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Research Article

Jordan Isomorphisms on Nest Subalgebras

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This paper is devoted to the study of Jordan isomorphisms on nest subalgebras of factor von Neumann algebras. It is shown that every Jordan isomorphism ϕ between the two nest subalgebras $\mathrm{alg}_M\beta$ and $\mathrm{alg}_M\gamma$ is either an isomorphism or an anti-isomorphism.

1. Introduction

Let $\mathscr{A}$ and $\mathscr{B}$ be two associative algebras. A Jordan isomorphism ϕ from $\mathscr{A}$ onto $\mathscr{B}$ is a bijective linear map such that $\phi(T^2) = \phi(T)^2$ for every $T \in \mathscr{A}$. Obviously, isomorphisms and anti-isomorphisms are basic examples of Jordan isomorphisms. Jordan isomorphisms have been studied by many authors for various rings and algebras (see [1–5]). The standard problem is to determine whether a Jordan isomorphism is either an isomorphism or an anti-isomorphism. Using linear algebraic techniques, Molnár and Šemrl [6] proved that automorphisms and antiautomorphisms are the only linear Jordan automorphisms of $\mathscr{T}_n(F)$, $n \geqslant 2$, where F is a field with at least three elements. Later, Beidar et al. [7] generalized this result and proved that every linear Jordan isomorphisms of $\mathscr{T}_n(C)$, $n \geqslant 2$, onto an arbitrary algebra over C, is either an isomorphism or an anti-isomorphism, where C is a 2-torsionfree commutative ring which is connected, that is, a ring in which the only idempotents are 0 and 1. Recently, Zhang [8] proved that every Jordan isomorphism ϕ between two nest algebras $\mathrm{alg}\beta$ and $\mathrm{alg}\gamma$ is either an isomorphism or an anti-isomorphism. The same result was concluded in [9] for Jordan isomorphisms on nest algebras. The motivation for this paper is the work by Zhang. The aim of the present paper is to characterize Jordan isomorphisms of nest subalgebras.

In fact, the characterization of Jordan isomorphisms is closely related to isometric problems (see [10–12]). However, Jordan algebras or structures are related to vertex (operator) algebras or superalgebras and to representations of Kac-Moody and Virasoro algebras [13]. Note that the vertex operators of string theory also give reps of Lie algebras and Kac-Moody and Virasoro algebras (both infinite dimensional Lie algebras) as well as reps of the Fischer-Griess Monster group algebra. Jordan algebras or structures are also related to QFT, CFT, SCAs, and WZW models [14]. Another paper [15] about M-theory with J3(O) and F4 makes use of the algebraic Kostant-Dirac operator, and a variant of this operator also appears on page 12 of a paper [16] about a gerbe based approach to supersymmetric WZW models (gerbes are useful in string theory; e.g., gerbes help in illuminating the geometry of mirror symmetry of CY threefolds, help in giving a noncommutative description of D-branes in the presence of topologically nontrivial background fields, and provide a geometric way to unify properties of p-form fields with gauge symmetries, etc.). There is more about how Jordan structures are involved with physics [17]. This paper studies the structure of Jordan isomorphisms of nest subalgebras. Our main result is that every Jordan isomorphism of nest subalgebras is either an isomorphism or an anti-isomorphism.

Now, let us introduce some notions and concepts that will be used later. Let M be a von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$. A nest β in M is totally a family of (self-adjoint) projections in M which is closed in the strong operator topology and included 0 and I. The nest subalgebra of M associated with a nest β, denoted by $\mathrm{alg}_M\beta$, is the set $\mathrm{alg}_M\beta = \{T \in M : PTP = TP \text{ for all } P \in \beta\}$. The diagonal $\mathscr{D}_M(\beta)$ of a nest subalgebra $\mathrm{alg}_M\beta$ is the von Neumann subalgebra $(\mathrm{alg}_M\beta) \cap (\mathrm{alg}_M\beta)^*$. Let $\mathscr{R}_M(\beta)$ denote the norm closed algebra generated by $\{PT(I-P) : T \in M, P \in \beta\}$. It is clear that $\mathscr{R}_M(\beta)$ is a norm closed ideal of the nest

subalgebra $\mathrm{alg}_M\beta$. If M is a factor von Neumann algebra, it follows from [18] that $\mathscr{D}_M(\beta)+\mathscr{R}_M(\beta)$ is weakly dense in $\mathrm{alg}_M\beta$. When $M=\mathscr{B}(\mathscr{H})$, $\mathrm{alg}_M\beta$ is called a nest algebra and is denoted by $\mathrm{alg}\beta$. As a notational convenience, if P is an idempotent, we let $P^\perp$ denote $I-P$ throughout this paper.

