Research Article

Explicit Solutions of the Boundary Value Problems for an Ellipse with Double Porosity

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The basic two-dimensional boundary value problems of the fully coupled linear equilibrium theory of elasticity for solids with double porosity structure are reduced to the solvability of two types of a problem. The first one is the BVPs for the equations of classical elasticity of isotropic bodies, and the other is the BVPs for the equations of pore and fissure fluid pressures. The solutions of these equations are presented by means of elementary (harmonic, metaharmonic, and biharmonic) functions. On the basis of the gained results, we constructed an explicit solution of some basic BVPs for an ellipse in the form of absolutely uniformly convergent series.

1. Introduction

In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.e., a mass of porous blocks separated from one another with an interconnected and continuously distributed system of fissures), most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or secondary porosity. When fluid flow and deformation processes occur simultaneously, three coupled partial differential equations can be derived [1, 2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid.

A theory of consolidation with double porosity structure was proposed by Wilson and Aifantis [1]. The physical and mathematical foundations of the theory of double porosity were considered in the papers [1–3], where analytical solutions of the relevant equations are also given. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in indeformable media with two degrees of porosity. The basic results and the historical information on the theory of porous media were summarized by

De Boer [4]. However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity structure by several authors [5–8].

Porous media theories play an important role in many branches of engineering, including material science, petroleum industry, chemical engineering, and soil mechanics, as well as biomechanics. In recent years, many authors investigated the BVPs of the 2D and 3D theories of elasticity for materials with double porosity structure, publishing a large number of papers (some of these results can be seen in [9–17] and references therein). In those works, the explicit solutions on some BVPs in the form of series are given in a form useful for the engineering practice.

In the present paper, the basic two-dimensional boundary value problems of the fully coupled linear equilibrium theory of elasticity for solids with double porosity structure are reduced to the solvability of two types of a problem. The first one is similar to the BVPs for the equations of classical elasticity of isotropic bodies, while the second one is the BVPs for the equations of the pore and fissure fluid pressures. The solutions of these equations are presented by means of elementary (harmonic, metaharmonic, and biharmonic) functions. On

the basis of the gained results, we constructed an explicit solution of some basic BVPs for an ellipse in the form of absolutely uniformly convergent series.

2. Basic Equations

Let $x = (x_1, x_2)$ be a point of the Euclidean 2D space R^2 . In what follows, we consider an ellipse with a double porosity structure that occupies the region Ω of R^2 .

The system of homogeneous equations of the linear equilibrium theory of elasticity for solids with double porosity structure can be written as follows [9]:

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} = \text{grad} (\beta_1 p_1 + \beta_2 p_2) \quad (1)$$

$$(k_1 \Delta - \gamma) p_1 + (k_{12} \Delta + \gamma) p_2 = 0, \quad (2)$$

$$(k_{21} \Delta + \gamma) p_1 + (k_2 \Delta - \gamma) p_2 = 0,$$

where $\mathbf{u} = (u_1(\mathbf{x}), u_2(\mathbf{x}))^T$ is the displacement vector in a solid, $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ are the pore and fissure fluid pressures, respectively, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to the fluid transfer rate with respect to the intensity of flow between the pores and fissures, λ, μ, k_1, k_2 are all constitutive coefficients, $k_j = \kappa_j / \mu'$ ($j = 1, 2$), $k_{12} = \kappa_{12} / \mu'$, $k_{21} = \kappa_{21} / \mu'$, μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, and Δ is the two-dimensional Laplace operator. We consider the vectors as column matrices, if necessary. Throughout this paper, it is assumed that $\beta_1^2 + \beta_2^2 > 0$. Superscript “ T ” denotes transposition. We will suppose that

$$\begin{aligned} k_1 &> 0, \\ k_2 &> 0, \\ k_1 k_2 - k_{12} k_{21} &> 0, \\ k_0 &= k_1 + k_2 + k_{12} + k_{21} > 0. \end{aligned} \quad (3)$$

Note that BVPs for system (2) which contain $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ can be investigated separately. Then if supposing $p_j(\mathbf{x})$ as known, we can study BVPs for system (1) with respect to $\mathbf{u}(\mathbf{x})$. By combining the obtained results, we arrive at explicit solutions of BVPs for systems (1)-(2).

Obviously, from system (2), we have the following equations for $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$:

$$\Delta (\Delta + \lambda_1^2) p_j = 0, \quad j = 1, 2, \quad (4)$$

where

$$\lambda_1 = i \sqrt{\frac{\gamma k_0}{k_1 k_2 - k_{12} k_{21}}}, \quad k_0 = k_1 + k_2 + k_{12} + k_{21}. \quad (5)$$

It is easy to see that the solutions of system (2) have the form

$$\begin{aligned} p_1 &= \phi + m_1 \phi_1, \\ p_2 &= \phi + \phi_1, \end{aligned} \quad (6)$$

where

$$\begin{aligned} m_1 &= -\frac{k_2 + k_{12}}{k_1 + k_{21}}, \\ \Delta \phi &= 0, \end{aligned} \quad (7)$$

$$(\Delta + \lambda_1^2) \phi_1 = 0.$$

First, we assume p_j to be known. Then, for $\mathbf{u}(\mathbf{x})$, we get the following nonhomogeneous equation:

$$\begin{aligned} \mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} &= \text{grad} [(\beta_1 + \beta_2) \phi + (\beta_1 m_1 + \beta_2) \phi_1]. \end{aligned} \quad (8)$$

