Research Article

Interactions of Delta Shock Waves for Zero-Pressure Gas Dynamics with Energy Conservation Law

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We study the interactions of delta shock waves and vacuum states for the system of conservation laws of mass, momentum, and energy in zero-pressure gas dynamics. The Riemann problems with initial data of three piecewise constant states are solved case by case, and four different configurations of Riemann solutions are constructed. Furthermore, the numerical simulations completely coinciding with theoretical analysis are shown.

1. Introduction

As is well known, the system of zero-pressure gas dynamics consisting of conservation laws of mass and momentum, which is also called the transport equations, or Euler equations for pressureless fluids, has been extensively investigated since the 90s of 20th century. It is derived from Boltzmann equations [1] and the flux-splitting scheme of the full compressible Euler equations [2, 3] and can be used to describe the motion process of free particles sticking together under collision [4] and the formation of large-scale structures in the universe [5, 6].

However, we have to mention that, as having no pressure, the energy transport must be taken into account for the considered media. Therefore, it is very necessary to consider the conservation law of energy in zero-pressure gas dynamics. To this end, we study the one-dimensional zero-pressure gas dynamics governed by the conservation laws of mass, momentum, and energy:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x &= 0, \\ \left(\frac{\rho u^2}{2} + H \right)_t + \left(\left(\frac{\rho u^2}{2} + H \right) u \right)_x &= 0, \end{aligned} \quad (1)$$

where ρ and u represent the density and velocity, respectively, $H = \rho \tau$ is the internal energy and assumed to be nonnegative, and τ is the internal energy per unit mass. The regions in the physical space where $\rho = 0$ and $H = 0$ are identified with the vacuum regions of the flow. Here, H is considered as an independent variable just for convenience.

System (1) was early studied by Kraiko [7]. In contrast to the traditional zero-pressure gas dynamics system that contains only the conservation laws of mass and momentum, in order to construct the solution of (1) for arbitrary initial data, a new type of discontinuities which are different from the classical ones and carry mass, impulse, and energy are needed. In [8, 9], system (1) was further discussed. Some special integral identities were introduced to define the delta-shock solutions and construct the Rankine-Hugoniot relation for delta shock waves. Moreover, using these integral identities, the balance laws describing mass, momentum, and energy transport from the area outside the delta shock wave front onto its front were derived. What is more, the delta shock wave type solutions for multidimensional zero-pressure gas dynamics with the energy conservation law were defined in [10].

A delta shock wave is a generalization of an ordinary shock wave. Roughly speaking, it is a kind of discontinuity, on which at least one of the state variables may develop

an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. It is more compressive than an ordinary shock wave and more characteristics enter the discontinuity line. Physically, the delta shock waves describe the process of formation of the galaxies in the universe and the process of concentration of particles. As for delta shock waves, there are numerous excellent papers, see [11–21] and so forth. Nevertheless, compared to these results, a distinctive feature for (1) is that the Dirac delta functions develop in both state variables ρ and H simultaneously, which is quite different from those aforementioned, in which only one state variable contains the Dirac delta function. In fact, the theory of delta shock waves with Dirac delta functions developing in both state variables has been established by Yang and Zhang [22, 23] for a class of 2×2 nonstrictly hyperbolic systems of conservation laws.

In the past over two decades, the investigation of interactions of delta shock waves has been increasingly active. This is important not only because of their significance in practical applications but also because of their basic role as building blocks for the general mathematical theory of quasi-linear hyperbolic equations. And the results on interactions are also touchstones for the numerical schemes. Specifically, Sheng and Zhang [18] discussed the overtaking of delta shock waves and vacuum states in one-dimensional zero-pressure gas dynamics. By solving the two-dimensional Riemann problems for zero-pressure gas dynamics with three constant states, Cheng et al. [24] studied the interactions among delta-shock waves, vacuums, and contact discontinuities. In addition, with the help of a generalized plane wave solution, Yang [25] studied a type of generalized plane delta-shock wave for the n -dimensional zero-pressure gas dynamics and investigated the overtaking of two plane delta shocks. For more works on the interactions of delta shock waves, we refer to [26–29] and so forth.

Motivated by the discussions above, in the present paper, we are concerned with the interactions among delta shock waves, vacuum states, and contact discontinuities in solutions. Therefore, we study the Riemann problem of (1) with initial data of three piecewise constant states as follows:

$$(\rho, u, H)(0, x) = \begin{cases} (\rho_-, u_-, H_-), & x < x_{01}, \\ (\rho_m, u_m, H_m), & x_{01} < x < x_{02}, \\ (\rho_+, u_+, H_+), & x > x_{02}, \end{cases} \quad (2)$$

where ρ_i, u_i, H_i ($i = \pm, m$) are arbitrary constants and x_{01}, x_{02} are any two fixed points on x -axis.

