_1 a_{11} b_{11} + \alpha r_2 a_{12} b_{11} \\
& + \alpha r_2 a_{11} b_{12} + \alpha r_1 a_{12} b_{12} + r_1 a_1 b_{13} + r_2 a_2 b_{13} \\
& + r_2 a_1 b_{14} + r_1 a_2 b_{14} + r_1 a_3 b_{15} + r_2 a_4 b_{15} + r_2 a_3 b_{16} \\
& + r_1 a_4 b_{16} + r_1 a_5 b_{17} + r_2 a_6 b_{17} + r_2 a_5 b_{18} + r_1 a_6 b_{18}) \\
& \cdot \eta_3.
\end{aligned} \tag{48}$$

As mentioned above, now we can compute that

$$\begin{aligned}
\{\bar{V}_1, \bar{U}_{1,\lambda}\} &= (r_2 a + r_1 b) \eta_1 + (r_2 a' + r_1 b') \eta_2 \\
&\quad + (r_2 a'' + r_1 b'') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu_1}\} &= (r_1 a + r_2 b) \eta_1 + (r_1 a' + r_2 b') \eta_2 \\
&\quad + (r_1 a'' + r_2 b'') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu_2}\} &= (r_1 c + r_2 d) \eta_1 + (r_1 c' + r_2 d') \eta_2 \\
&\quad + (r_1 c'' + r_2 d'') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu_3}\} &= (r_2 c + r_1 d) \eta_1 + (r_2 c' + r_1 d') \eta_2 \\
&\quad + (r_2 c'' + r_1 d'') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu_4}\} &= (r_1 f + r_2 g) \eta_1 + (r_1 f' + r_2 g') \eta_2 \\
&\quad + (r_1 f'' + r_2 g'') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu_5}\} &= (r_2 f + r_1 g) \eta_1 + (r_2 f' + r_1 g') \eta_2 \\
&\quad + (r_2 f'' + r_1 g'') \eta_3,
\end{aligned}$$

$$\begin{aligned}
\{\bar{V}_1, \bar{U}_{1,\mu'_1}\} &= (r_1 a + r_2 b) \eta_2 + \alpha (r_1 a' + r_2 b') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu''_1}\} &= (r_2 b + r_1 a) \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu'_2}\} &= (r_1 c + r_2 d) \eta_2 + \alpha (r_1 c' + r_2 d') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu''_2}\} &= (r_1 c + r_2 d) \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu'_3}\} &= (r_2 c + r_1 d) \eta_2 + \alpha (r_2 c' + r_1 d') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu''_3}\} &= (r_2 c + r_1 d) \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu'_4}\} &= (r_1 f + r_2 g) \eta_2 + \alpha (r_1 f' + r_2 g') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu''_4}\} &= (r_1 f + r_2 g) \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu'_5}\} &= (r_2 f + r_1 g) \eta_2 + \alpha (r_2 f' + r_1 g') \eta_3, \\
\{\bar{V}_1, \bar{U}_{1,\mu''_5}\} &= (r_2 f + r_1 g) \eta_3.
\end{aligned} \tag{49}$$

and furthermore we use the formula [17]

$$\gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln \left| \{\bar{V}, \bar{V}\} \right|, \tag{50}$$

to obtain that $\gamma = 0$. Applying corresponding variational identity (40), we obtain the following Hamiltonian structures for the hierarchy of bi-integrable couplings (35):

$$\bar{u}_{t_m} = \bar{K}_{1,m}(\bar{u}) = \bar{J}_1 \frac{\delta \bar{H}_{1,m}}{\delta \bar{u}}; \quad m \geq 1, \tag{51}$$

with the Hamiltonian operator

$$\bar{J}_1 = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha \eta_3 & 0 \\ \eta_3 & 0 & 0 \end{pmatrix}^{-1} \otimes J_1, \tag{52}$$

where J_1 is defined by (15) and the Hamiltonian functionals

$$\bar{H}_{1,m} = - \int \frac{(r_2 a_{m+1} + r_1 b_{m+1}) \eta_1 + (r_1 a'_{m+1} + r_2 b'_{m+1}) \eta_2 + (r_2 a''_{m+1} + r_1 b''_{m+1}) \eta_3}{m} dx, \quad m \geq 1. \tag{53}$$