It is well known that the general solution of (8) has the form

$$\mathbf{u} = \mathbf{v} + \mathbf{v}_0, \quad (9)$$

where $\mathbf{v}$ is a general solution of the equation

$$\mu \Delta \mathbf{v} + (\lambda + \mu) \text{grad div } \mathbf{v} = 0 \quad (10)$$

and $\mathbf{v}_0$ is a particular solution of the nonhomogeneous equation

$$\mathbf{v}_0 = \frac{1}{\lambda + 2\mu} \text{grad} \left[(\beta_1 + \beta_2) \varphi_0 - \frac{\beta_1 m_1 + \beta_2}{\lambda_1^2} \phi_1 \right]; \quad (11)$$

here,

$$\begin{aligned} \Delta \phi &= 0, \\ \Delta \varphi_0 &= \phi, \\ (\Delta + \lambda_1^2) \phi_1 &= 0. \end{aligned} \quad (12)$$

3. Basic Equations in the Elliptic Coordinate System

Let us rewrite equalities (6), (9), and (10)-(12) in elliptic coordinates system ξ, η ($0 \leq \xi < \infty$, $0 \leq \eta < 2\pi$) (see Appendix A). We obtain the following.

Formula (9) takes the form

$$\tilde{\mathbf{u}} = \tilde{\mathbf{v}} + \tilde{\mathbf{v}}_0. \quad (13)$$

Formula (11) takes the form

$$\tilde{\mathbf{v}}_0 = \frac{1}{\lambda + 2\mu} \frac{1}{h} \left(\frac{\partial \Psi}{\partial \xi} \tilde{\mathbf{e}}_\xi + \frac{\partial \Psi}{\partial \eta} \tilde{\mathbf{e}}_\eta \right), \quad (14)$$

where

$$\begin{aligned} \Psi &:= (\beta_1 + \beta_2) \bar{\varphi}_0 - \frac{\beta_1 m_1 + \beta_2}{\lambda_1^2} \bar{\phi}_1, \\ h &= \sqrt{\sinh^2 \xi + \sin^2 \eta}, \\ \bar{\varphi}_0 &= \bar{\varphi}_0(\xi, \eta), \\ \bar{\phi}_1 &= \bar{\phi}_1(\xi, \eta). \end{aligned} \quad (15)$$

Here f_{jk} and g_{jk} are given functions.

FIGURE 2: Semiellipse.

Note that when stress components $(h_0^2/\mu)\sigma_{\xi\xi}$ and $(h_0^2/\mu)\tau_{\xi\eta}$ are given on boundary $\xi = \xi_1$, then the BVP is solvable if the principal vector and the principal moment of external stresses are equal to zero.

Clearly, BVPs for (2) can be investigated separately (Problem B). Then, by admitting $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ as known, we can study BVPs for system (1) with respect to $\mathbf{u}(\mathbf{x})$ (Problem A). By combining the obtained results, we arrive at the explicit solutions of BVPs for systems (1)-(2) and (21).

Problem A. First we state the BVPs for the semiellipse, when the conditions of uninterrupted continuation of solutions (i.e., the conditions of symmetry or antisymmetry) are given on $\eta = 0$, $\eta = \pi$, and $\xi = 0$ [22]. Thus, we obtain the solutions of BVPs for the ellipse.

Now let us formulate the boundary value problems for semiellipse as follows: find in domain $\Omega_1 = \{0 \leq \xi < \xi_1, 0 \leq \eta < \pi\}$ a regular solution $\tilde{\mathbf{v}}(\bar{u}, \bar{v}) \in C^2(\Omega)$ of (19) satisfying the following boundary conditions (see Figure 2):

for $\eta = 0$:

$$\begin{aligned} \text{(a)} \quad & \bar{v} = 0, \\ & \tau_{\xi\eta} = 0 \iff \\ & \bar{v} = 0, \\ & \frac{\partial \bar{u}}{\partial \eta} = 0 \text{ or} \end{aligned} \quad (22)$$

$$\begin{aligned} \text{(b)} \quad & \bar{u} = 0, \\ & \sigma_{\eta\eta} = 0 \iff \\ & \bar{u} = 0, \\ & \frac{\partial \bar{v}}{\partial \eta} = 0, \end{aligned}$$

for $\eta = \pi$:

$$\begin{aligned} \text{(a)} \quad & \bar{v} = 0, \\ & \tau_{\xi\eta} = 0 \iff \\ & \bar{v} = 0, \\ & \frac{\partial \bar{u}}{\partial \eta} = 0 \text{ or} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \bar{u} = 0, \\ & \sigma_{\eta\eta} = 0 \iff \\ & \bar{u} = 0, \\ & \frac{\partial \bar{v}}{\partial \eta} = 0, \end{aligned} \quad (23)$$

for $\xi = 0$:

$$\begin{aligned} \text{(a)} \quad & u = 0, \\ & \tau_{\xi\eta} = 0 \iff \\ & u = 0, \\ & \frac{\partial v}{\partial \xi} = 0 \text{ or} \end{aligned} \quad (24)$$

$$\begin{aligned} \text{(b)} \quad & v = 0, \\ & \sigma_{\xi\xi} = 0 \iff \\ & v = 0, \\ & \frac{\partial u}{\partial \xi} = 0, \end{aligned}$$

for $\xi = \xi_1$:

$$\begin{aligned} \text{(a)} \quad & \bar{u} = f_{11}, \\ & \bar{v} = f_{12} \text{ or} \\ \text{(b)} \quad & \frac{h_0^2}{\mu} \sigma_{\xi\xi} = g_{11}, \\ & \frac{h_0^2}{\mu} \tau_{\xi\eta} = g_{12}. \end{aligned} \quad (25)$$