We refer the readers to [19] for background information about von Neumann algebras and to [18] for the theory of nest algebras and nest subalgebras.

2. Main Result

The following theorem is our main result.

Theorem 1. *Let $alg_M\beta$, $alg_M\gamma$ be two nontrivial subalgebras in factor von Neumann algebra M, and let $\phi: alg_M\beta\to alg_M\gamma$ be a Jordan isomorphism. Then, ϕ is either an isomorphism or an anti-isomorphism.*

The proof is purely algebraic and will be organized in a series of lemmas. As a notational convenience, throughout this paper, we assume that β and γ are nontrivial nests in a factor von Neumann algebra M and that $\phi: alg_M\beta\to alg_M\gamma$ is a Jordan isomorphism.

Lemma 2. *Let β be a nest in factor von Neumann algebra M, and then, for $X,Y\in M$, $X(alg_M\beta)Y=0$ if and only if there exists a projection $P\in\beta$ such that $X=XP^\perp$, $Y=PY$.*

Proof. It is clear that if there exists a projection $P\in\beta$ such that $X=XP^\perp$, $Y=PY$, then $X(\mathrm{alg}_M\beta)Y=0$.

Clearly, let K be the closure of the space $(\mathrm{alg}_M\beta)YH$, and let P be the projection onto K. Then, $P\in M$ and $P^\perp TP=0$ for all $T\in\mathrm{alg}_M\beta$, and so $P\in\beta$. It is clear that $XP=0$, $P^\perp Y=0$. Thus, $X=XP^\perp$, $Y=PY$. The proof is complete. □

Lemma 3. *$\phi(I)=I$, and, for any $A,B,T\in alg_M\beta$, the following are equivalent:*

(a) $\phi(A^2)=\phi(A)^2$;

(b) $\phi(AB+BA)=\phi(A)\phi(B)+\phi(B)\phi(A)$;

(c) $\phi(ABA)=\phi(A)\phi(B)\phi(A)$;

(d) $\phi(ABT+TBA)=\phi(A)\phi(B)\phi(T)+\phi(T)\phi(B)\phi(A)$.

Proof. As (a) ⇒ (b), let $X=A+B$, and then, by (a), we have

$$\begin{aligned}\varphi\left(X^2\right)&=\phi\left(A^2\right)+\phi(AB+BA)+\phi\left(B^2\right)\\&=\left[\phi(A)+\phi(B)\right]^2.\end{aligned}\tag{1}$$

This shows that (b) is established.

As (b) ⇒ (c), by (b),

$$\begin{aligned}&2\phi(ABA)\\&\quad=\phi\left[(AB+BA)A+A(AB+BA)\right]-\phi\left(A^2B+BA^2\right)\\&\quad=\left[\phi(A)\phi(B)+\phi(B)\phi(A)\right]\phi(A)\\&\qquad+\phi(A)\left[\phi(A)\phi(B)+\phi(B)\phi(A)\right]\\&\qquad-\phi(A)^2\phi(B)-\phi(B)\phi(A)^2\\&\quad=2\phi(A)\phi(B)\phi(A).\end{aligned}\tag{2}$$

As (c) ⇒ (d),

$$\begin{aligned}&\phi(ABT+TBA)\\&\quad=\phi\left[(A+T)B(A+T)\right]-\phi(ABA)-\phi(TBT)\\&\quad=\left[\phi(A)+\phi(T)\right]\phi(B)\left[\phi(A)+\phi(T)\right]\\&\qquad-\phi(A)\phi(B)\phi(A)-\phi(T)\phi(B)\phi(T)\\&\quad=\phi(A)\phi(B)\phi(T)+\phi(T)\phi(B)\phi(A).\end{aligned}\tag{3}$$

