Research Article

A Nonlinear Schrödinger Equation Resonating at an Essential Spectrum

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We consider the nonlinear Schrödinger equation $-\Delta u + f(u) = V(x)u$ in $\mathbb{R}^N$. The potential function V satisfies that the essential spectrum of the Schrödinger operator $-\Delta - V$ is $[0, +\infty)$ and this Schrödinger operator has infinitely many negative eigenvalues accumulating at zero. The nonlinearity f satisfies the resonance type condition $\lim_{|t| \rightarrow \infty} f(t)/t = 0$. Under some additional conditions on V and f , we prove that this equation has infinitely many solutions.

1. Introduction and Statement of Results

In this paper, the following nonlinear Schrödinger problem in $\mathbb{R}^N$ ($N \geq 3$) is considered:

$$-\Delta u = V(x)u + g(u) \quad \text{in } \mathbb{R}^N. \quad (1)$$

The nonlinearity g satisfies the resonance type condition:

$$\lim_{|t| \rightarrow \infty} \frac{g(t)}{t} \in \sigma_{\text{ess}}(T). \quad (2)$$

Here, $\sigma_{\text{ess}}(T)$ denotes the essential spectrum (see [1, Chapter 7] or [2, Chapter 7.4]) of the Schrödinger operator T defined as follows:

$$\begin{aligned} T : L^2(\mathbb{R}^N) &\longrightarrow L^2(\mathbb{R}^N), \\ u &\longmapsto -\Delta u - Vu, \end{aligned} \quad (3)$$

with domain $D(T) := \{u \in L^2(\mathbb{R}^N) \mid Tu \in L^2(\mathbb{R}^N)\}$.

Equation (1) arises in quantum mechanics and is related to the study of the nonlinear Schrödinger equation for a particle in an electromagnetic field and it has attracted considerable attention from researchers in recent decades. One can see [3–6] and the references therein. However, there are very few results on (1) with the resonance type condition (2).

The following nonlinear elliptic problem with resonance type conditions in bounded domain has been studied by many authors and numerous existence and multiplicity results have been obtained in the past forty years:

$$-\Delta u = h(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (4)$$

Here, Ω is a bounded domain in $\mathbb{R}^N$, and g satisfies the resonance type condition:

$$\lim_{|t| \rightarrow \infty} \frac{h(x, t)}{t} = \nu_k. \quad (5)$$

ν_k is the k th eigenvalue of $-\Delta$ with 0-Dirichlet boundary condition on $\partial\Omega$. We refer to [7–10] and references therein for more detailed discussions of some historical results. This problem has deeply inspired developments in critical point theory in the last forty years, such as the Landesman-Lazer type conditions (e.g., [8, 10]), Morse theory (e.g., [9]), and the variational reduction method (e.g., [11]). The difficulty of this problem lies in the proof of the boundedness of the Palais-Smale sequence or the $(C)_c$ sequence (see Definition 5 in Section 2) of its corresponding functional.

The resonant Schrödinger problem (1) is much more difficult than the bounded domain case and it has fewer studies. Unlike the case of bounded domains for which the linear operators $-\Delta$ in bounded domain are compact, there are continuous spectra of the linear operator T . Moreover, the

proofs of boundedness and compactness of the $(C)_c$ sequence of the corresponding functional of (4) are greatly different from and more difficult than the case of bounded domain (see [4, 5, 12]).

More precisely, this paper considers the nonlinear Schrödinger equation:

$$-\Delta u + f(u) = V(x)u, \quad u \in H^1(\mathbb{R}^N). \quad (6)$$

Here, $N \geq 3$ and $H^1(\mathbb{R}^N)$ is the Sobolev space:

$$H^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \mid \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} u^2 dx < +\infty \right\}. \quad (7)$$

For $V : \mathbb{R}^N \rightarrow \mathbb{R}$, the following is assumed:

(V₁) Consider $V \in C^1(\mathbb{R}^N \setminus \{0\})$, $V \geq 0$ in $\mathbb{R}^N \setminus \{0\}$, $\lim_{|x| \rightarrow \infty} V(x) = 0$, $V \in L^s_{\text{loc}}(\mathbb{R}^N)$ with $s = \max\{2, N/2\}$ if $N \geq 3$ and $N \neq 4$, $V \in L^s_{\text{loc}}(\mathbb{R}^N)$ for some $s > 2$ if $N = 4$, and there exist $R > 0$, $0 < \beta < 2$, and $C > 0$ such that, for $|x| \geq R$,

$$V(x) \geq C|x|^{-\beta}. \quad (8)$$

(V₂) For any $x \in \mathbb{R}^N \setminus \{0\}$, $x \cdot \nabla V(x) \leq 0$ and there exists $0 < \gamma < 2$ such that

$$\gamma V(x) + x \cdot \nabla V(x) \geq 0, \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \quad (9)$$

For f , the following is assumed:

(f₁) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function:

$$a_0 := \lim_{t \rightarrow 0} \frac{f(t)}{t} > 0, \quad (10)$$

$$\lim_{|t| \rightarrow \infty} \frac{f(t)}{t} = 0, \quad (11)$$

$$tf(t) \geq 0, \quad \forall t \in \mathbb{R}. \quad (12)$$

(f₂) There exists $D > 0$ such that, for any $t \in \mathbb{R}$,

$$F(t) \leq Dtf(t). \quad (13)$$

Here,

$$F(t) = \int_0^t f(s) ds, \quad t \in \mathbb{R}. \quad (14)$$

For $t \in \mathbb{R}$, let

$$g(t) = a_0 t - f(t), \quad (15)$$

$$\tilde{F}(t) = F(t) - \frac{1}{2}tf(t). \quad (16)$$

(f₃) For any $t \in \mathbb{R}$, $\tilde{F}(t) \geq 0$ and there exists $\delta \in (0, a_0)$ such that if $|g(t)| \geq (a_0 - \delta)|t|$, then

$$\tilde{F}(t) \geq \delta. \quad (17)$$

(f₄) For any $t \in \mathbb{R}$, $f(-t) = -f(t)$.