The solutions of (19) and (22)–(25) are represented by means of two harmonic functions $\varphi_1(\xi, \eta)$ and $\varphi_2(\xi, \eta)$. Consider

$$\begin{aligned} \bar{u} = & \left[\sinh^2 \xi_1 \coth \xi \left(\frac{\partial \varphi_1}{\partial \eta} + \frac{\partial \varphi_2}{\partial \xi} \right) + (\kappa - 1) \varphi_2 \right] \sinh \xi \\ & \cdot \cos \eta - \left[\cosh^2 \xi_1 \tanh \xi \left(\frac{\partial \varphi_1}{\partial \xi} - \frac{\partial \varphi_2}{\partial \eta} \right) \right. \\ & \left. - (\kappa - 1) \varphi_1 \right] \cosh \xi \sin \eta, \\ \bar{v} = & - \left[\cosh^2 \xi_1 \tanh \xi \left(\frac{\partial \varphi_1}{\partial \eta} + \frac{\partial \varphi_2}{\partial \xi} \right) + (\kappa - 1) \varphi_2 \right] \\ & \cdot \cosh \xi \sin \eta - \left[\sinh^2 \xi_1 \coth \xi \left(\frac{\partial \varphi_1}{\partial \xi} - \frac{\partial \varphi_2}{\partial \eta} \right) \right. \\ & \left. - (\kappa - 1) \varphi_1 \right] \sinh \xi \cos \eta, \end{aligned}$$

$$D = \frac{\kappa\mu}{\cosh(2\xi) - \cos(2\eta)} \left[\left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \cosh\xi \sin\eta + \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi \cos\eta \right],$$

$$K = \frac{\kappa\mu}{\cosh(2\xi) - \cos(2\eta)} \left[\left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \sinh\xi \cos\eta - \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \cosh\xi \sin\eta \right].$$

(26)

The components of stress vector (17) can be written as

$$\begin{aligned} \frac{h_0^2}{\mu} \sigma_{\xi\xi} &= \left[2\sinh^2\xi_1 \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \cosh\xi - \left((\kappa-2) \frac{\partial\varphi_1}{\partial\eta} - \kappa \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi \right] \cos\eta + \left[2 \cdot \cosh^2\xi_1 \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi + \left(\kappa \frac{\partial\varphi_1}{\partial\xi} + (\kappa-2) \frac{\partial\varphi_2}{\partial\eta} \right) \cosh\xi \right] \sin\eta \\ &\quad - \frac{4 \sinh(\xi_1 + \xi) \sinh(\xi_1 - \xi)}{\cosh(2\xi) - \cos(2\eta)} \left\{ \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \cdot \cosh\xi \sin\eta + \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi \cos\eta \right\}, \\ \frac{h_0^2}{\mu} \sigma_{\eta\eta} &= - \left[2\sinh^2\xi_1 \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \cosh\xi - \left(\kappa \frac{\partial\varphi_1}{\partial\eta} - (\kappa-2) \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi \right] \cos\eta - \left[2 \cdot \cosh^2\xi_1 \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi + \left((\kappa-2) \frac{\partial\varphi_1}{\partial\xi} + \kappa \frac{\partial\varphi_2}{\partial\eta} \right) \cosh\xi \right] \sin\eta \\ &\quad + \frac{4 \sinh(\xi_1 + \xi) \sinh(\xi_1 - \xi)}{\cosh(2\xi) - \cos(2\eta)} \left\{ \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \cdot \cosh\xi \sin\eta + \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \sinh\xi \cos\eta \right\}, \\ \frac{h_0^2}{\mu} \tau_{\xi\eta} &= - \left[2\cosh^2\xi_1 \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \sinh\xi - \left((\kappa-2) \frac{\partial\varphi_1}{\partial\eta} - \kappa \frac{\partial\varphi_2}{\partial\xi} \right) \cosh\xi \right] \sin\eta + \left[2 \cdot \sinh^2\xi_1 \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \cosh\xi \right. \end{aligned}$$

$$\begin{aligned} &+ \left(\kappa \frac{\partial\varphi_1}{\partial\xi} + (\kappa-2) \frac{\partial\varphi_2}{\partial\eta} \right) \sinh\xi \left. \right] \cos\eta \\ &+ \frac{4 \sinh(\xi_1 + \xi) \sinh(\xi_1 - \xi)}{\cosh(2\xi) - \cos(2\eta)} \left\{ \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) \cdot \sinh\xi \cos\eta - \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) \cosh\xi \sin\eta \right\}. \end{aligned}$$

(27)