Taking $B=I$ in (b), we get

$$2\phi(A)=\phi(A)\phi(I)+\phi(I)\phi(A).\tag{4}$$

Multiplying the above equation on both sides by $\phi(I)$ and noticing that $\phi(I)^2=\phi(I)$, we have

$$\begin{aligned}\phi(I)\phi(A)&=\phi(I)\phi(A)\phi(I),\\\phi(A)\phi(I)&=\phi(I)\phi(A)\phi(I).\end{aligned}\tag{5}$$

Thus, for any $A\in\mathrm{alg}_M\beta$, $\phi(A)\phi(I)=\phi(I)\phi(A)$. It follows from $M\cap(\mathrm{alg}_M\beta)'=\mathbb{C}I$ that $\phi(I)=I$. Hence, (a)–(d) are equivalent. □

Lemma 4. *If $P\in\beta/\{0,I\}$, then, for all $T,S\in alg_M\beta$, one has*

$$\begin{aligned}\phi(P)\phi(T)\phi\left(P^\perp\right)\phi(S)\phi(P)&=0;\\\phi\left(P^\perp\right)\phi(S)\phi(P)\phi(T)\phi\left(P^\perp\right)&=0;\\\phi(P)\phi(S)\phi\left(P^\perp\right)\phi(T)\phi(P)&=0;\\\phi\left(P^\perp\right)\phi(T)\phi(P)\phi(S)\phi\left(P^\perp\right)&=0.\end{aligned}\tag{6}$$

Proof. By Lemma 3(b), for all $T\in\mathrm{alg}_M\beta$,

$$\phi\left(PTP^\perp\right)=\phi(P)\phi(T)\phi\left(P^\perp\right)+\phi\left(P^\perp\right)\phi(T)\phi(P).\tag{7}$$

Let $S\in\mathrm{alg}_M\beta$, and then

$$\begin{aligned}&\phi(P)\phi(T)\phi\left(P^\perp\right)\phi(S)\phi(P)\\&\qquad+\phi\left(P^\perp\right)\phi(T)\phi(P)\phi(S)\phi\left(P^\perp\right)\\&\qquad+\phi(P)\phi(S)\phi\left(P^\perp\right)\phi(T)\phi(P)\\&\qquad+\phi\left(P^\perp\right)\phi(S)\phi(P)\phi(T)\phi\left(P^\perp\right)\end{aligned}$$

$$\begin{aligned}
&= [\phi(P)\phi(T)\phi(P^{\perp}) + \phi(P^{\perp})\phi(T)\phi(P)] \\
&\quad \cdot [\phi(P)\phi(S)\phi(P^{\perp}) + \phi(P^{\perp})\phi(S)\phi(P)] \\
&\quad + [\phi(P)\phi(S)\phi(P^{\perp}) + \phi(P^{\perp})\phi(S)\phi(P)] \\
&\quad \cdot [\phi(P)\phi(T)\phi(P^{\perp}) + \phi(P^{\perp})\phi(T)\phi(P)] \\
&= \phi(PTP^{\perp})\phi(PSP^{\perp}) + \phi(PSP^{\perp})\phi(PTP^{\perp}) = 0.
\end{aligned} \tag{8}$$

This implies that

$$\begin{aligned}
&\phi(P)\phi(T)\phi(P^{\perp})\phi(S)\phi(P) \\
&\qquad + \phi(P)\phi(S)\phi(P^{\perp})\phi(T)\phi(P) \\
&\qquad = 0, \\
&\phi(P^{\perp})\phi(T)\phi(P)\phi(S)\phi(P^{\perp}) \\
&\qquad + \phi(P^{\perp})\phi(S)\phi(P)\phi(T)\phi(P^{\perp}) \\
&\qquad = 0.
\end{aligned} \tag{9}$$