We will deal with the Riemann problem (1), (2) case by case along with constructing the solutions. For this purpose, it is necessary to consider whether two adjacent waves intersect and interact with each other when constructing the global solution. However, it is often not so easy to see whether two delta shock waves meet and how they interact with each other. Therefore, some technical treatments are needed.

This paper is arranged as follows. In Section 2, the delta shock solution of (1) is reviewed and a general case when the delta shock wave is emitted at the beginning with a nonzero initial data is considered. Section 3 discusses the

interactions of the delta shock waves and vacuum states. The Riemann solutions of (1), (2) are constructed globally. Finally, four kinds of numerical simulations coinciding with the theoretical analysis are presented in Section 4.

2. Delta-Shock Solution

This section briefly reviews the delta shock solution of (1) with the initial data

$$(\rho, u, H)(0, x) = (\rho_{\pm}, u_{\pm}, H_{\pm}), \quad \pm x > 0, \quad (3)$$

and the detailed study can be found in [30].

System (1) has a triple eigenvalue $\lambda = u$ and two right eigenvectors $\vec{r}_1 = (1, 0, 0)^T$ and $\vec{r}_2 = (0, 0, 1)^T$. Since $\nabla \lambda \cdot \vec{r}_i = 0$ for $i = 1, 2$, λ is linearly degenerate, which means that the elementary waves involve only contact discontinuities. The self-similar solution $(\rho, v, H)(\xi)$ ($\xi = x/t$) is constructed by two cases.

For the case $u_- \leq u_+$, the solution containing two contact discontinuities and a vacuum state besides two constants is expressed as

$$(\rho, u, H)(\xi) = \begin{cases} (\rho_-, u_-, H_-), & -\infty < \xi < u_-, \\ (0, u(\xi), 0), & u_- \leq \xi \leq u_+, \\ (\rho_+, u_+, H_+), & u_+ < \xi < +\infty, \end{cases} \quad (4)$$

where $u(\xi)$ is a smooth function satisfying $u(u_-) = u_-$ and $u(u_+) = u_+$.

For the case $u_- > u_+$, the singularity of solutions must develop because of the overlap of characteristic lines. Therefore, the solution involving a delta shock wave is introduced.

Let $(\rho, u, H; w, u_\delta, h)$ be the delta shock solution of the form

$$\begin{aligned} & (\rho, u, H)(t, x) \\ &= \begin{cases} (\rho_-, u_-, H_-)(t, x), & x < x(t), \\ (w(t) \delta(x - x(t)), u_\delta(t), h(t) \delta(x - x(t))), & x = x(t), \\ (\rho_+, u_+, H_+)(t, x), & x > x(t), \end{cases} \end{aligned} \quad (5)$$

and then the following generalized Rankine-Hugoniot relation holds

$$\begin{aligned} \frac{dx}{dt} &= u_\delta, \\ \frac{dw}{dt} &= -[\rho] u_\delta + [\rho u], \\ \frac{d(wu_\delta)}{dt} &= -[\rho u] u_\delta + [\rho u^2], \end{aligned}$$

$$\frac{d(wu_\delta^2/2 + h)}{dt} = - \left[\frac{\rho u^2}{2} + H \right] u_\delta + \left[\left(\frac{\rho u^2}{2} + H \right) u \right], \quad (6)$$

where $[\rho] = \rho_- - \rho_+$. In order to ensure the uniqueness, the delta shock wave should satisfy the entropy condition

$$u_+ < u_\delta < u_-, \quad (7)$$

which means that the characteristics on both sides of the discontinuity are in-coming.