Let us rewrite conditions (25) when $\xi = \xi_1$ in the following equivalent form:

$$\begin{aligned} \text{(a)} \quad & \frac{2}{h_0^2} (\cosh\xi_1 \sin\eta\bar{u} + \sinh\xi_1 \cos\eta\bar{v}) \\ &= -\frac{1}{2} \sinh(2\xi_1) \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) + (\kappa-1) \varphi_1, \\ & \frac{2}{h_0^2} (\sinh\xi_1 \cos\eta\bar{u} - \cosh\xi_1 \sin\eta\bar{v}) \\ &= \frac{1}{2} \sinh(2\xi_1) \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) + (\kappa-1) \varphi_2 \text{ or} \\ \text{(b)} \quad & \frac{2}{\mu} (\cosh\xi_1 \sin\eta\sigma_{\xi\xi} + \sinh\xi_1 \cos\eta\tau_{\xi\eta}) \\ &= \sinh(2\xi_1) \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\xi} \right) + \kappa \frac{\partial\varphi_1}{\partial\xi} \\ &\quad + (\kappa-2) \frac{\partial\varphi_2}{\partial\eta}, \\ & \frac{2}{\mu} (\sinh\xi_1 \cos\eta\sigma_{\xi\xi} - \cosh\xi_1 \sin\eta\tau_{\xi\eta}) \\ &= \sinh(2\xi_1) \frac{\partial}{\partial\eta} \left(\frac{\partial\varphi_1}{\partial\xi} - \frac{\partial\varphi_2}{\partial\eta} \right) - (\kappa-2) \frac{\partial\varphi_1}{\partial\eta} \\ &\quad + \kappa \frac{\partial\varphi_2}{\partial\xi}. \end{aligned}$$

(28)

In (28), functions φ_1 and φ_2 are harmonic functions, which, by using the method of separation of variables [23, 24], can be presented as follows:

$$\varphi_i = \sum_{n=1}^{\infty} \varphi_{in}, \quad i = 1, 2, \quad (29)$$

where

$$\begin{aligned} \varphi_{1n} &= B_{1n} \frac{\sinh(n\xi)}{\cosh(n\xi_1)} \sin(n\eta), \\ \varphi_{2n} &= B_{2n} \frac{\cosh(n\xi)}{\cosh(n\xi_1)} \cos(n\eta) \text{ or} \\ \varphi_{1n} &= B_{1n} \frac{\cosh(n\xi)}{\cosh(n\xi_1)} \cos(n\eta), \\ \varphi_{2n} &= B_{2n} \frac{\sinh(n\xi)}{\cosh(n\xi_1)} \sin(n\eta). \end{aligned}$$

(30)

FIGURE 3: Diagram of the main matrix of the algebraic equation system.

When $n = 0$, then $\varphi_{10} = B_{10} + a_{02}\xi + a_{03}\eta + a_{04}\xi\eta$ and $\varphi_{20} = B_{20} + b_{02}\xi + b_{03}\eta + b_{04}\xi\eta$, where $B_{10}, a_{02}, \dots, b_{04}$ are constants. From the condition of limitation of $\text{grad } \varphi_{i0} = (1/h)(\partial\varphi_{i0}/\partial\xi + \partial\varphi_{i0}/\partial\alpha)$ in the focuses of ellipse and condition of periodicity of functions φ_{i0} along η coordinate, there ensues the following: $a_{0j} = 0$, $b_{0j} = 0$, $j = 2, 3, 4$. Therefore, $\varphi_{10} = 0$ and $\varphi_{20} = A_{20}$ or $\varphi_{10} = A_{10}$, and $\varphi_{20} = 0$. When on boundary $\xi = \xi_0$ the values of the stress vector are given, then the rigid displacements of the ellipse are defined by the equalities. $\varphi_{10} = 0$ and $\varphi_{20} = A_{20}$ and $\varphi_{10} = A_{10}$, and $\varphi_{20} = 0$.

By substituting (22)–(24) and (29) into (28), for determining unknown B_{1n} and B_{2n} , we get the following infinite system of algebraic equations, whose main matrix has a block-diagonal form (see Figure 3).

The dimension of all matrices D_i ($i = 1, 2, \dots$) is 2×2 and $\det D_i \neq 0$. When $i \rightarrow \infty$, $\det D_i \rightarrow M$, where $M \neq 0$.

Thus, the system is uniquely solvable. Let us substitute the obtained values, B_{1n} and B_{2n} , into φ_1 and φ_2 . As a result, we obtain the vector $\bar{v}(\bar{u}, \bar{v})$.

Problem B. Find regular solutions p_1 and p_2 to (2) in domain $\Omega = \{0 \leq \xi < \xi_1, 0 \leq \eta < 2\pi\}$ satisfying one of the following boundary conditions (see Figure 1):

when $\xi = \xi_1$:

$$\begin{aligned} \text{(a)} \quad p_1 &= f_{13}, \\ p_2 &= f_{14} \quad \text{or} \\ \text{(b)} \quad \frac{\partial p_1}{\partial \eta} &= f_{23}, \\ \frac{\partial p_2}{\partial \eta} &= f_{24}. \end{aligned} \quad (31)$$

The solution of (2) is (18), where function $\bar{\phi}_1$ is a solution of (16). The solution of (16) by applying the method of separation of variables is reduced to solutions of the Mathieu differential equation or modified Mathieu differential equations (see Appendix B) [25]. These solutions are known as Mathieu functions. Thus, $\bar{\phi}_1(\xi, \eta) = R(\xi) \cdot \Phi_2(\eta)$, where $\Phi_2(\eta)$ is the

solution of Mathieu differential equation $d^2\Phi_2(\eta)(\eta)/d\eta^2 + [a - 2q \cos(2\eta)]\Phi_2(\eta)(\eta) = 0$ (see Appendix C), and R is the solution of modified Mathieu differential equations $d^2R(\xi)/d\xi^2 - [a - 2q \cosh(2\xi)]R(\xi) = 0$ (see Appendix D), where $a := c + (1/2)\lambda_1^2$, $q := (1/4)\lambda_1^2$, $c = \text{const}$ is an arbitrary constant.