Based on (11) and (34), a direct computation yields a recursion relation

$$P_{1,m} = \bar{L}_1 P_{1,m-1}, \tag{54}$$

where

$$\bar{L}_1 = M_2^T (L_1, L_1^1, L_2^1) = \begin{pmatrix} L_1 & 0 & 0 \\ L_1^1 & L_1 & 0 \\ L_2^1 & \alpha L_1^1 & L_1 \end{pmatrix}, \tag{55}$$

with L_1 being defined by (17) L_1^1 and L_2^1 are defined as follows:

$$L_1^1 = \begin{pmatrix} 0 & \partial^{-1}(-u_4 u_1' - u_4' u_1 - u_3' \partial) & l_{13}' & l_{14}' & l_{15}' \\ u_3'' & u_2 \partial^{-1} u_4'' + u_2' \partial^{-1} u_4 & l_{23}'' & l_{24}'' & l_{25}'' \\ u_2'' & u_3 \partial^{-1} u_4'' + u_3' \partial^{-1} u_4 - u_1' & l_{33}'' & l_{34}'' & l_{35}'' \\ u_5'' & u_4 \partial^{-1} u_4'' + u_4' \partial^{-1} u_4 & l_{43}'' & l_{44}'' & l_{45}'' \\ u_4'' & u_5 \partial^{-1} u_4'' + u_5' \partial^{-1} u_4 & l_{53}'' & l_{54}'' & l_{55}'' \end{pmatrix}, \quad (56)$$

$$\begin{aligned} l_{13}' &= \partial^{-1}(-u_5 u_1' - u_5' u_1 - u_2' \partial), \\ l_{14}' &= \partial^{-1}(u_2 u_1' + u_2' u_1 - u_5' \partial), \\ l_{15}' &= \partial^{-1}(u_3 u_1' + u_3' u_1 - u_4' \partial), \\ l_{23}' &= u_2 \partial^{-1} u_5'' + u_2' \partial^{-1} u_5 - u_1', \\ l_{24}' &= -u_2 \partial^{-1} u_2'' - u_2' \partial^{-1} u_2, \\ l_{25}' &= -u_2 \partial^{-1} u_3'' - u_2' \partial^{-1} u_3, \\ l_{33}' &= u_3 \partial^{-1} u_5'' + u_3' \partial^{-1} u_5, \\ l_{34}' &= -u_3 \partial^{-1} u_2'' - u_3' \partial^{-1} u_2, \\ l_{35}' &= -u_3 \partial^{-1} u_3'' - u_3' \partial^{-1} u_3, \\ l_{43}' &= u_4 \partial^{-1} u_5'' + u_4' \partial^{-1} u_5, \\ l_{44}' &= -u_4 \partial^{-1} u_2'' - u_4' \partial^{-1} u_2, \\ l_{45}' &= -u_4 \partial^{-1} u_3'' - u_4' \partial^{-1} u_3 - u_1', \\ l_{53}' &= u_5 \partial^{-1} u_5'' + u_5' \partial^{-1} u_5, \\ l_{54}' &= -u_5 \partial^{-1} u_2'' - u_5' \partial^{-1} u_2 - u_1', \\ l_{55}' &= -u_5 \partial^{-1} u_3'' - u_5' \partial^{-1} u_3, \end{aligned} \quad (57)$$

$$L_2^1 = \begin{pmatrix} 0 & \partial^{-1}(-u_4 u_1'' - u_3'' \partial - u_4'' u_1 - \alpha u_4' u_1') & l_{13}'' & l_{14}'' & l_{15}'' \\ u_3'' & u_2 \partial^{-1} u_4'' + u_2' \partial^{-1} u_4 + \alpha u_2' \partial^{-1} u_4' & l_{23}'' & l_{24}'' & l_{25}'' \\ u_2'' & u_3 \partial^{-1} u_4'' + u_3' \partial^{-1} u_4 + \alpha u_3' \partial^{-1} u_4' - u_1'' & l_{33}'' & l_{34}'' & l_{35}'' \\ u_5'' & u_4 \partial^{-1} u_4'' + u_4' \partial^{-1} u_4 + \alpha u_4' \partial^{-1} u_4' & l_{43}'' & l_{44}'' & l_{45}'' \\ u_4'' & u_5 \partial^{-1} u_4'' + u_5' \partial^{-1} u_4 + \alpha u_5' \partial^{-1} u_4' & l_{53}'' & l_{54}'' & l_{55}'' \end{pmatrix}, \quad (58)$$

