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Research Article

Bound-State Solution of s-Wave Klein-Gordon Equation for Woods-Saxon Potential

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The bound-state solution of s-wave Klein-Gordon equation is calculated for Woods-Saxon potential by using the asymptotic iteration method (AIM). The energy eigenvalues and eigenfunctions are obtained for the required condition of bound-state solutions.

1. Introduction

In relativistic quantum mechanics, the Klein-Gordon (KG) equation has a wide range of applications. Recently, there have been many studies on KG equation with various types of potentials by using different methods to describe the corresponding relativistic physical systems, namely, asymptotic iteration method [1], formal variable separation method [2], supersymmetric quantum mechanics [3–7], algebraic method [8–13], and Nikiforov-Uvarov method [14]. Most of these studies have been considered for equal or pure scalar $S(x)$ and vector potentials $V(x)$ cases. In this study, to have a bound-state solution, we use the proposed transformation in [15] with parameters β which adjust the relation between scalar and vector Woods-Saxon potentials. Woods-Saxon potential [16] is a short range nuclear potential and it is important in all branches of physics. The bound-state solution of KG equation for Woods-Saxon potential has been of a great interest in the last decades with different methods [17–21]. It has been solved with Nikiforov-Uvarov method by using the Pekeris approximation to the centrifugal potential for any l-states for constant mass [22, 23] and for effective mass [24, 25].

In this paper, we use an alternative method which is called the asymptotic iteration method (AIM) [26] for obtaining the whole spectra of Woods-Saxon potentials. AIM has been used in many physical systems to obtain the corresponding eigenvalues and eigenfunctions [26–31]. Still, there are a few applications on KG equation with AIM [32–34] with a general transformation between vector and scalar potentials. To obtain the corresponding potential the vector potential is chosen in exponential potential form. Therefore, the KG equation reduces to Schrödinger-like equation with Woods-Saxon potential.

The organization of this paper is as follows: Section 2 deals with the formalism of the one-dimensional KG equation with scalar potential greater than vector potential. A general description of AIM is outlined in Section 3. The solution of Woods-Saxon potential in KG equation is treated in the subsequent section by means of AIM. Finally, Section 5 is devoted to conclusions.

2. Formalism of the Klein-Gordon Equation

Generally, the s-wave Klein-Gordon equation with scalar potential $S(x)$ and vector potential $V(x)$ can be written as [9]

$$\left\{\frac{d^2}{dx^2}+\frac{1}{\hbar^2c^2}\left[(E-V(x))^2-\left(mc^2+S(x)\right)^2\right]\right\}f(x) = 0, \tag{1}$$

where E represents the energy and m represents the mass of the particle. In this equation, the radial wave function is expressed as $R(x) = f(x)/x$. We know that when the relation between vector and scalar potentials $S(x) \geq V(x)$,

it yields real bound-state solutions. In this work, we consider the relation between potentials in the condition for bound-state solution which was introduced in [15] as

$$S(x) = V(x)(\beta - 1), \quad \beta \geq 0, \tag{2}$$

where β parameter is arbitrary constant. When $\beta = 0, 1, 2$, the scalar potential leads to case of $S(x) = -V(x)$, $S(x) = 0$ (purely vector potential), and $S(x) = V(x)$ (equal potentials), respectively. When $\beta > 2$, we get the required condition for the case of $S(x) > V(x)$. Substituting (2) into (1) yields

$$\left\{\frac{d^2}{dx^2} - \frac{1}{\hbar^2 c^2}\left[V_{\text{eff}}(x) - \left(E^2 - m^2\right)\right]\right\} f(x) = 0, \tag{3}$$

where the effective potential is

$$V_{\text{eff}} = V(x)\left[2E + 2c^2 m(\beta - 1) + V(x)(\beta - 2)\beta\right]. \tag{4}$$

Now, the final equation is the transformed KG equation in the condition of unequal vector and scalar potentials. The following section deals with the corresponding method to solve the KG equation.