The main result of this paper is as follows.

Theorem 1. Suppose that (V₁)-(V₂) and (f₁)-(f₄) are satisfied; then, problem (6) has infinitely many solutions.

Remark 2. (i) Under the assumptions (V₁) and (V₂), the essential spectrum of T defined by (3) is $[0, +\infty)$ (see Lemma 3 in Section 2). Therefore, condition (11) indicates that f satisfies a resonance type condition at the essential spectrum of T .

(ii) A typical example for V which satisfies (V₁)-(V₂) is

$$V(r, \theta) = \frac{P(\theta)}{r^\beta}. \quad (18)$$

Here, $r = |x|$, $\theta = x/|x| \in S^{N-1} := \{y \in \mathbb{R}^N \mid |y| = 1\}$, $0 < \beta < \min\{2, N/2\}$, $P \in C^1(S^{N-1})$, and $P > 0$ in S^{N-1} .

(iii) There are many functions satisfying (f₁)-(f₄). Let $G(t) = \int_0^t g(s) ds$. Using

$$\frac{d}{dt} \left(\frac{G(t)}{t^2} \right) = \frac{2\tilde{F}(t)}{t^3}, \quad (19)$$

we can construct a function f satisfying (f₁)-(f₄) from a given $\tilde{F}$. For example, let

$$\tilde{F}(t) = \frac{\sigma|t|^3}{1 + |t|^{5/2}}, \quad t \in \mathbb{R}, \quad (20)$$

where

$$\sigma = \frac{a_0}{4\tau}, \quad \tau = \int_0^{+\infty} \frac{ds}{1 + |s|^{5/2}}. \quad (21)$$

Then, by (19), for $t \geq 0$,

$$G(t) = 2\sigma t^2 \int_0^t \frac{ds}{1 + |s|^{5/2}} \quad (22)$$

and, for $t < 0$, $G(t) = G(-t)$. It follows that

$$g(t) = 4\sigma t \int_0^t \frac{ds}{1 + |s|^{5/2}} + \frac{2\sigma t^2}{1 + |t|^{5/2}} \quad \text{if } t \geq 0, \quad (23)$$

$$g(t) = -g(-t) \quad \text{if } t < 0,$$

and $f(t) = a_0 t - g(t)$ for any $t \in \mathbb{R}$. Because the even function $g(t)/t$ is strictly increasing in $(0, +\infty)$, $g(t)/t \rightarrow a_0$ as $t \rightarrow +\infty$, it can be deduced that f satisfies $tf(t) > 0$, $\forall t \in \mathbb{R} \setminus \{0\}$. Moreover, it is easy to verify that $f(t) = a_0 t + o(t)$ as $t \rightarrow 0$, $f(t) = (2/3)\sigma t^{-1/2} + o(t^{-1/2})$ as $|t| \rightarrow \infty$, $F(t) = (1/2)a_0 t^2 + o(t^2)$ as $t \rightarrow 0$, and $F(t) = (4/3)\sigma|t|^{1/2} + o(|t|^{1/2})$ as $|t| \rightarrow \infty$. Because the even function $g(t)/t$ is strictly increasing in $(0, +\infty)$, $g(t)/t \rightarrow a_0$ as $t \rightarrow +\infty$, and $\tilde{F}(t) \rightarrow +\infty$ as $|t| \rightarrow +\infty$, it can be deduced that, for sufficiently small $\delta > 0$, if $|g(t)|/|t| \geq a_0 - \delta$, then $\tilde{F}(t) \geq \delta$. Therefore, f satisfies (f₁)-(f₄).

Let

$$X = H^1(\mathbb{R}^N). \quad (24)$$

The inner product and the norm in X are

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + a_0 uv) dx, \quad \|u\| = (u, u)^{1/2}, \quad (25)$$

respectively. Here, a_0 is the constant in (10).

Let

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ &\quad + \int_{\mathbb{R}^N} F(u) dx, \quad u \in X. \end{aligned} \quad (26)$$

Then, for any $u, v \in X$, the Gateaux derivative of Φ is as follows:

$$\begin{aligned} \langle \Phi'(u), v \rangle &= \int_{\mathbb{R}^N} \nabla u \nabla v dx - \int_{\mathbb{R}^N} V(x) uv dx \\ &\quad + \int_{\mathbb{R}^N} f(u) v dx. \end{aligned} \quad (27)$$

It follows that $u \in X$ is a solution of (6) if and only if u is a critical point of Φ . In Section 2, it is shown that Φ satisfies the Cerami condition. In Section 3, an abstract critical point theorem from [13] is used to show that Φ has infinitely many critical points.