By (9),

$$\begin{aligned}
&\phi^{-1}[\phi(P)\phi(T)\phi(P^{\perp})\phi(S)\phi(P) \\
&\quad + \phi(P^{\perp})\phi(S)\phi(P)\phi(T)\phi(P^{\perp})] \\
&= \phi^{-1}(\phi(P)\phi(T)\phi(P^{\perp}))\phi^{-1}(\phi(P^{\perp})\phi(S)\phi(P)) \\
&\quad + \phi^{-1}(\phi(P^{\perp})\phi(S)\phi(P))\phi^{-1}(\phi(P)\phi(T)\phi(P^{\perp})) \\
&= [PTP^{\perp} - \phi^{-1}(\phi(P^{\perp})\phi(T)\phi(P))] \\
&\quad \cdot [PSP^{\perp} - \phi^{-1}(\phi(P)\phi(S)\phi(P^{\perp}))] \\
&\quad + [PSP^{\perp} - \phi^{-1}(\phi(P)\phi(S)\phi(P^{\perp}))] \\
&\quad \cdot [PTP^{\perp} - \phi^{-1}(\phi(P^{\perp})\phi(T)\phi(P))] \\
&= \phi^{-1}[\phi(P^{\perp})\phi(T)\phi(P)\phi(S)\phi(P^{\perp}) \\
&\qquad + \phi(P)\phi(S)\phi(P^{\perp})\phi(T)\phi(P)] + PAP^{\perp} \\
&= -\phi^{-1}[\phi(P)\phi(T)\phi(P^{\perp})\phi(S)\phi(P) \\
&\qquad + \phi(P^{\perp})\phi(S)\phi(P)\phi(T)\phi(P^{\perp})] + PAP^{\perp},
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
A = -[&PTP^{\perp}(\phi(P)\phi(S)\phi(P^{\perp})) \\
&+ \phi^{-1}(\phi(P^{\perp})\phi(T)\phi(P))PSP^{\perp} \\
&+ \phi^{-1}(\phi(P)\phi(S)\phi(P^{\perp}))PTP^{\perp} \\
&+ PSP^{\perp}\phi^{-1}(\phi(P^{\perp})\phi(T)\phi(P))].
\end{aligned} \tag{11}$$

Clearly, $A \in \mathrm{alg}_M\beta$. Thus, by the above equation, we have

$$\begin{aligned}
&\phi(P)\phi(T)\phi(P^{\perp})\phi(S)\phi(P) \\
&\qquad + \phi(P^{\perp})\phi(S)\phi(P)\phi(T)\phi(P^{\perp}) \\
&= \frac{1}{2}\phi(PAP^{\perp}) \\
&= \frac{1}{2}[\phi(P)\phi(A)\phi(P^{\perp}) + \phi(P^{\perp})\phi(A)\phi(P)].
\end{aligned} \tag{12}$$

This shows that, for any $T, S \in \mathrm{alg}_M\beta$,

$$\begin{aligned}
&\phi(P)\phi(T)\phi(P^{\perp})\phi(S)\phi(P) \\
&\quad = \phi(P^{\perp})\phi(S)\phi(P)\phi(T)\phi(P^{\perp}) = 0.
\end{aligned} \tag{13}$$

□

Lemma 5. *For any $T \in alg_M\beta$ and any projection $P \in \beta$, either $\phi(PTP^{\perp}) = \phi(P)\phi(T)\phi(P^{\perp})$ or $\phi(PTP^{\perp}) = \phi(P^{\perp})\phi(T)\phi(P)$.*

Proof. If $P = 0$ or $P = I$, the result is clear. Suppose that $P \in \beta/\{0, I\}$. Let

$$A = \phi(P)\phi(T)\phi(P^{\perp}), \qquad B = \phi(P^{\perp})\phi(T)\phi(P). \tag{14}$$

Then, by Lemma 4, for any $X \in \mathrm{alg}_M\gamma$,

$$\phi(P^{\perp})XA = \phi(P)XB = 0. \tag{15}$$

By Lemma 2 and (15), there exists a projection $P_1 \in \gamma$ such that

$$\phi(P^{\perp}) = \phi(P^{\perp})P_1^{\perp}; \tag{16}$$

$$A = P_1A. \tag{17}$$

There exists a projection $P_2 \in \gamma$ such that

$$\phi(P) = \phi(P)P_2^{\perp}; \tag{18}$$

$$B = P_2B. \tag{19}$$

By (16) and (18), we have

$$\phi(P^{\perp})P_1^{\perp} + \phi(P)P_2^{\perp} = \phi(I) = I. \tag{20}$$

Multiplying the above equation on both sides from the right side by P_1P_2, we get $P_1P_2 = 0$. If $A \neq 0$, then by (17) $P_1 \neq 0$. So, $P_2 = 0$. Hence, by (19), $B = 0$. Similarly, if $B \neq 0$, then $A = 0$. This implies that $A = 0$ or $B = 0$. From the fact that $\phi(PTP^{\perp}) = A + B$, for all $T \in \mathrm{alg}_M\beta$, one of the following is set up:

$$\begin{aligned}
\phi(PTP^{\perp}) &= \phi(P)\phi(T)\phi(P^{\perp}), \\
\phi(PTP^{\perp}) &= \phi(P^{\perp})\phi(T)\phi(P).
\end{aligned} \tag{21}$$