Let us assume that harmonic function $\bar{\phi}$ from (18) is represented in the form of series:

$$\begin{aligned} \bar{\phi} &= \sum_{n=0}^{\infty} C_n \cosh(n\xi) \cos(n\eta) \\ &+ \sum_{n=1}^{\infty} D_n \sinh(n\xi) \sin(n\eta). \end{aligned} \quad (32)$$

Taking into account the boundary conditions (31), to determine unknown coefficients C_n and D_n , we obtain the system of infinite algebraic equations, whose main matrix has a block-diagonal form (Figure 3). By solving this system, we find $\bar{\phi}(\xi, \eta)$.

For determining $\bar{\varphi}_0$ from (17), let us consider the particular cases: keeping in mind the homogeneous boundary conditions, we get $C_n = 0$ or $D_n = 0$. Let us assume that $D_n = 0$.

Substituting (32) in (17), we obtain the following equation:

$$\begin{aligned} \frac{\partial^2 \bar{\varphi}_0}{\partial \xi^2} + \frac{\partial^2 \bar{\varphi}_0}{\partial \eta^2} &= \sum_{k=0}^{\infty} C_k \cosh(k\xi) \cos(k\eta) (\cosh(2\xi) - \cos(2\eta)) \\ &= \sum_{n=0}^{\infty} E_n \cosh(n\xi) \cos(n\eta). \end{aligned} \quad (33)$$

The solution of (33) is sought in the following form:

$$\bar{\varphi}_0 = \bar{\varphi}_{0g} + \bar{\varphi}_{0p}, \quad (34)$$

where $\bar{\varphi}_{0g}$ is a harmonic function and it can be obtained in the same way as $\bar{\phi}$. Thus, we have

$$\begin{aligned} \Delta \bar{\varphi}_{0g} &= 0, \\ \bar{\varphi}_{0g} &= \sum_{n=0}^{\infty} T_n \cosh(n\xi) \cos(n\eta). \end{aligned} \quad (35)$$

$\bar{\varphi}_{0p}$ is the particular solution of the following equation:

$$\frac{\partial^2 \bar{\varphi}_{0p}}{\partial \xi^2} + \frac{\partial^2 \bar{\varphi}_{0p}}{\partial \eta^2} = \sum_{n=0}^{\infty} E_n \cosh(n\xi) \cos(n\eta). \quad (36)$$

The solution of (36) is sought in the form

$$\bar{\varphi}_{0p} = \sum_{n=0}^{\infty} Q_n \cosh(n\xi) \cos(n\eta). \quad (37)$$

Substituting (37) in (36), we obtain the relations between Q_n and E_n and hence it follows that

$$\begin{aligned} \bar{\varphi}_0 &= \sum_{n=0}^{\infty} T_n \cosh(n\xi) \cos(n\eta) \\ &+ \sum_{n=0}^{\infty} Q_n \cosh(n\xi) \cos(n\eta) \end{aligned} \quad (38)$$

or

$$\bar{\varphi}_0 = \sum_{n=0}^{\infty} M_n \cosh(n\xi) \cos(n\eta), \quad (39)$$

where $M_n = T_n + Q_n$.

Quite similarly, we obtain the solution, when $C_n = 0$. By combining the obtained results, we obtain an explicit solution of (17).

Remark 1. When we have the nonhomogeneous boundary conditions, then the coefficients in (32) are different from zero, $C_n \neq 0$, $D_n \neq 0$, and function $\bar{\varphi}_0$ will be obtained by above-mentioned method for each series, separately.

By substituting $\bar{\varphi}_0$ and $\bar{\phi}_1$ in (14), we obtain function $\bar{v}_0$. Finally, from (13), we obtain displacement $\bar{u}$ in Ω at arbitrary point. Stress components are defined from (20).

5. Conclusions

The main results of this work can be formulated as follows:

- (1) The system of equations of the linear equilibrium theory of elasticity for solids with double porosity structure is written in terms of elliptic coordinates.
- (2) The problems are reduced to the solvability of two types of problem. The first one is similar to the BVPs for the equations of classical elasticity of isotropic bodies, while the second one is the BVPs for the equations of the porous and fissure fluid pressures.
- (3) Analytical (exact) solutions are obtained for 2D BVPs for the ellipse with double porosity structure.

By using the above-mentioned method, the following is possible:

- (4) It is possible to construct explicitly the solutions of basic BVPs for systems (1) and (2) for simple cases of 2D domains (circle, plane with circular hole.) in the form of absolutely and uniformly convergent series that are useful in the engineering practice.
- (5) It is possible to obtain numerical solutions of the boundary value problems.
- (6) It is possible to construct explicitly the solutions of basic BVPs of the systems of equations in the modern linear theories of elasticity, thermoelasticity, and poroelasticity for materials with microstructures and for elastic materials with double porosity for a circle, and so forth.