Notation. Let E be Banach space. We denote the dual space of E by E' and denote strong and weak convergence in E by $\rightarrow$ and $\rightharpoonup$, respectively. For $\varphi \in C^1(E, \mathbb{R})$, we denote the Fréchet derivative of φ at u by $\varphi'(u)$. The Gateaux derivative of φ is denoted by $\langle \varphi'(u), v \rangle$, $\forall u, v \in E$. $L^p(\mathbb{R}^N)$ denotes the standard L^p space ($1 \leq p \leq \infty$). We use $O(h)$ and $o(h)$ to mean $|O(h)| \leq C|h|$ and $o(h)/|h| \rightarrow 0$, respectively.

2. The Cerami Condition for Φ

Lemma 3. *If V satisfies the assumptions (V_1) , then the essential spectrum of the Schrödinger operator T defined by (3) is $[0, +\infty)$ and T has infinitely many negative eigenvalues accumulating at zero.*

Proof. Because V satisfies (V_1) , by [14, Theorem XIII.15(b)] or [2, Theorem 10.29(b)], the essential spectrum of the Schrödinger operator T defined by (3) is $[0, +\infty)$. The second result of this lemma comes from [2, Theorem 10.31] or [14, Theorem XIII.6(a)]. $\square$

Lemma 4. *If V satisfies the assumptions (V_1) and (V_2) , then T has no eigenvalue in $[0, +\infty)$.*

Proof. Because V satisfies the assumptions (V_1) and (V_2) , by [15, Corollary 2], to prove this lemma, it suffices to prove that

V satisfies the conditions I and II in [15, Corollary 2]. For this, it suffices to prove that, for $0 < \alpha < 2$, the functions

$$M(x) = \int_{|x-y| \leq 1} |V(y)| \cdot |x-y|^{4-N-\alpha} dy, \quad (28)$$

$$L(x) = \int_{|x-y| \leq 1} |y \nabla V(y)| \cdot |x-y|^{4-N-\alpha} dy$$

are bounded functions in $\mathbb{R}^N$. Because $\lim_{|x| \rightarrow \infty} V(x) = 0$ and $0 < \alpha < 2$, there exists $R' > 0$ such that, for $|x| > R'$,

$$\begin{aligned} M(x) &\leq \int_{|x-y| \leq 1} |x-y|^{4-N-\alpha} = \int_{|y| \leq 1} |y|^{4-N-\alpha} dy \\ &< +\infty. \end{aligned} \quad (29)$$

Because $V \in L_{\text{loc}}^{N/2}(\mathbb{R}^N)$, for $|x| \leq R'$, the following is true:

$$\begin{aligned} M(x) &\leq \left(\int_{|x-y| \leq 1} |V(y)|^{N/2} dy \right)^{2/N} \\ &\quad \cdot \left(\int_{|x-y| \leq 1} |x-y|^{N(4-N-\alpha)/(N-2)} dy \right)^{(N-2)/N} \\ &= \left(\int_{|y| \leq R'+1} |V(y)|^{N/2} dy \right)^{2/N} \\ &\quad \cdot \left(\int_{|y| \leq 1} |y|^{N(4-N-\alpha)/(N-2)} dy \right)^{(N-2)/N}. \end{aligned} \quad (30)$$

Because $0 < \alpha < 2$, the following holds:

$$\int_{|y| \leq 1} |y|^{N(4-N-\alpha)/(N-2)} dy < +\infty. \quad (31)$$

Combining (29) and (30) yields that M is a bounded function in $\mathbb{R}^N$. From (V_2) , we have $L(x) \leq \gamma M(x)$, $\forall x \in \mathbb{R}^N$. It follows that L is also a bounded function in $\mathbb{R}^N$. $\square$

Definition 5 (see [16]). Let E be a Banach space. Let $J \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. One can call the fact that J satisfies the Cerami condition at c , denoted by $(C)_c$ condition, if for any $\{u_n\} \subset E$ satisfying

$$\begin{aligned} \lim_{n \rightarrow \infty} J(u_n) &= c, \\ \lim_{n \rightarrow \infty} (1 + \|u_n\|) \|J'(u_n)\| &= 0, \end{aligned} \quad (32)$$

there exists $u \in E$ such that, up to a subsequence, $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$. If J satisfies the $(C)_c$ condition for every $c \in \mathbb{R}$, then J is said to satisfy the Cerami condition. A sequence $\{u_n\} \subset E$ satisfying (32) is called a $(C)_c$ sequence of J .

It was shown in [13] that the Cerami condition actually suffices to get a deformation theorem (see [8]) and then, by standard minimax arguments, it allows rather general minimax results.

Lemma 6. Suppose that V satisfies that $V \in L_{loc}^{N/2}(\mathbb{R}^N)$ and $\lim_{|x| \rightarrow \infty} V(x) = 0$. If $u_n \rightharpoonup u$ in X , then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) u_n^2 dx = \int_{\mathbb{R}^N} V(x) u^2 dx. \quad (33)$$

Proof. Because $u_n \rightharpoonup u$ in X , $\{\|u_n\|\}$ is bounded. It follows that

$$M := \sup_n \int_{\mathbb{R}^N} |u_n - u|^2 dx < +\infty. \quad (34)$$

From $\lim_{|x| \rightarrow \infty} V(x) = 0$, we deduce that, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\sup_{|x| \geq R_\epsilon} |V(x)| < \frac{\epsilon}{M}. \quad (35)$$

It follows that

$$\begin{aligned} & \int_{|x| \geq R_\epsilon} |V(x)| \cdot |u_n - u|^2 dx \\ & \leq \sup_{|x| \geq R_\epsilon} |V(x)| \cdot \int_{\mathbb{R}^N} |u_n - u|^2 dx < \epsilon. \end{aligned} \quad (36)$$