Since M is factor, then there exists a partial isometry operator $V \in M$ such that $V = PVP^{\perp}$, thus, $V \in \mathrm{alg}_M\beta$. Therefore,

either $\phi(V) = \phi(P)\phi(V)\phi(P^{\perp})$ or $\phi(V) = \phi(P^{\perp})\phi(V)\phi(P)$. Suppose $\phi(V) = \phi(P)\phi(V)\phi(P^{\perp})$, if there exists $S \in \mathrm{alg}_M\beta$ such that $\phi(PSP^{\perp}) \neq \phi(P)\phi(S)\phi(P^{\perp})$, then $\phi(PSP^{\perp}) = \phi(P^{\perp})\phi(S)\phi(P)$. On the other hand, one of the following is set up:

$$\begin{aligned} \phi\left(V + PSP^{\perp}\right) &= \phi(P)\,\phi(V+S)\,\phi\left(P^{\perp}\right), \\ \phi\left(V + PSP^{\perp}\right) &= \phi\left(P^{\perp}\right)\phi(V+S)\,\phi(P). \end{aligned} \tag{22}$$

And $\phi(V) = \phi(P)\phi(V)\phi(P^{\perp})$, $\phi(PSP^{\perp}) \neq \phi(P)\phi(S)\phi(P^{\perp})$; hence,

$$\phi\left(V + PSP^{\perp}\right) = \phi\left(P^{\perp}\right)\phi(V+S)\,\phi(P). \tag{23}$$

So, $\phi(V) = \phi(P^{\perp})\phi(V)\phi(P) = \phi(P)\phi(V)\phi(P^{\perp})$. This shows that $\phi(V) = 0$; thus, $V = 0$. A contradiction. In conclusion, for any $T \in \mathrm{alg}_M\beta$, we have $\phi(PTP^{\perp}) = \phi(P)\phi(T)\phi(P^{\perp})$. Similarly, suppose that $\phi(V) = \phi(P^{\perp})\phi(V)\phi(P)$, and then, for any $T \in \mathrm{alg}_M\beta$, we have $\phi(PTP^{\perp}) = \phi(P^{\perp})\phi(T)\phi(P)$. Consequently, for any $T \in \mathrm{alg}_M\beta$ and any projection $P \in \beta$, either $\phi(PTP^{\perp}) = \phi(P)\phi(T)\phi(P^{\perp})$ or $\phi(PTP^{\perp}) = \phi(P^{\perp})\phi(T)\phi(P)$. □

Proof of Theorem 1. By Lemma 5, if, for any $T \in \mathrm{alg}_M\beta$,

$$\phi\left(PTP^{\perp}\right) = \phi(P)\,\phi(T)\,\phi\left(P^{\perp}\right), \tag{24}$$

let $A \in \mathrm{alg}_M\beta$, and then

$$\begin{aligned} &\phi\left(PAPTP^{\perp}\right) \\ &\quad = \phi(P)\,\phi(PAP)\,\phi\left(PTP^{\perp}\right)\phi\left(P^{\perp}\right) \\ &\qquad + \phi(P)\,\phi\left(PTP^{\perp}\right)\phi(PAP)\,\phi\left(P^{\perp}\right). \end{aligned} \tag{25}$$

Thus, for all $T, A \in \mathrm{alg}_M\beta$, we have

$$\phi\left(PAPTP^{\perp}\right) = \phi(PAP)\,\phi\left(PTP^{\perp}\right). \tag{26}$$

Clearly,

$$\begin{aligned} \phi\left(PAP^{\perp}PTP^{\perp}\right) &= \phi\left(PAP^{\perp}\right)\phi\left(PTP^{\perp}\right) = 0, \\ \phi\left(P^{\perp}AP^{\perp}PTP^{\perp}\right) &= \phi\left(P^{\perp}AP^{\perp}\right)\phi\left(PTP^{\perp}\right) = 0. \end{aligned} \tag{27}$$