Under the entropy condition (7), by solving the ordinary differential equations (6) with the initial data $t = 0: x(0) = 0, w(0) = 0, h(0) = 0, u_\delta(0) = 0$, one has

$$\begin{aligned} x(t) &= \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}} t, \\ w(t) &= \sqrt{\rho_- \rho_+} (u_- - u_+) t, \\ u_\delta(t) &= \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ h(t) &= \frac{\rho_- \rho_+ (u_- - u_+)^2 + 2(\sqrt{\rho_-} + \sqrt{\rho_+})(H_- \sqrt{\rho_+} + H_+ \sqrt{\rho_-})}{2(\sqrt{\rho_-} + \sqrt{\rho_+})^2} (u_- - u_+) t. \end{aligned} \quad (8)$$

For convenience, we now consider a special case when a delta shock wave is emitted at the beginning with the initial data

$$\begin{aligned} t &= 0: \\ x(0) &= 0, \\ w(0) &= m_0, \\ h(0) &= h_0, \\ u_\delta(0) &= u_0 \end{aligned} \quad (9)$$

satisfying $u_- > u_0 > u_+$. It yields from (6) and (9) that

$$x(t) = \begin{cases} \frac{m_0 u_0 t + [\rho u^2] t^2 / 2}{m_0 + [\rho u] t}, & \rho_- = \rho_+, \\ \frac{[\rho u] t + m_0 - w(t)}{[\rho]}, & \rho_- \neq \rho_+, \end{cases}$$

FIGURE 1: $u_- > u_m > u_+$.

$$\begin{aligned} w(t) &= (m_0^2 + 2m_0([\rho u] - [\rho]u_0)t \\ &\quad + \rho_- \rho_+ (u_- - u_+)^2 t^2)^{1/2}, \\ u_\delta(t) &= \frac{m_0 u_0 + [\rho u^2] t - [\rho u] x}{w(t)}, \\ h(t) &= - \left[\frac{\rho u^2}{2} + H \right] x + \left[\left(\frac{\rho u^2}{2} + H \right) u \right] t + \frac{m_0 u_0^2}{2} \\ &\quad + h_0 - \frac{w(t) u_\delta^2}{2}. \end{aligned} \quad (10)$$

One can check that the delta shock solution (10) satisfies the following:

- (1) $x'(t)$ is a monotone function of t .
- (2) If $\rho_- + \rho_+ > 0$, then

$$\lim_{t \rightarrow +\infty} u_\delta(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}. \quad (11)$$

While if $\rho_- = \rho_+ = 0$, then $u_\delta = u_0$.

- (3) $u_+ < u_\delta < u_-$.

3. Interactions of Delta Shock Waves

In this section, we analyze the interactions of delta shock waves. To ensure that all the cases are covered completely, according to the relation among u_- , u_m , u_+ , our discussion is divided into four cases:

- (1) $u_- > u_m > u_+$;
- (2) $u_- > u_+ > u_m$;
- (3) $u_+ > u_- > u_m$;
- (4) $u_+ > u_m > u_-$.

Case 1 ($u_- > u_m > u_+$). In this case, two delta shock waves δ_1 and δ_2 will be emitted from $(x_{01}, 0)$ and $(x_{02}, 0)$, respectively, as shown in Figure 1.

According to what has been discussed in Section 2, these two delta shock waves are uniquely determined by

$$\delta_1: \begin{cases} x_1(t) = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_m}u_m}{\sqrt{\rho_-} + \sqrt{\rho_m}}t + x_{01}, \\ u_{\delta_1} = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_m}u_m}{\sqrt{\rho_-} + \sqrt{\rho_m}}, \\ w_{\delta_1}(t) = \sqrt{\rho_- \rho_m}(u_- - u_m)t, \\ h_{\delta_1}(t) = \frac{\rho_- \rho_m (u_- - u_m)^2 + 2(\sqrt{\rho_-} + \sqrt{\rho_m})(H_- \sqrt{\rho_m} + H_m \sqrt{\rho_-})}{2(\sqrt{\rho_-} + \sqrt{\rho_m})^2} (u_- - u_m)t, \end{cases} \quad (12)$$

$$\delta_2: \begin{cases} x_2(t) = \frac{\sqrt{\rho_m}u_m + \sqrt{\rho_+}u_+}{\sqrt{\rho_+} + \sqrt{\rho_m}}t + x_{02}, \\ u_{\delta_2} = \frac{\sqrt{\rho_m}u_m + \sqrt{\rho_+}u_+}{\sqrt{\rho_m} + \sqrt{\rho_+}}, \\ w_{\delta_2}(t) = \sqrt{\rho_m \rho_+}(u_m - u_+)t, \\ h_{\delta_2}(t) = \frac{\rho_m \rho_+ (u_m - u_+)^2 + 2(\sqrt{\rho_m} + \sqrt{\rho_+})(H_m \sqrt{\rho_+} + H_+ \sqrt{\rho_m})}{2(\sqrt{\rho_m} + \sqrt{\rho_+})^2} (u_m - u_+)t. \end{cases} \quad (13)$$