For $y \in \mathbb{R}^N$ and $r > 0$, let

$$B_{\mathbb{R}^N}(y, r) = \{x \in \mathbb{R}^N \mid |x - y| < r\}. \quad (37)$$

Because $V \in L^{N/2}(B_{\mathbb{R}^N}(0, R_\epsilon))$ and $u_n \rightharpoonup u$ in X , by Lemma 2.13 of [6], we have

$$\lim_{n \rightarrow \infty} \int_{|x| < R_\epsilon} V(x) u_n^2 dx = \int_{|x| < R_\epsilon} V(x) u^2 dx. \quad (38)$$

Combining (36) and (38) yields (33). $\square$

Lemma 7. Let $\{u_n\} \subset X$ be a $(C)_c$ sequence of Φ . Then, up to a subsequence, $\{\|u_n\|\}$ is bounded.

Proof. By Definition 5, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \Phi(u_n) = c, \\ & \lim_{n \rightarrow \infty} (1 + \|u_n\|) \|\Phi'(u_n)\| = 0. \end{aligned} \quad (39)$$

Arguing indirectly, assume that $\|u_n\| \rightarrow \infty$ and set $w_n = u_n / \|u_n\|$.

Case 1. $\{w_n\}$ is nonvanishing; that is, there exist $y_n \in \mathbb{R}^N$ and $\alpha > 0$ such that

$$\int_{B_{\mathbb{R}^N}(y_n, 1)} w_n^2 dx \geq \alpha, \quad n = 1, 2, \dots \quad (40)$$

Set $\tilde{w}_n = w_n(\cdot + y_n)$. Then, by (40), $\tilde{w}_n \rightharpoonup w \neq 0$ in X .

By (39) and (27), for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have, as $n \rightarrow \infty$,

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx - \int_{\mathbb{R}^N} V(x) u_n \phi dx + \int_{\mathbb{R}^N} f(u_n) \phi dx \\ & = o(1). \end{aligned} \quad (41)$$

It follows that, as $n \rightarrow \infty$,

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla \tilde{w}_n \nabla \phi dx - \int_{\mathbb{R}^N} V(x + y_n) \tilde{w}_n \phi dx \\ & + \int_{\mathbb{R}^N} \frac{f(\tilde{w}_n)}{\tilde{w}_n} \tilde{w}_n \phi dx = o(1), \end{aligned} \quad (42)$$

where $\tilde{w}_n = u_n(\cdot + y_n)$. If $\lim |y_n| = +\infty$, sending $n \rightarrow \infty$ in (42), by (11), $\lim_{|x| \rightarrow \infty} V(x) = 0$ and the fact that, for $x \in \mathbb{R}^N \mid w \neq 0$, $|\tilde{w}_n(x)| \rightarrow +\infty$, we get that

$$\int_{\mathbb{R}^N} \nabla w \nabla \phi dx = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \quad (43)$$

This is impossible, because $0 \neq w \in X$. If $\{y_n\}$ has a subsequence bounded in $\mathbb{R}^N$, up to a subsequence, it can be assumed that $y_0 = \lim_{n \rightarrow \infty} y_n$, and sending $n \rightarrow \infty$ in (42), we get that

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla w \nabla \phi dx - \int_{\mathbb{R}^N} V(x + y_0) w \phi dx = 0, \\ & \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \quad (44)$$

It follows that

$$\begin{aligned} & \int_{\mathbb{R}^N} \nabla w(\cdot - y_0) \nabla \phi dx - \int_{\mathbb{R}^N} V(x) w(\cdot - y_0) \phi dx \\ & = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \end{aligned} \quad (45)$$

This implies that $w(\cdot - y_0)$ is a nonzero eigenfunction of T corresponding to eigenvalue 0. It contradicts Lemma 4.

Case 2. $\{w_n\}$ is vanishing; that is,

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_{\mathbb{R}^N}(y, 1)} w_n^2 dx = 0. \quad (46)$$

The Lions lemma (see, e.g., [6, Lemma 1.21]) shows that, for any $p \in (2, 2N/(N-2))$,

$$\lim_{n \rightarrow \infty} \|w_n\|_{L^p} = 0. \quad (47)$$

Here, $\|\cdot\|_{L^p}$ denotes $L^p(\mathbb{R}^N)$ norm. Equations (39) and (27) show that, as $n \rightarrow \infty$,

$$\begin{aligned} o(1) &= \langle \Phi'(u_n), u_n \rangle \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} V(x) u_n^2 dx \\ &\quad + \int_{\mathbb{R}^N} f(u_n) u_n dx \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + a_0 \int_{\mathbb{R}^N} u_n^2 dx - \int_{\mathbb{R}^N} V(x) u_n^2 dx \\ &\quad - \int_{\mathbb{R}^N} g(u_n) u_n dx \\ &= \|u_n\|^2 - \int_{\mathbb{R}^N} V(x) u_n^2 dx - \int_{\mathbb{R}^N} g(u_n) u_n dx \\ &= \|u_n\|^2 \left(1 - \int_{\mathbb{R}^N} V(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{g(u_n)}{u_n} w_n^2 dx \right), \end{aligned} \quad (48)$$

where g is defined in (15). It follows that

$$\int_{\mathbb{R}^N} \frac{g(u_n)}{u_n} w_n^2 dx + \int_{\mathbb{R}^N} V(x) w_n^2 dx \longrightarrow 1, \quad (49)$$

$$n \longrightarrow \infty.$$

By Lemma 6 and $w_n \rightharpoonup 0$ in X , we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) w_n^2 dx = 0. \quad (50)$$