By (26)-(27), for all $A, T \in \mathrm{alg}_M\beta$, we have

$$\phi\left(APTP^{\perp}\right) = \phi(A)\,\phi\left(PTP^{\perp}\right). \tag{28}$$

Similar to the proof of (28), for any $A, T \in \mathrm{alg}_M\beta$,

$$\phi\left(PTP^{\perp}A\right) = \phi\left(PTP^{\perp}\right)\phi(A). \tag{29}$$

Let $A, B \in \mathrm{alg}_M\beta$, and by (28), for any $T \in \mathrm{alg}_M\beta$,

$$\begin{aligned} \phi\left(ABPTP^{\perp}\right) &= \phi(AB)\,\phi\left(PTP^{\perp}\right) \\ &= \phi(A)\,\phi\left(BPTP^{\perp}\right) \\ &= \phi(A)\,\phi(B)\,\phi\left(PTP^{\perp}\right). \end{aligned} \tag{30}$$

Thus, for all $A, B \in \mathrm{alg}_M\beta$, we have

$$\left[\phi(AB) - \phi(A)\,\phi(B)\right]\phi(P)\,\mathrm{alg}_M\gamma\phi\left(P^{\perp}\right) = 0. \tag{31}$$

Similarly, by (29),

$$\phi(P)\,\mathrm{alg}_M\gamma\phi\left(P^{\perp}\right)\left[\phi(AB) - \phi(A)\,\phi(B)\right] = 0. \tag{32}$$

By the above two equations and Lemma 2, there exists a projection $Q_1 \in \gamma$ such that

$$\begin{aligned} &\left[\phi(AB) - \phi(A)\,\phi(B)\right]\phi(P) \\ &\quad = \left[\phi(AB) - \phi(A)\,\phi(B)\right]\phi(P)\,Q_1^{\perp}; \end{aligned} \tag{33}$$

$$\phi\left(P^{\perp}\right) = Q_1\phi\left(P^{\perp}\right). \tag{34}$$

And there exists a projection $Q_2 \in \gamma$ such that

$$\phi(P) = \phi(P)\,Q_2^{\perp}; \tag{35}$$

$$\begin{aligned} &\phi\left(P^{\perp}\right)\left[\phi(AB) - \phi(A)\,\phi(B)\right] \\ &\quad = Q_2\phi\left(P^{\perp}\right)\left[\phi(AB) - \phi(A)\,\phi(B)\right]. \end{aligned} \tag{36}$$

Since ϕ is a Jordan isomorphism and $\phi(PTP^{\perp}) = \phi(P)\phi(T)\phi(P^{\perp})$, then, for any $T \in \mathrm{alg}_M\beta$, we have

$$\phi\left(P^{\perp}\right)\phi(T)\,\phi(P) = 0. \tag{37}$$

Especially, $\phi(P^{\perp})Q_1^{\perp}\phi(P) = 0$; that is, $\phi(P^{\perp})Q_1^{\perp}\phi(P^{\perp}) = \phi(P^{\perp})Q_1^{\perp}$. Thus, by formula (34), we have

$$\phi\left(P^{\perp}\right)Q_1^{\perp} = Q_1^{\perp}\phi\left(P^{\perp}\right) = 0. \tag{38}$$

This implies that $Q_1^{\perp}[\phi(P^{\perp})] = 0$, where $[\phi(P^{\perp})]$ is an orthogonal projection onto $\phi(P^{\perp})\mathscr{H}$. Since P is a nontrivial projection, then $[\phi(P^{\perp})] \neq 0$, thus, $Q_1^{\perp} = 0$. Therefore, by (33),

$$\left[\phi(AB) - \phi(A)\,\phi(B)\right]\phi(P) = 0. \tag{39}$$

By (35) and similar discussion, we get

$$\phi(P)\,Q_2 = Q_2\phi(P) = 0. \tag{40}$$

This shows that $[\phi(P)]Q_2 = 0$; thus, $Q_2 = 0$. So, by (36),

$$\phi\left(P^{\perp}\right)\left[\phi(AB) - \phi(A)\,\phi(B)\right] = 0. \tag{41}$$