We have $u_+ < u_{\delta_2} < u_m < u_{\delta_1} < u_-$ by entropy condition (6), which means that δ_1 will overtake δ_2 at a finite time. The intersection point (x_0, t_0) is calculated by

$$\begin{aligned} x_0 - x_{01} &= u_{\delta_1} t_0, \\ x_0 - x_{02} &= u_{\delta_2} t_0, \end{aligned} \quad (14)$$

which yields that

$$(x_0, t_0) = \left(\frac{x_{02} - x_{01}}{u_{\delta_1} - u_{\delta_2}} u_{\delta_1} + x_{01}, \frac{x_{02} - x_{01}}{u_{\delta_1} - u_{\delta_2}} \right). \quad (15)$$

At the intersection (x_0, t_0) , the new initial data are formed as follows:

$$t = t_0: \begin{cases} x(t_0) = x_1(t_0) = x_2(t_0), \\ w(t_0) = w_{\delta_1}(t_0) + w_{\delta_2}(t_0) := w_0, \\ h(t_0) = h_{\delta_1}(t_0) + h_{\delta_2}(t_0) := h_0, \\ u_{\delta}(t_0) = \frac{w_{\delta_1}(t_0)u_{\delta_1} + w_{\delta_2}(t_0)u_{\delta_2}}{w_{\delta_1}(t_0) + w_{\delta_2}(t_0)} := u_{\delta}^0 \end{cases} \quad (16)$$

satisfying $u_{\delta_1} > u_{\delta}^0 > u_{\delta_2}$.

In view of $u_- > u_+$, a new delta shock wave will generate after interaction and we denote it with $\delta: x = x(t)$. The trajectory, velocity, and weights $(x(t), u_{\delta}(t), w(t), h(t))$ of δ can be uniquely obtained by solving the ordinary differential equations (6) with the initial date (16). The detail is omitted.

Thus, the result of interaction of two delta shock waves is still a single delta shock wave. This fact can be formulated as

$$\delta + \delta \longrightarrow \delta. \quad (17)$$

Case 2 ($u_- > u_+ > u_m$ (when $u_m > u_+ > u_-$, the structure of solution is similar)). In this situation, a delta shock wave δ_1 determined by (12) is emitted from $(x_{01}, 0)$ and two contact discontinuities $J_1: u = u_m$ and $J_2: u = u_+$ with a vacuum in between are emitted from $(x_{02}, 0)$, as shown in Figure 2.

Since the propagating speed of δ_1 satisfies $u_m < u_{\delta_1} < u_-$, so δ_1 must meet the contact discontinuity $J_1: u = u_m$ at $t_1 = (x_{02} - x_{01})/(u_{\delta_1} - u_m)$, and a new delta shock wave $\delta_2: x = x_2(t)$ forms, which is subjected to the generalized Rankine-Hugoniot relation

$$\begin{aligned} \frac{dx}{dt} &= u_{\delta}, \\ \frac{dw}{dt} &= \rho_- u_- - \rho_- u_{\delta}, \\ \frac{d(wu_{\delta})}{dt} &= \rho_- u_-^2 - \rho_- u_- u_{\delta}, \\ \frac{d}{dt} \left(\frac{wu_{\delta}^2}{2} + h \right) &= \left(\frac{\rho_- u_-^2}{2} + H_- \right) u_- \\ &\quad - \left(\frac{\rho_- u_-^2}{2} + H_- \right) u_{\delta} \end{aligned} \quad (18)$$

with the initial data

$$t = t_1: \begin{cases} x(t_1) = x_1(t_1) := x_1, \\ w(t_1) = w_{\delta_1}(t_1) := w_1, \\ h(t_1) = h_{\delta_1}(t_1) := h_1, \\ u_{\delta}(t_1) = u_{\delta_1}(t_1) := u_{\delta_1}. \end{cases} \quad (19)$$

FIGURE 2: $u_- > u_+ > u_m$.