Together with (49), this implies

$$\int_{\mathbb{R}^N} \frac{g(u_n)}{u_n} w_n^2 dx \longrightarrow 1, \quad n \longrightarrow \infty. \quad (51)$$

By (39),

$$\begin{aligned} o(1) + c &= \Phi(u_n) - \frac{1}{2} \langle \Phi'(u_n), u_n \rangle \\ &= \int_{\mathbb{R}^N} \tilde{F}(u_n) dx. \end{aligned} \quad (52)$$

Let

$$E_n = \left\{ x \in \mathbb{R}^N \mid \left| \frac{g(u_n(x))}{u_n(x)} \right| \geq a_0 - \delta \right\}. \quad (53)$$

By $\tilde{F} \geq 0$, (17), and (52), the following is true:

$$\begin{aligned} \delta \text{mes}(E_n) &\leq \int_{E_n} \tilde{F}(u_n) dx \leq \int_{\mathbb{R}^N} \tilde{F}(u_n) dx \\ &\leq c + o(1). \end{aligned} \quad (54)$$

Here, $\text{mes}(A)$ denotes the Lebesgue measure of A . It follows that

$$\sup_n \text{mes}(E_n) < +\infty. \quad (55)$$

Let

$$F_n = \left\{ x \in \mathbb{R}^N \mid \left| \frac{g(u_n(x))}{u_n(x)} \right| < a_0 - \delta \right\}. \quad (56)$$

Because $\delta \in (0, a_0)$ (see (f_3)), we have

$$\begin{aligned} \int_{F_n} \left| \frac{g(u_n)}{u_n} w_n^2 \right| dx &\leq (a_0 - \delta) \int_{\mathbb{R}^N} w_n^2 dx \\ &\leq (a_0 - \delta) \|w_n\|_{L^2}^2 \leq \frac{a_0 - \delta}{a_0} \|w_n\|^2 \\ &= \frac{a_0 - \delta}{a_0} < 1. \end{aligned} \quad (57)$$

Under the assumption (f_1) , there exists $C > 0$ such that

$$|f(t)| \leq C|t|, \quad \forall t \in \mathbb{R}. \quad (58)$$

It follows that $|g(u_n)| \leq C|u_n|$ for some constant $C > 0$. Then, by (55) and (47), for $2 < p < 2N/(N-2)$,

$$\begin{aligned} \int_{E_n} \left| \frac{g(u_n)}{u_n} w_n^2 \right| dx &\leq C \int_{E_n} w_n^2 dx \\ &\leq C (\text{mes}(E_n))^{(p-2)/p} \|w_n\|_{L^p}^2 \\ &\longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \quad (59)$$

Combining (57) and (59) yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| \frac{g(u_n)}{u_n} w_n^2 \right| dx \\ \leq \limsup_{n \rightarrow \infty} \int_{F_n} \left| \frac{g(u_n)}{u_n} w_n^2 \right| dx \\ + \limsup_{n \rightarrow \infty} \int_{E_n} \left| \frac{g(u_n)}{u_n} w_n^2 \right| dx < 1. \end{aligned} \quad (60)$$

This contradicts (51).

From the two cases given above, it can be deduced that $\{u_n\}$ is bounded in X . $\square$

Lemma 8. *The functional Φ satisfies the $(C)_c$ condition for every $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}$ be a $(C)_c$ sequence of Φ . By Lemma 7, $\{\|u_n\|\}$ is bounded. It follows that there exists $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$ in X . Because the embedding $X \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^N)$ is compact, we get that, up to a subsequence,

$$u_n \longrightarrow u \quad \text{a.e. in } \mathbb{R}^N. \quad (61)$$

As the proof of Lemma 6.15 of [6], we have $\Phi'(u) = 0$. It follows that

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} f(u) u dx - \int_{\mathbb{R}^N} V(x) u^2 dx = 0. \quad (62)$$

We are going to prove that, up to a subsequence, $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$.

By (39), as $n \rightarrow \infty$,

$$\begin{aligned} \langle \Phi'(u_n), u_n \rangle &= \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} f(u_n) u_n dx \\ &\quad - \int_{\mathbb{R}^N} V(x) u_n^2 dx = o(1). \end{aligned} \quad (63)$$

By Lemma 6,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) u_n^2 dx = \int_{\mathbb{R}^N} V(x) u^2 dx. \quad (64)$$

Combining (62)–(64) yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \int_{\mathbb{R}^N} f(u_n) u_n dx \right) \\ = \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} f(u) u dx. \end{aligned} \quad (65)$$

Because $u_n \rightharpoonup u$ in X , by [17, Theorem 1.6], we have

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u|^2 dx. \quad (66)$$

Because $tf(t) \geq 0$ for any $t \in \mathbb{R}$ (see (12)), by (61) and the Fatou lemma, we have

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n) u_n dx \geq \int_{\mathbb{R}^N} f(u) u dx. \quad (67)$$

Using (65)–(67), it can be deduced that, up to a subsequence,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad (68)$$

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n) u_n dx = \int_{\mathbb{R}^N} f(u) u dx. \quad (69)$$