In addition, by Lemma 3(c), we have

$$\begin{aligned} &\phi(P)\,\phi(AB)\,\phi\left(P^{\perp}\right) \\ &\quad = \phi\left(PAPBP^{\perp}\right) + \phi\left(PAP^{\perp}BP^{\perp}\right) \\ &\quad = \phi(PAP)\,\phi\left(PBP^{\perp}\right) + \phi\left(PAP^{\perp}\right)\phi\left(P^{\perp}BP^{\perp}\right) \\ &\quad = \phi(P)\,\phi(A)\,\phi(P)\,\phi(B)\,\phi\left(P^{\perp}\right) \\ &\qquad + \phi(P)\,\phi(A)\,\phi\left(P^{\perp}\right)\phi(B)\,\phi\left(P^{\perp}\right). \end{aligned} \tag{42}$$

Thus, for any $A, B \in \mathrm{alg}_M\beta$, we have

$$\phi(P)\left[\phi(AB)-\phi(A)\phi(B)\right]\phi\left(P^{\perp}\right)=0. \tag{43}$$

By (39) and (41), for all $A, B \in \mathrm{alg}_M\beta$, we have $\phi(AB) = \phi(A)\phi(B)$. Hence, ϕ is an isomorphism.

If for any $T \in \mathrm{alg}_M\beta$, we have $\phi(PTP^{\perp}) = \phi(P^{\perp})\phi(T)\phi(P)$, then, for any $X \in \mathrm{alg}_M\beta^{\perp}$, we define $\psi(X) = \phi(JX^*J)$, where J is a conjugate linear involution operator defined in Lemma 2.3 of [11]. It is not difficult to verify that $\psi : \mathrm{alg}_M\beta^{\perp} \to \mathrm{alg}_M\gamma$ is a Jordan isomorphism and, for any projection $Q \in \beta^{\perp}/\{0, I\}$, we have

$$\psi\left(QXQ^{\perp}\right)=\psi(Q)\psi(X)\psi\left(Q^{\perp}\right). \tag{44}$$

Thus, from the above discussion, ψ is an isomorphism. Consequently, ϕ is an anti-isomorphism.

By Lemma 5, for all $T \in \mathrm{alg}_M\beta$, one of the following holds:

$$\begin{aligned}&\phi\left(PTP^{\perp}\right)=\phi(P)\phi(T)\phi\left(P^{\perp}\right) \quad \text{or}\\ &\phi\left(PTP^{\perp}\right)=\phi\left(P^{\perp}\right)\phi(T)\phi(P).\end{aligned} \tag{45}$$

If $\phi(PTP^{\perp}) = \phi(P)\phi(T)\phi(P^{\perp})$, then Lemma 3(c) and the fact that $BP = PBP$ imply that

$$\begin{aligned}\phi(AB)\phi\left(PTP^{\perp}\right)&=\phi\left(ABPTP^{\perp}\right)\\ &=\phi(A)\phi\left(BPTP^{\perp}\right)\\ &=\phi(A)\phi(B)\phi\left(PTP^{\perp}\right)\end{aligned} \tag{46}$$

for all $A, B \in \mathrm{alg}_M\beta$. Hence, for all $A, B \in \mathrm{alg}_M\beta$, we have

$$\left[\phi(AB)-\phi(A)\phi(B)\right]\phi(P)\,\mathrm{alg}_M\gamma\phi\left(P^{\perp}\right)=0. \tag{47}$$

Similarly, it follows from Lemma 3(c) and $P^{\perp}A = P^{\perp}AP^{\perp}$ that, for all $A, B \in \mathrm{alg}_M\beta$, we have

$$\phi(P)\,\mathrm{alg}_M\gamma\phi\left(P^{\perp}\right)\left[\phi(AB)-\phi(A)\phi(B)\right]=0. \tag{48}$$