Therefore, by solving (18) and (19), we have

$$\begin{aligned}
 x_2(t) &= \frac{1}{\rho_-} \left(w_1 + \rho_- u_- (t - t_1) \right. \\
 &\quad \left. - (w_1^2 + 2w_1 \rho_- (u_- - u_{\delta_1}) (t - t_1))^{1/2} \right) + x_1, \\
 w_2(t) &= (w_1^2 + 2w_1 \rho_- (u_- - u_{\delta_1}) (t - t_1))^{1/2}, \\
 u_{\delta_2}(t) &= u_- - \frac{w_1 (u_- - u_{\delta_1})}{w_2(t)}, \\
 h_2(t) &= \left(\frac{\rho_- u_-^2}{2} + H_- \right) u_- (t - t_1) - \left(\frac{\rho_- u_-^2}{2} + H_- \right) \\
 &\quad \cdot (x_2(t) - x_1) + \frac{w_1 u_{\delta_1}^2}{2} + h_1 - \frac{w_2(t) u_{\delta_2}^2(t)}{2}.
 \end{aligned} \tag{20}$$

It is clear that δ_2 will cross the vacuum region with a varying propagation speed. Noting that $\lim_{t \rightarrow +\infty} u_{\delta_2}(t) = u_- > u_+$, so δ_2 will penetrate over the whole vacuum region and then meet J_2 : $u = u_+$ at a finite time. The intersection (x_2, t_2) is determined by

$$\begin{aligned}
 x_2 &= \frac{1}{\rho_-} \left(w_1 + \rho_- u_- (t_2 - t_1) \right. \\
 &\quad \left. - (w_1^2 + 2w_1 \rho_- (u_- - u_{\delta_1}) (t_2 - t_1))^{1/2} \right) + x_1, \\
 x_2 &= u_+ t_2 + x_{02}.
 \end{aligned} \tag{21}$$

At (x_2, t_2) , a new initial value problem is formed and can be solved similar to Case 1. We denote the delta shock wave connecting two constant states (ρ_-, u_-, H_-) and (ρ_+, u_+, H_+) with δ_3 after the interaction of δ_2 and J_2 .

The conclusion of this case is that the delta shock wave will penetrate over the whole vacuum region between two contact discontinuities. This fact is expressed as

$$\delta + J + \text{Vac} + J \longrightarrow \delta. \tag{22}$$

Case 3 ($u_+ > u_- > u_m$ (when $u_m > u_- > u_+$, the structure of solution is similar)). Similar to Case 2, there are a delta shock wave, two contact discontinuities, and a vacuum near $t = 0$ on the (x, t) -plane, as shown in Figure 3.

FIGURE 3: $u_+ > u_- > u_m$. FIGURE 4: $u_+ > u_m > u_-$.

The delta shock wave δ_1 collides with J_1 at first and a new delta shock wave δ_2 generates. However, since $\lim_{t \rightarrow +\infty} u_{\delta_2}(t) = u_- < u_+$, δ_2 cannot penetrate over the vacuum region and finally has $x_2(t) = u_- t + x_{02}$ as its asymptote. This fact is symbolized as

$$\delta + J + \text{Vac} + J \longrightarrow \delta + \text{Vac} + J. \tag{23}$$

Case 4 ($u_+ > u_m > u_-$). In this situation, both the contact discontinuities with a vacuum state in between are emitted from $(x_{01}, 0)$ and $(x_{02}, 0)$, respectively. Noting that the contact discontinuities J_2 and J_3 own the same propagating speed, thus there is no collision of waves and the solution is expressed as

$$\begin{aligned}
 &(\rho, u, H)(t, x) \\
 &= \begin{cases} (\rho_-, u_-, H_-), & x < u_- t + x_{01}, \\ \text{Vac}, & u_- t + x_{01} < x < u_m t + x_{01}, \\ (\rho_m, u_m, H_m), & u_m t + x_{01} < x < u_m t + x_{02}, \\ \text{Vac}, & u_m t + x_{02} < x < u_+ t + x_{02}, \\ (\rho_+, u_+, H_+), & x > u_+ t + x_{02}, \end{cases} \tag{24}
 \end{aligned}$$

which is called a collisionless solution, as shown in Figure 4.

FIGURE 5: Numerical results of u at $t = 1$ (a) and $t = 6.5$ (b) for Data 1. FIGURE 6: Numerical results of ρ at $t = 1$ (a) and $t = 6.5$ (b) for Data 1.

4. Numerical Simulations

In order to verify the validity of the interactions of delta shock waves and vacuum states mentioned in Section 3, we present some representative numerical simulations in this section. Many more numerical tests have been performed to make sure that what are presented are not numerical artifacts.

To discretize the system, we employ the second-order nonoscillatory central schemes [31] with 300×300 cells and $CFL = 0.475$. In what follows, by taking $x_{01} = -0.2$ and $x_{02} = 0.2$, we simulate the interaction of waves by four cases. For

convenience, each situation will be simulated at two different times.