By (13), $\{Du_n f(u_n) - F(u_n)\}$ is a sequence of nonnegative functions. Because

$$Du_n f(u_n) - F(u_n) \longrightarrow Duf(u) - F(u), \quad (70)$$

a.e. in $\mathbb{R}^N$,

by the Fatou lemma and (69), the following is true:

$$\begin{aligned} & \int_{\mathbb{R}^N} (Du(x) f(u(x)) - F(u(x))) dx \\ & \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (Du_n(x) f(u_n(x)) - F(u_n(x))) dx \\ & = D \int_{\mathbb{R}^N} u(x) f(u(x)) dx \\ & \quad - \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(u_n(x)) dx. \end{aligned} \quad (71)$$

It follows that

$$\int_{\mathbb{R}^N} F(u(x)) dx \geq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(u_n(x)) dx. \quad (72)$$

By (12), we deduce that $F \geq 0$ in $\mathbb{R}$. Then, by

$$F(u_n) \longrightarrow F(u), \quad \text{a.e. in } \mathbb{R}^N, \quad (73)$$

and the Fatou lemma, the following holds:

$$\int_{\mathbb{R}^N} F(u(x)) dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(u_n(x)) dx. \quad (74)$$

Combining (72) and (74) leads to

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(u_n(x)) = \int_{\mathbb{R}^N} F(u(x)) dx. \quad (75)$$

By (10) and (11), it can be deduced that there exists $C > 0$ such that

$$|f(t)| \leq C|t|, \quad \forall t \in \mathbb{R}. \quad (76)$$

Then, by the mean value theorem, the Hölder inequality, and (76), we get that, for any $a, b \in \mathbb{R}$ and for any sufficiently small $\epsilon > 0$, there exists $0 < \theta < 1$ such that

$$\begin{aligned} |F(a+b) - F(a)| &= |bf(a+\theta b)| \leq C|b| \cdot |a+\theta b| \\ &\leq C|a| \cdot |b| + Cb^2 \\ &\leq \frac{C\epsilon^2}{2} a^2 + C \left(1 + \frac{1}{2\epsilon^2}\right) b^2. \end{aligned} \quad (77)$$

Because (61), (75), and (77) hold and $\{\|u_n\|\}$ is bounded, Theorem 2 of [18] shows that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(u_n - u) dx = 0. \quad (78)$$

It follows that, for every sufficiently small $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \int_{E_{n,\epsilon}} F(u_n - u) dx = 0. \quad (79)$$

Here,

$$E_{n,\epsilon} = \{x \mid |u_n(x) - u(x)| \leq \epsilon^{-1}\}. \quad (80)$$

Equations (10) and (12) show that

$$F(t) > 0, \quad t \neq 0, \quad (81)$$

$$\liminf_{|t| \rightarrow 0} \frac{F(t)}{t^2} > 0. \quad (82)$$

Combining (81) and (82) yields that, for any sufficiently small $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$t^2 \leq C_\epsilon F(t), \quad \forall t \in [-\epsilon^{-1}, \epsilon^{-1}]. \quad (83)$$

By (79) and (83), we get that, for every sufficiently small $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \int_{E_{n,\epsilon}} |u_n - u|^2 dx = 0. \quad (84)$$

From

$$\begin{aligned} \epsilon^{-2} \text{mes}(\mathbb{R}^N \setminus E_{n,\epsilon}) &\leq \int_{\mathbb{R}^N \setminus E_{n,\epsilon}} |u_n - u|^2 dx \leq M \\ &:= \sup_n \int_{\mathbb{R}^N} |u_n - u|^2 dx < \infty, \end{aligned} \quad (85)$$

we get that

$$\text{mes}(\mathbb{R}^N \setminus E_{n,\epsilon}) \leq M\epsilon^2, \quad n = 1, 2, \dots \quad (86)$$

By (86), the following holds:

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus E_{n,\epsilon}} |u_n - u|^2 dx \\ & \leq (\text{mes}(\mathbb{R}^N \setminus E_{n,\epsilon}))^{2/N} \left(\int_{\mathbb{R}^N} |u_n - u|^{2^*} dx \right)^{2/2^*} \\ & \leq M' (\text{mes}(\mathbb{R}^N \setminus E_{n,\epsilon}))^{2/N} \leq M' M^{2/N} \epsilon^{4/N}, \end{aligned} \quad (87)$$

where $2^* = 2N/(N-2)$ and

$$M' = \sup_n \left(\int_{\mathbb{R}^N} |u_n - u|^{2^*} dx \right)^{2/2^*} < \infty. \quad (88)$$

Combining (84) and (87) leads to

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n - u|^2 dx = 0. \quad (89)$$

Together with (68), this implies $u_n \rightarrow u$ in X . This completes the proof. $\square$

3. Proof of Theorem 1

For the proof of Theorem 1, the following abstract theorem, which is a corollary of [13, Theorem 2.4], is used.

Theorem 9. *Let E be a real Hilbert space with norm $\|\cdot\|$. Suppose that J is a C^1 -functional defined in E and satisfies the following conditions:*

- (A₁) J satisfies the $(C)_c$ condition for every $c > 0$, and $J(0) \geq 0$.
- (A₂) J is an even functional; that is, $J(-u) = J(u)$ for every $u \in E$.
- (A₃) There exist $c_0 > J(0)$ and a closed subspace Z of E with finite codimension such that there exists $\rho > 0$ such that

$$\inf_{S_\rho \cap Z} J \geq c_0. \quad (90)$$

Here, $S_\rho = \{u \in E \mid \|u\| = \rho\}$.