By (47), (48), and Lemma 2, there exists a projection $Q_1, Q_2 \in \gamma$ such that

$$\begin{aligned}&\left[\phi(AB)-\phi(A)\phi(B)\right]\phi(P)\\ &\quad=\left[\phi(AB)-\phi(A)\phi(B)\right]\phi(P)Q_1^{\perp};\end{aligned} \tag{49}$$

$$\phi\left(P^{\perp}\right)=Q_1\phi\left(P^{\perp}\right); \tag{50}$$

$$\phi(P)=\phi(P)Q_2^{\perp}; \tag{51}$$

$$\begin{aligned}&\phi\left(P^{\perp}\right)\left[\phi(AB)-\phi(A)\phi(B)\right]\\ &\quad=Q_2\phi\left(P^{\perp}\right)\left[\phi(AB)-\phi(A)\phi(B)\right].\end{aligned} \tag{52}$$

Since $\phi(PTP^{\perp}) = \phi(P)\phi(T)\phi(P^{\perp})$, we have $\phi(P^{\perp})\phi(T)\phi(P) = 0$ for all $T \in \mathrm{alg}_M\beta$. Let $[\phi(P)] \in M$, and $[\phi(P)]$ denote the orthogonal projection from $\mathscr{H}$ to $\phi(P)\mathscr{H}$. In particular, $\phi(P^{\perp})Q_1^{\perp}\phi(P) = 0$; that is, $\phi(P^{\perp})Q_1^{\perp}\phi(P^{\perp}) = \phi(P^{\perp})Q_1^{\perp}$. Thus, by formula (50),

$$\phi\left(P^{\perp}\right)Q_1^{\perp}=Q_1^{\perp}\phi\left(P^{\perp}\right)=0. \tag{53}$$

This shows that $Q_1^{\perp}[\phi(P^{\perp})] = 0$. Since P is a nontrivial projection, then $[\phi(P^{\perp})] \neq 0$, so $Q_1^{\perp} = 0$. By (49),

$$\left[\phi(AB)-\phi(A)\phi(B)\right]\phi(P)=0. \tag{54}$$

In similar discussion, we have $\phi(P)Q_2 = Q_2\phi(P) = 0$. This shows that $[\phi(P)]Q_2 = 0$, so $Q_2 = 0$. Thus, by (52),

$$\phi\left(P^{\perp}\right)\left[\phi(AB)-\phi(A)\phi(B)\right]=0. \tag{55}$$

Because $P^{\perp}BP^{\perp}PAP^{\perp} = 0$, by Lemma 3, we have

$$\begin{aligned}\phi\left(PAP^{\perp}BP^{\perp}\right)&=\phi\left(P^{\perp}BP^{\perp}\circ PAP^{\perp}\right)\\ &=\phi\left(P^{\perp}BP^{\perp}\right)\circ\phi\left(PAP^{\perp}\right)\\ &=\phi\left(P^{\perp}BP^{\perp}\right)\phi\left(PAP^{\perp}\right)\end{aligned} \tag{56}$$

for all $A, B \in \mathrm{alg}_M\beta$. Since $PBP^{\perp}PAP = 0$, similarly,

$$\phi\left(PAPBP^{\perp}\right)=\phi(PAP)\phi\left(PBP^{\perp}\right). \tag{57}$$

By (56), (57), and Lemma 4,

$$\begin{aligned}&\phi(P)\phi(AB)\phi\left(P^{\perp}\right)\\ &\quad=\phi\left(PAPBP^{\perp}\right)+\phi\left(PAP^{\perp}BP^{\perp}\right)\\ &\quad=\phi(PAP)\phi\left(PBP^{\perp}\right)+\phi\left(PAP^{\perp}\right)\phi\left(P^{\perp}BP^{\perp}\right)\\ &\quad=\phi(P)\phi(A)\phi(P)\phi(B)\phi\left(P^{\perp}\right)\\ &\qquad+\phi(P)\phi(A)\phi\left(P^{\perp}\right)\phi(B)\phi\left(P^{\perp}\right)\\ &\quad=\phi(P)\phi(A)\phi(B)\phi\left(P^{\perp}\right).\end{aligned} \tag{58}$$

Thus, for all $A, B \in \mathrm{alg}_M\beta$, we have

$$\phi(P)\left[\phi(AB)-\phi(A)\phi(B)\right]\phi\left(P^{\perp}\right)=0. \tag{59}$$

By (54), (55), and (59), we have $\phi(AB) = \phi(A)\phi(B)$ for any $A, B \in \mathrm{alg}_M\beta$. Similarly, $\phi(AB) = \phi(B)\phi(A)$.

Consequently, ϕ is either an isomorphism or anti-isomorphism. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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