3. Asymptotic Iteration Method

The AIM is proposed to solve the second-order differential equations and the details can be found in [26]. First, we consider the following second-order homogeneous differential equation of the form

$$y''(x) = \lambda_0 y'(x) + s_0 y(x), \tag{5}$$

where λ_0 and s_0 are functions and $y'(x)$ and $y''(x)$ denote derivatives of y with respect to x. It is easy to show that $(n+2)$th derivative of the function $y(x)$ can be written as

$$y^{(n+2)}(x) = \lambda_n y'(x) + s_n y(x). \tag{6}$$

The functions λ_n and s_n are given by the recurrence relations

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_{n-1}\lambda_0, \\ s_n &= s'_{n-1} + \lambda_{n-1} s_0. \end{aligned} \tag{7}$$

For sufficiently large n, the ratio of the functions satisfies the following equations:

$$\frac{\lambda_n}{s_n} = \frac{\lambda_{n-1}}{s_{n-1}} = \alpha(x); \tag{8}$$

then the solution of (5) can be written as [26]

$$\begin{aligned} y(x) = {} & \exp\left(-\int^x \alpha\, dt\right) \\ & \cdot \left[C_1 + C_2 \int^x \exp\left(\int^s (\lambda_0 + 2\alpha)\, dt\right) ds\right]. \end{aligned} \tag{9}$$

In calculating the parameters in (7), for $n = 0$, we take the initial conditions as $\lambda_{-1} = 1$ and $s_{-1} = 0$ [29] and $\Delta_n(x) = 0$ for

$$\Delta_n(x) = \lambda_n(x) s_{n-1}(x) - \lambda_{n-1}(x) s_n(x), \tag{10}$$

where $\Delta_n(x)$ is the termination condition in (8).

If the second-order homogeneous linear differential equation can be written in the form of [27]

$$\begin{aligned} y''(x) = {} & 2\left(\frac{ax^{N+1}}{1 - bx^{N+2}} - \frac{(m+1)}{x}\right) y'(x) \\ & - \frac{wx^N}{1 - bx^{N+2}}, \end{aligned} \tag{11}$$

using the function generator of AIM, it has an exact solution for $y_n(x)$ [27] as

$$\begin{aligned} y_n(x) = {} & (-1)^n C_1 (\sigma)_n (N+2)^n (\sigma)_n \\ & \cdot {}_2F_1\left(-n, \rho + n; \sigma; bx^{N+2}\right) \end{aligned} \tag{12}$$

with the defined parameters $(\sigma)_n = \Gamma(\sigma + n)/\sigma$, $\sigma = (2m + N + 3)/(N + 2)$, and $\rho = ((2m+1)b + 2a)/(N+2)b$.

4. Solution for Woods-Saxon Potential

Let us consider the Woods-Saxon vector potential

$$V(x) = \frac{-V_0}{1 + e^{(x-R)/\alpha}} \quad \text{and then } S(x) = \frac{V_0(1-\beta)}{1 + e^{(x-R)/\alpha}}. \tag{13}$$

After substituting these potentials into (3) and changing the variables $y = 1/(1 + e^{(x-R)/\alpha})$, we obtain the second-order differential equation

$$\begin{aligned} & \left\{\frac{d^2}{dy^2} + \frac{(1-2y)}{y(1-y)}\frac{d}{dy} - \frac{1}{y^2(1-y)^2}\left[\xi + A + B\right]\right\} \\ & \cdot f(y) = 0 \end{aligned} \tag{14}$$

with the corresponding parameters

$$\begin{aligned} \xi^2 &= -\left(\frac{E^2 - m^2c^4}{\hbar^2 c^2}\right), \\ A &= \left(\frac{2EV_0\alpha^2 + 2(\beta - 1)c^2 m V_0 \alpha^2}{\hbar^2 c^2}\right), \\ B &= -\left(\frac{(\beta - 2)\beta V_0^2 \alpha^2}{\hbar^2 c^2}\right). \end{aligned} \tag{15}$$

As we expected, (13) is transformed to the suitable form to apply the AIM. Therefore, (14) should have a solution in the form of "normalized" wave functions

$$f(y) = y^{\xi}(1-y)^{\nu}\chi(y) \tag{16}$$

with $\nu = \sqrt{\xi + A + B}$. Substituting (16) into (14), one obtains

$$\begin{aligned} \chi''(y) = {} & \left[\frac{2\xi(y-1) + 2(1+\nu)y - 1}{y(1-y)}\right]\chi'(y) \\ & + \left[\frac{(\xi + \nu)(1 + 2\xi) + A}{y(1-y)}\right]\chi(y). \end{aligned} \tag{17}$$