- (A₄) For any $n \in \mathbb{N}$, there exists a subspace $V \subset X$ with dimension larger than or equal to n such that

$$\sup_V J < +\infty. \quad (91)$$

Then, J has infinitely many pairs of critical points.

Proof. Arguing indirectly, assume that J has only a finite number of critical points, say n_0 . The codimension of Z is here denoted by m_0 . By (A₄), there exists a subspace $V \subset Y$ with dimension larger than or equal to $n_0 + m_0 + 1$ such that

$$\sup_V J < +\infty. \quad (92)$$

This indicates that there exists $c_\infty > c_0$ such that

$$\sup_V J < c_\infty. \quad (93)$$

Then, by [13, Theorem 2.4], J has at least $(n_0 + m_0 + 1) - m_0 = n_0 + 1$ critical points. This contradicts the assumption that J has only n_0 critical points. $\square$

Let

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \leq \dots \quad (94)$$

be the eigenvalues of T below the essential spectrum of T . Each λ_n has been repeated in the sequence according to its finite multiplicity. Then, by Lemma 3, the following is true:

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad \lambda_n < 0, \quad \forall n. \quad (95)$$

Let

$$n_* = \min \{n \in \mathbb{N} \mid a_0 + \lambda_n > 0\}. \quad (96)$$

Here, a_0 is the constant in (10). From this definition,

$$a_0 + \lambda_{n_*} > 0 \geq a_0 + \lambda_{n_*-1}, \quad \text{if } n_* > 1. \quad (97)$$

Let $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ denote the spectral family of the operator T defined by (3).

Let

$$Z = \begin{cases} X, & \text{if } n_* = 1 \\ (\text{id} - E_\nu)(L^2(\mathbb{R}^N)) \cap X, & \text{if } n_* > 1. \end{cases} \quad (98)$$

Here, $\text{id} : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is the identity map and

$$\nu = \frac{1}{2} (\lambda_{n_*} + \lambda_{n_*-1}). \quad (99)$$

It is easy to see that the codimension of Z equals $n_* - 1$.

Lemma 10. *There exists $\varsigma > 0$ such that*

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 dx + a_0 \int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} V(x) u^2 dx \\ & \geq \varsigma \|u\|^2, \quad \forall u \in Z. \end{aligned} \quad (100)$$

Proof. From (96) and (98), the following holds:

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u|^2 dx + a_0 \int_{\mathbb{R}^N} u^2 dx - \int_{\mathbb{R}^N} V(x) u^2 dx \\ & \geq \kappa \int_{\mathbb{R}^N} u^2 dx, \quad \forall u \in Z. \end{aligned} \quad (101)$$

Here,

$$\kappa = a_0 + \lambda_{n_*} > 0. \quad (102)$$

It follows that

$$\begin{aligned} \varsigma &:= \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx + a_0 \int_{\mathbb{R}^N} u^2 dx \right. \\ & \quad \left. - \int_{\mathbb{R}^N} V(x) u^2 dx \mid u \in Z, \|u\| = 1 \right\} \geq 0. \end{aligned} \quad (103)$$

If $\varsigma = 0$, then there exists $\{u_n\} \subset Z$ satisfying $\|u_n\| = 1$, $n = 1, 2, \dots$, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + a_0 \int_{\mathbb{R}^N} u_n^2 dx \right. \\ & \quad \left. - \int_{\mathbb{R}^N} V(x) u_n^2 dx \right) = 0. \end{aligned} \quad (104)$$

Up to a subsequence, it can be assumed that $u_n \rightharpoonup u_0$ in X and $u_n \rightarrow u_0$ a.e. in $\mathbb{R}^N$. Since Z is a closed subspace of X , by $u_n \rightharpoonup u_0$ in X , we have $u_0 \in Z$. Because $u_n \rightharpoonup u_0$ in X , by [17, Theorem 1.6], we have

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u_0|^2 dx. \quad (105)$$

Because $u_n \rightarrow u_0$ a.e. in $\mathbb{R}^N$, by the Fatou lemma, the following holds:

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} u_n^2 dx \geq \int_{\mathbb{R}^N} u_0^2 dx. \quad (106)$$

Moreover, by Lemma 6,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) u_n^2 dx = \int_{\mathbb{R}^N} V(x) u_0^2 dx. \quad (107)$$

Combining (104)–(107) yields

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + a_0 \int_{\mathbb{R}^N} u_0^2 dx - \int_{\mathbb{R}^N} V(x) u_0^2 dx \leq 0. \quad (108)$$

Because $u_0 \in Z$, by (101) and (108), we have $u_0 = 0$. Then, by (104) and (107),

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx + a_0 \int_{\mathbb{R}^N} u_n^2 dx \right) = 0. \quad (109)$$

This contradicts $\|u_n\| = 1$, $n = 1, 2, \dots$. Therefore, $\varsigma > 0$. $\square$

Proof of Theorem 1. We are going to prove that the functional Φ satisfies the conditions (A_1) – (A_4) in Theorem 9.

By Lemma 8, the functional Φ satisfies the condition (A_1) . By (f_4) , Φ satisfies the condition (A_2) .