FIGURE 4: Elliptic coordinate system.

(7) In practice, such BVPs are quite common in many areas of science. The potential users of the obtained results will be the scientists and engineers working on the problems of solid mechanics, micromechanics and nanomechanics, mechanics of materials, engineering mechanics, engineering medicine, biomechanics, engineering geology, geomechanics, hydro-engineering, applied and computing mechanics, and applied mathematics.

Appendix

A. Some Basic Formulas in Elliptic Coordinates

If x, y ($-\infty < x < +\infty$, $-\infty < y < +\infty$) are Cartesian coordinates and ξ, η are elliptic coordinates ($0 \leq \xi < \infty$, $0 \leq \eta < 2\pi$), then $x = c \cosh \xi \cos \eta$, $y = c \sinh \xi \sin \eta$, where c is scale factor (Figure 4) [26].

The coordinate curves are ellipses and hyperbolas:

$$\begin{aligned} \frac{x^2}{c^2 \cosh^2 \xi_0} + \frac{y^2}{c^2 \sinh^2 \xi_0} &= 1, \quad \xi_0 = \text{const} \neq 0, \\ \frac{x^2}{c^2 \cos^2 \eta_0} - \frac{y^2}{c^2 \sin^2 \eta_0} &= 1, \end{aligned} \quad (A.1)$$

$$\eta_0 = \text{const} \neq 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi.$$

$h := h_\xi = h_\eta = (c/\sqrt{2}) \sqrt{\cosh(2\xi) - \cos(2\eta)} = c \sqrt{\sinh^2 \xi + \sin^2 \eta}$ are metric coefficients (Lamé coefficients).

The Laplacian operator in the elliptic coordinates has the form

$$\Delta f = \frac{1}{h^2} \left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} \right). \quad (A.2)$$

The operator grad has the form

$$\nabla f := \text{grad}(f) = \frac{1}{h} \left(\frac{\partial f}{\partial \xi} \bar{e}_\xi + \frac{\partial f}{\partial \eta} \bar{e}_\eta \right), \quad (A.3)$$

where $f = f(\xi, \eta)$.

The Helmholtz equation can be written as

$$\frac{1}{h^2} \left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} \right) + \lambda_1^2 f = 0, \quad (\text{A.4})$$

where $f = f(\xi, \eta)$,

and biharmonic equation has the form

$$\begin{aligned} \Delta \Delta f &= (\Delta f)^2 = \left[\frac{1}{h^2} \left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} \right) \right]^2 \\ &= \frac{1}{h^4} \left(\frac{\partial^4 f}{\partial \xi^4} + 2 \frac{\partial^4 f}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 f}{\partial \eta^4} \right) = 0. \end{aligned} \quad (\text{A.5})$$

B. Solution of Helmholtz Equation in the Elliptic Coordinates

Helmholtz equation in the elliptic coordinates has the following form:

$$\frac{1}{\sinh^2 u + \sin^2 v} \left(\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} \right) + k^2 F = 0. \quad (\text{B.1})$$

Let us solve this equation by using the method of separation of variables.

Let function F be sought in the form

$$F(u, v) = U(u) \cdot V(v), \quad (\text{B.2})$$

and then Helmholtz equation takes the following form:

$$\frac{1}{\sinh^2 u + \sin^2 v} \left(V \frac{d^2 U}{du^2} + U \frac{d^2 V}{dv^2} \right) + k^2 UV = 0. \quad (\text{B.3})$$

From here, we obtain

$$\frac{1}{\sinh^2 u + \sin^2 v} \left(\frac{1}{U} \frac{d^2 U}{du^2} + \frac{1}{V} \frac{d^2 V}{dv^2} \right) + k^2 = 0. \quad (\text{B.4})$$

Let us rewrite the last equation in the following form:

$$\left(\frac{1}{U} \frac{d^2 U}{du^2} + k^2 \sinh^2 u \right) + \left(\frac{1}{V} \frac{d^2 V}{dv^2} + k^2 \sin^2 v \right) = 0. \quad (\text{B.5})$$

From here, we get

$$\begin{aligned} \frac{1}{U} \frac{d^2 U}{du^2} + k^2 \sinh^2 u &= c, \\ c + \frac{1}{V} \frac{d^2 V}{dv^2} + k^2 \sin^2 v &= 0. \end{aligned} \quad (\text{B.6})$$

Thus,

$$\begin{aligned} \frac{d^2 U}{du^2} - (c - k^2 \sinh^2 u) U &= 0, \\ \frac{d^2 V}{dv^2} + (c + k^2 \sin^2 v) V &= 0. \end{aligned} \quad (\text{B.7})$$

Take into account the following identities:

$$\begin{aligned} \sinh^2 u &= \frac{1}{2} [\cosh(2u) - 1], \\ \sin^2 v &= \frac{1}{2} [1 - \cos(2v)]. \end{aligned} \quad (\text{B.8})$$