Within the framework of procedures of AIM, the functions $\lambda_0(y)$ and $s_0(y)$ can be written by comparing (17) with (5). It can be seen that $\lambda_0(y)$ and $s_0(y)$ functions are obtained by using the termination relation in (10) as

$$\lambda_0 = \frac{1 - 2\xi(y-1) - 2(v+1)y}{y(y-1)},$$
$$s_0 = \frac{2 + (A - 3v - 6)y + \left(4v^2 + 9v - A + 6\right)y^2 + 3\xi(y-1)\left(-2 + (3+2v)y\right) + 2\xi^2\left(y^2 - 3y + 2\right)}{y^2(y-1)^2}. \tag{18}$$

By substituting these functions into the quantization condition, we get

$$\begin{aligned}
\frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} &\Longrightarrow \xi_0 + \sqrt{\xi + A + B} = -\frac{1}{2} - \frac{1}{2}\sqrt{1+4A}, \quad n = 0,\\
\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} &\Longrightarrow \xi_1 + \sqrt{\xi + A + B} = -\frac{3}{2} - \frac{1}{2}\sqrt{1+4A}, \quad n = 1,\\
\frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} &\Longrightarrow \xi_2 + \sqrt{\xi + A + B} = -\frac{5}{2} - \frac{1}{2}\sqrt{1+4A}, \quad n = 2,\\
&\vdots
\end{aligned} \tag{19}$$

Generalizing the above expressions, it is possible to write the general formula of ξ as

$$\xi_n + \sqrt{\xi + A + B} = -\frac{(2n+1)}{2} - \frac{1}{2}\sqrt{1+4A}. \tag{20}$$

Using the definitions of parameters ξ and ε, it can be shown that

$$E_n^2 - m^2c^4 = -\hbar^2c^2\left(-\frac{(2n+1)}{2} - \frac{1}{2}\sqrt{1 + 4\frac{2EV_0\alpha^2 + 2(\beta-1)c^2mV_0\alpha^2}{\hbar^2c^2}} - v\right). \tag{21}$$

Rearranging this expression, we get a more explicit expression for eigenvalues

$$E_n = \pm\left[m^2c^4 - \hbar^2c^2\left(-\frac{(2n+1)}{2} - \frac{1}{2}\sqrt{1 + 4\frac{2EV_0\alpha^2 + 2(\beta-1)c^2mV_0\alpha^2}{\hbar^2c^2}} - v\right)\right]^{1/2} \tag{22}$$

which is exactly the same as with the eigenvalue equation obtained in [21] through a proper choice of parameters.

4.1. Calculation of Eigenfunction. To get the normalized wave function, we have to compare (17) with the differential equation defined in (11). After some algebraic calculations, we get the required parameters as

$$\begin{aligned}
a &= \frac{2v+1}{2},\\
b &= 1,\\
N &= -1,\\
m &= \frac{2E-1}{2},\\
\sigma &= 2\xi + 1,\\
\rho &= 2\xi - 2v + 1.
\end{aligned} \tag{23}$$

Thus, these parameters yield a solution in the form of

$$\chi_n(y) = (-1)^n C_2 \cdot \frac{\Gamma(2\xi_n + n + 1)\ {}_2F_1\left(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; y\right)}{\Gamma(2\xi_n + 1)} \tag{24}$$

with the Gamma function Γ and the Gauss hypergeometric function ${}_2F_1$. With the aid of (24), the corresponding radial function becomes

$$R_n(x) = (-1)^n N_n x\left(y^{\xi}(1-y)^{v}\right) \cdot {}_2F_1\left(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; y\right), \tag{25}$$

where the normalization constant is

$$N_n = C_2\frac{\Gamma(2\xi_n + n + 1)}{\Gamma(2\xi_n + 1)}. \tag{26}$$

Substituting the values of y in (25), the total radial differential equation transforms to

$$R_n(y) = (-1)^n N_n x\left(\frac{\left(e^{(x-R)/\alpha}\right)^{v}}{\left(1 + e^{(x-R)/\alpha}\right)^{v+\xi}}\right) \cdot {}_2F_1\left(-n, 2(\xi_n - v) + 1 + n; 2\xi_n + 1; \frac{1}{1 + e^{(x-R)/\alpha}}\right). \tag{27}$$

After rearranging the parameters, this result is exactly the same as with those of calculated for Woods-Saxon potentials in literature [17–19, 21].

5. Conclusion

In this study, we have obtained the whole spectrum of Woods-Saxon potential for s-wave KG equation by using AIM method. During the calculation, we have used the adjusting parameter β between scalar and vector potentials to obey the bound-state solution condition. The eigenvalues and eigenfunction are calculated directly and exactly by using the procedure of AIM in simple way.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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