Case 1 ($u_- > u_m > u_+$). We take the initial data as follows:

Data 1: $(\rho, u, H)(0, x)$

$$= \begin{cases} (0.19, 0.15, 0.50), & x < -0.2, \\ (0.05, 0.00, 0.30), & -0.2 < x < 0.2, \\ (0.25, -0.13, 0.01), & x > 0.2. \end{cases} \quad (25)$$

The numerical results are presented by Figures 5–7.

FIGURE 7: Numerical results of H at $t = 1$ (a) and $t = 6.5$ (b) for Data 1. FIGURE 8: Numerical results of u at $t = 0.8$ (a) and $t = 2.9$ (b) for Data 2.

We observe from Figures 5–7 that when $t = 1$, two delta shock waves appear at $(-0.2, 0)$ and $(0.2, 0)$, respectively. As t increases, they will overtake each other and finally unify into a new delta shock wave at $t = 6.5$.

Case 2 ($u_- > u_+ > u_m$). We choose the following initial data

$$\begin{aligned} \text{Data 2: } & (\rho, u, H)(0, x) \\ & = \begin{cases} (0.20, 0.60, 0.16), & x < -0.2, \\ (0.30, -0.10, 0.30), & -0.2 < x < 0.2, \\ (0.25, 0.015, 0.20), & x > 0.2. \end{cases} \end{aligned} \quad (26)$$

The numerical results are shown in Figures 8–10.

From Figures 8–10, we can clearly see that, at $t = 0.8$, a delta shock wave and two contact discontinuities with a vacuum state in between are emitted from $(-0.2, 0)$ and $(0.2, 0)$, respectively. However, at $t = 2.9$, the delta shock wave penetrates over the whole vacuum region, and a new delta shock wave generates.

Case 3 ($u_+ > u_- > u_m$). The initial data are

$$\begin{aligned} \text{Data 3: } & (\rho, u, H)(0, x) \\ & = \begin{cases} (0.36, 0.18, 0.60), & x < -0.2, \\ (0.30, -0.05, 0.30), & -0.2 < x < 0.2, \\ (0.20, 0.40, 0.70), & x > 0.2. \end{cases} \end{aligned} \quad (27)$$

The numerical results are shown in Figures 11–13.

FIGURE 9: Numerical results of ρ at $t = 0.8$ (a) and $t = 2.9$ (b) for Data 2. FIGURE 10: Numerical results of H at $t = 0.8$ (a) and $t = 2.9$ (b) for Data 2.

Figures 11–13 imply that a delta shock wave is emitted from $(-0.2, 0)$, and two contact discontinuities with a vacuum in between are emitted from $(0.2, 0)$ at $t = 0.5$. But the delta shock wave can not penetrate over the whole vacuum region even though time is on the increase. In this process, the region of vacuum state keeps expanding.

Case 4 ($u_+ > u_m > u_-$). We select the initial data to be

$$\begin{aligned} \text{Data 4: } & (\rho, u, H)(0, x) \\ & = \begin{cases} (1.05, -0.10, 0.70), & x < -0.2, \\ (0.70, 0.00, 0.30), & -0.2 < x < 0.2, \\ (1.06, 0.10, 0.90), & x > 0.2. \end{cases} \end{aligned} \quad (28)$$

FIGURE 11: Numerical results of u at $t = 0.5$ (a) and $t = 5.5$ (b) for Data 3. FIGURE 12: Numerical results of ρ at $t = 0.5$ (a) and $t = 5.5$ (b) for Data 3.

The numerical results are presented by Figures 14–16.

From Figures 14–16, we observe that both the contact discontinuities with a vacuum state in between are emitted from $(-0.2, 0)$ and $(0.2, 0)$ at $t = 0.8$, respectively. As time

goes on, the vacuum state keeps continuously expanding and never disappears.

To sum up, all of the above numerical results clearly reveal the interactions of delta shock waves and vacuum states discussed in Section 3. We also indicate that because of the occurrence of singularity as the weighted Dirac delta functions, some oscillations appear in the numerical

FIGURE 13: Numerical results of H at $t = 0.5$ (a) and $t = 5.5$ (b) for Data 3.

FIGURE 14: Numerical results of u at $t = 0.8$ (a) and $t = 5$ (b) for Data 4.

experiments as shown in Figures 8 and 11. It may be a challenge for numerical schemes when delta shock waves develop in solutions.

Competing Interests

The authors declare that they have no competing interests.

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FIGURE 15: Numerical results of ρ at $t = 0.8$ (a) and $t = 5$ (b) for Data 4. FIGURE 16: Numerical results of H at $t = 0.8$ (a) and $t = 5$ (b) for Data 4.

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