We obtain

$$\begin{aligned} \frac{d^2 U}{du^2} - \left\{ c - \frac{1}{2} k^2 [\cosh(2u) - 1] \right\} U &= 0, \\ \frac{d^2 V}{dv^2} + \left\{ c + \frac{1}{2} k^2 [1 - \cos(2v)] \right\} V &= 0. \end{aligned} \quad (\text{B.9})$$

From here, we get

$$\begin{aligned} \frac{d^2 U}{du^2} - \left[\left(c + \frac{1}{2} k^2 \right) - \frac{1}{2} k^2 \cosh(2u) \right] U &= 0, \\ \frac{d^2 V}{dv^2} + \left[\left(c + \frac{1}{2} k^2 \right) - \frac{1}{2} k^2 \cos(2v) \right] V &= 0. \end{aligned} \quad (\text{B.10})$$

Let us introduce the notations $a := c + (1/2)k^2$ and $q := (1/4)k^2$; then we have $d^2 U/du^2 - [a - 2q \cosh(2u)]U = 0$; this is the modification differential equation of Mathieu. Consider $d^2 V/dv^2 + [a - 2q \cos(2v)]V = 0$; this is the differential equation of Mathieu.

C. The Solutions of Differential Equation of Mathieu

Let us assume that $c = 0$; then $a := (\lambda_1/\sqrt{2})^2$ and $q := (1/4)\lambda_1^2$. The differential equation of Mathieu takes the form

$$\frac{\partial^2 \Phi_2(\eta)}{\partial \eta^2} + (a - 2q \cos 2\eta) \Phi_2(\eta) = 0. \quad (\text{C.1})$$

The solutions of this equation have the following forms [25]:

(a) When $q > 0$,

$$\begin{aligned} ce_{2m}(q, \eta) &= \sum_{n=0}^{\infty} A_{2n}^{(2m)}(q) \cos(2n\eta), \\ se_{2m+2}(q, \eta) &= \sum_{n=0}^{\infty} B_{2n+2}^{(2m+2)}(q) \sin[(2n+2)\eta], \\ ce_{2m+1}(q, \eta) &= \sum_{n=0}^{\infty} A_{2n+1}^{(2m+1)}(q) \cos[(2n+1)\eta], \\ se_{2m+1}(q, \eta) &= \sum_{n=0}^{\infty} B_{2n+1}^{(2m+1)}(q) \sin[(2n+1)\eta]. \end{aligned} \quad (\text{C.2})$$

(b) When $q < 0$,

$$\begin{aligned}
 ce_{2m}(-q, \eta) &= (-1)^m \sum_{n=0}^{\infty} (-1)^n A_{2n}^{(2m)}(q) \cos(2m\eta), \\
 se_{2m+2}(-q, \eta) \\
 &= (-1)^m \sum_{n=0}^{\infty} (-1)^n B_{2n+2}^{(2m+2)}(q) \sin[(2n+2)\eta], \\
 ce_{2m+1}(-q, \eta) \\
 &= (-1)^m \sum_{n=0}^{\infty} (-1)^n A_{2n+1}^{(2m+1)}(q) \cos[(2n+1)\eta], \\
 se_{2m+1}(-q, \eta) \\
 &= (-1)^m \sum_{n=0}^{\infty} (-1)^n B_{2n+1}^{(2m+1)}(q) \sin[(2n+1)\eta],
 \end{aligned} \tag{C.3}$$

where, for $ce_{2m}(q, \eta)$,

$$\begin{aligned}
 aA_0 - qA_2 &= 0, \\
 (a-4)A_2 - q(A_4 + 2A_0) &= 0,
 \end{aligned} \tag{C.4}$$

$$(a-4n^2)A_{2n} - q(A_{2n+2} + A_{2n-2}) = 0, \quad n \geq 2;$$

for $ce_{2m+1}(q, \eta)$,

$$\begin{aligned}
 (a-1-q)A_1 - qA_3 &= 0, \\
 [a-(2n+1)^2]A_{2n+1} - q(A_{2n+3} + A_{2n-1}) &= 0, \\
 n &\geq 1;
 \end{aligned} \tag{C.5}$$

for $se_{2m+1}(q, \eta)$,

$$\begin{aligned}
 (a-1+q)B_1 - qB_3 &= 0, \\
 [a-(2n+1)^2]B_{2n+1} - q(B_{2n+3} + B_{2n-1}) &= 0, \\
 n &\geq 1;
 \end{aligned} \tag{C.6}$$

and, for $se_{2m+2}(q, \eta)$,

$$\begin{aligned}
 (a-4)B_2 - qB_4 &= 0, \\
 (a-4n^2)B_{2n} - q(B_{2n+2} + B_{2n-2}) &= 0, \quad n \geq 2
 \end{aligned} \tag{C.7}$$

(here $A_{2n}^{(2m)} \equiv A_{2n}, \dots, B_{2n+2}^{(2m+2)} \equiv B_{2n+2}$).