$$\begin{aligned} l_{13}'' &= \partial^{-1}(-u_5 u_1'' - u_5'' u_1 - u_2'' \partial - \alpha u_5' u_1'), \\ l_{14}'' &= \partial^{-1}(u_2 u_1'' + u_2'' u_1 - u_5'' \partial + \alpha u_2' u_1'), \\ l_{15}'' &= \partial^{-1}(u_3 u_1'' + u_3'' u_1 - u_4'' \partial + \alpha u_3' u_1'), \\ l_{23}'' &= u_2 \partial^{-1} u_5'' + u_2' \partial^{-1} u_5 + \alpha u_2' \partial^{-1} u_5' - u_1'', \\ l_{24}'' &= -u_2 \partial^{-1} u_2'' - u_2' \partial^{-1} u_2 - \alpha u_2' \partial^{-1} u_2', \\ l_{25}'' &= -u_2 \partial^{-1} u_3'' - u_2' \partial^{-1} u_3 - \alpha u_2' \partial^{-1} u_3', \\ l_{33}'' &= u_3 \partial^{-1} u_5'' + u_3' \partial^{-1} u_5 + \alpha u_3' \partial^{-1} u_5', \\ l_{34}'' &= -u_3 \partial^{-1} u_2'' - u_3' \partial^{-1} u_2 - \alpha u_3' \partial^{-1} u_2', \\ l_{35}'' &= -u_3 \partial^{-1} u_3'' - u_3' \partial^{-1} u_3 - \alpha u_3' \partial^{-1} u_3', \end{aligned}$$

$$\begin{aligned} l_{43}'' &= u_4 \partial^{-1} u_5'' + u_4' \partial^{-1} u_5 + \alpha u_4' \partial^{-1} u_5', \\ l_{44}'' &= -u_4 \partial^{-1} u_2'' - u_4' \partial^{-1} u_2 - \alpha u_4' \partial^{-1} u_2', \\ l_{45}'' &= -u_4 \partial^{-1} u_3'' - u_4' \partial^{-1} u_3 - \alpha u_4' \partial^{-1} u_3' - u_1'', \\ l_{53}'' &= u_5 \partial^{-1} u_5'' + u_5' \partial^{-1} u_5 + \alpha u_5' \partial^{-1} u_5', \\ l_{54}'' &= -u_5 \partial^{-1} u_2'' - u_5' \partial^{-1} u_2 - \alpha u_5' \partial^{-1} u_2' - u_1'', \\ l_{55}'' &= -u_5 \partial^{-1} u_3'' - u_5' \partial^{-1} u_3 - \alpha u_5' \partial^{-1} u_3'. \end{aligned} \quad (59)$$

4. Conclusion

In this paper, new Lax integrable hierarchy (14) is obtained by constructing an isospectral problem associated with $SO(4)$. Furthermore, we have obtained a new class of bi-integrable couplings (35) for soliton hierarchy (14) based on non-semisimple Lie algebra (25).

There are many soliton hierarchies obtained by using matrix spectral problems based on certain real Lie algebras such as $SO(3)$ and $SO(4)$. Recently seeking for integrable couplings of systems has become more and more meaningful in the area of mathematics and physics. Certainly, there are many interesting aspects of integrable couplings we have not solved, for example, the hierarchies associated with $SO(5)$ and $SO(6)$, the relations between the hierarchy of tri-integrable couplings associated with $SO(4)$, and the hierarchy of tri-integrable couplings associated with $SO(3)$. These problems are very interesting.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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