D. The Solutions of the Modification Differential Equation of Mathieu

The modification differential equation of Mathieu has the form

$$\frac{\partial^2 R(\xi)}{\partial \eta^2} - (a - 2q \cosh 2\xi) R(\xi) = 0. \tag{D.1}$$

The solutions of this equation are as follows [25]:

(a) If $q > 0$,

$$\begin{aligned}
 Ce_{2m}(q, \xi) &= \sum_{k=0}^{\infty} A_{2k} J_{2k}(2\sqrt{q} \sinh \xi), \\
 Ce_{2m+1}(q, \xi) &= \sum_{k=0}^{\infty} (-1)^{k+1} A_{2k+1} J_{2k+1}(2\sqrt{q}) \cosh \xi, \\
 Se_{2m+1}(q, \xi) &= \sum_{k=0}^{\infty} B_{2k+1} J_{2k+1}(2\sqrt{q} \sinh \xi), \\
 Se_{2m+2}(q, \xi) \\
 &= \tanh \xi \sum_{k=0}^{\infty} (-1)^k 2k B_{2k} J_{2k}(2\sqrt{q} \cosh \xi).
 \end{aligned} \tag{D.2}$$

(b) If $q < 0$, then the argument of Bessel function is $(2\sqrt{-q} = i2\sqrt{q})$.

E. Physical Motivation of Double Porosity Model

The double porosity model has received a lot of attention from mathematicians and from engineers. In such a model, there are two pore systems with different permeability. There is first a set of isolated porous blocks of low permeability (sometimes called matrix), surrounded by network of high permeability connected porous medium (usually called fractures network). Pores are pervasive in most of the igneous, metamorphic, and sedimentary rocks in the earth's crust. In fact, porosity found in the earth may have many shapes and sizes, but two types of porosity are more important. One is the matrix porosity, and the other is fracture or crack porosity which may occupy very little volume, but fluid flow occurs primarily through the fracture network.

In physical terms, the theory of poroelasticity postulates that when a porous material is subjected to stress, the resulting matrix deformation leads to volumetric changes in the pores. The pores are filled with fluid. The presence of the fluid results in the flow of the pore fluid between regions of higher and lower pore pressure.

The physical process of coupled deformation is governed by the following equations:

(1) The equations of motion [27, 28]:

$$t_{lj,j} = \rho(\ddot{u}_l - F_l), \quad l, j = 1, 2, 3, \tag{E.1}$$

where t_{lj} is the components of total stress tensor, $\rho > 0$ is the reference mass density, and F is the body force per unit mass. We assume that subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3), and the dot denotes differentiation with respect to t (here t denotes the time variable; $t \geq 0$).

(2) The equations of fluid mass conservation [5]:

$$\begin{aligned}\operatorname{div} \mathbf{v}^{(1)} + \dot{\zeta}_1 + \beta_1 \dot{e}_{rr} + \nu(p_1 - p_2) &= 0, \\ \operatorname{div} \mathbf{v}^{(2)} + \dot{\zeta}_2 + \beta_2 \dot{e}_{rr} - \nu(p_1 - p_2) &= 0,\end{aligned}\quad (\text{E.2})$$

where $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are the fluid flux vectors for pores and fissures, respectively.

e_{ij} is the components of strain tensor:

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3. \quad (\text{E.3})$$

β_1 and β_2 are the effective stress parameters and ν is the internal transport coefficients and corresponds to a fluid transfer rate respecting the intensity of the flow between pores and fissures, $\nu \geq 0$, and ζ_1 and ζ_2 are the increments of fluid in pores and fissures, respectively, and are defined by

$$\begin{aligned}\zeta_1 &= \alpha_1 p_1 + \alpha_{12} p_2, \\ \zeta_2 &= \alpha_{21} p_1 + \alpha_2 p_2.\end{aligned}\quad (\text{E.4})$$

α_1 and α_2 measure the compressibility of pore and fissure systems, respectively, and α_{12} and α_{21} are the cross-coupling compressibility for fluid flow at the interface between the two pore systems at a microscopic level.

(3) The constitutive equations (extending Terzaghi's effective stress concept to double porosity) [1, 5]:

$$\begin{aligned}t_{ij} &= 2\mu e_{ij} + \lambda e_{rr} \delta_{ij} - (\beta_1 p_2 + \beta_2 p_1) \delta_{ij}, \\ l, j &= 1, 2, 3.\end{aligned}\quad (\text{E.5})$$

λ and μ are the Lamé constants, and δ_{ij} is Kronecker's delta.

(4) Darcy's law for material with double porosity [7, 8]:

$$\begin{aligned}\mathbf{v}^{(1)} &= -\frac{1}{\mu'} (\kappa_1 \operatorname{grad} p_1 + \kappa_{12} \operatorname{grad} p_2) - \rho_1 s^{(1)}, \\ \mathbf{v}^{(2)} &= -\frac{1}{\mu'} (\kappa_{21} \operatorname{grad} p_1 + \kappa_2 \operatorname{grad} p_2) - \rho_2 s^{(2)},\end{aligned}\quad (\text{E.6})$$

where μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with the matrix and fissure porosity, respectively, and κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; $\rho_1, s^{(1)}$ and $\rho_2, s^{(2)}$ are the densities of fluid and the external forces for the pore and fissure phases, respectively.

Substituting (E.3)–(E.6) into (E.1) and (E.2), assuming that $\mathbf{u} = (u_1, u_2)$, $p_j = p_j(x_1, x_2)$, $u_j = u_j(x_1, x_2)$, $\rho = 0$, $\rho_1 = 0$, $s^{(j)} = 0$ and $F_j = 0$, we obtain the system of homogeneous equations of motion in the coupled linear theory of elasticity for solids with double porosity structure (see (1) and (2)).

Competing Interests

The authors declare that they have no competing interests.

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