By (10) and (11), it can be deduced that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\left| F(t) - \frac{a_0 t^2}{2} \right| \leq \epsilon t^2 + C_\epsilon |t|^\mu, \quad \forall t \in \mathbb{R}. \quad (110)$$

Here,

$$\mu = \frac{2N-2}{N-2} \in (2, 2^*). \quad (111)$$

Let Z be the space defined by (98). It has a finite codimension. By Lemma 10 and (110), we have, for any $u \in Z$,

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{a_0}{2} \int_{\mathbb{R}^N} u^2 dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx \\ &\quad + \int_{\mathbb{R}^N} \left(F(u) - \frac{a_0}{2} u^2 \right) dx \\ &\geq \frac{\varsigma}{2} \|u\|^2 - \epsilon \|u\|_{L^2}^2 - C_\epsilon \|u\|_{L^\mu}^\mu \\ &\geq \left(\frac{\varsigma}{2} - \epsilon \right) \|u\|^2 - C_\epsilon \|u\|_{L^\mu}^\mu \\ &\geq \left(\frac{\varsigma}{2} - \epsilon \right) \|u\|^2 - C' C_\epsilon \|u\|^\mu, \end{aligned} \quad (112)$$

where the positive constant C' comes from the Sobolev inequality $\|u\|_{L^\mu} \leq (C')^{1/\mu} \|u\|$, $\forall u \in X$. Choosing $\epsilon = \varsigma/4$ and $\rho = (\varsigma/8C'C_\epsilon)^{1/(\mu-2)}$, by (112), the following holds:

$$\inf_{S_\rho \cap Z} \Phi \geq c_0 := \frac{\varsigma}{8} \rho^2 > \Phi(0). \quad (113)$$

It follows that Φ satisfies the assumption (A_3) .

Let $\vartheta \in C_0^\infty(\mathbb{R}^N)$ such that $\|\vartheta\|_{L^2} = 1$ and $\text{supp } \vartheta \subset \{x \in \mathbb{R}^N \mid 1 < |x| < 2\}$. Then, for the function $\vartheta_\lambda(x) = \lambda^{-N/2} \vartheta(\lambda^{-1}x)$, $\|\vartheta_\lambda\|_{L^2} = 1$ and $\text{supp } \vartheta_\lambda \subset \{x \in \mathbb{R}^N \mid \lambda < |x| < 2\lambda\}$. Then, by (8), it follows that, for $\lambda > R$,

$$\begin{aligned} & - \int_{\mathbb{R}^N} V(x) |\vartheta_\lambda|^2 dx + \int_{\mathbb{R}^N} |\nabla \vartheta_\lambda|^2 dx \\ & \leq -C\lambda^{-\beta} \|\vartheta\|_{L^2}^2 + C\lambda^{-2} \|\nabla \vartheta\|_{L^2}^2, \end{aligned} \quad (114)$$

where C is a positive constant independent of λ . Because $\beta < 2$, it follows that there exists $\lambda_0 > R$ such that

$$\int_{\mathbb{R}^N} |\nabla \vartheta_\lambda|^2 dx - \int_{\mathbb{R}^N} V(x) |\vartheta_\lambda|^2 dx < 0 \quad \text{for } \lambda \geq \lambda_0. \quad (115)$$

Let $\psi_n = \vartheta_{2^n \lambda_0}$, $n \in \mathbb{N}$. These functions have mutually disjoint compact supports. Let

$$V_n = \text{span} \{\psi_1, \psi_2, \dots, \psi_n\}. \quad (116)$$

Then, V_n is an n -dimensional subspace of X . By (10) and (11), it can be deduced that, for any $\epsilon > 0$, there exist $L_\epsilon > 0$ and $K_\epsilon > 0$ such that

$$\begin{aligned} |F(t)| &\leq \epsilon |t|^2, \quad \text{if } |t| > L_\epsilon, \\ |F(t)| &\leq K_\epsilon |t|^{3/2}, \quad \text{if } |t| \leq L_\epsilon. \end{aligned} \quad (117)$$

Because $\{\psi_i \mid i \in \mathbb{N}\}$ have mutually disjoint compact supports, by (115), we have, for $u = \sum_{i=1}^n t_i \psi_i \in V_n$,

$$\Phi(u) = \frac{1}{2} \sum_{i=1}^n \alpha_i t_i^2 + \int_{\mathbb{R}^N} F(u) dx, \quad (118)$$

where

$$\begin{aligned} \alpha_i &= \int_{\mathbb{R}^N} |\nabla \psi_i|^2 dx - \int_{\mathbb{R}^N} V(x) |\psi_i|^2 dx < 0, \\ & i = 1, 2, \dots, n. \end{aligned} \quad (119)$$

Let

$$\Omega_n = \bigcup_{i=1}^n \text{supp } \psi_i. \quad (120)$$

Then, Ω_n is a bounded set in $\mathbb{R}^N$, since every ψ_i has a compact support. By (117),

$$\begin{aligned}
 \int_{\mathbb{R}^N} F(u) dx &= \int_{\{x \mid |u(x)| > L_\epsilon\}} F(u) dx \\
 &\quad + \int_{\{x \mid |u(x)| \leq L_\epsilon\}} F(u) dx \\
 &\leq \epsilon \int_{\mathbb{R}^N} u^2 dx + K_\epsilon \int_{\Omega_n} |u|^{3/2} dx \\
 &\leq \epsilon \sum_{i=1}^n \omega_i t_i^2 \\
 &\quad + K_\epsilon (\text{mes}(\Omega_n))^{1/4} \left(\int_{\Omega_n} u^2 dx \right)^{3/4} \\
 &= \epsilon \sum_{i=1}^n \omega_i t_i^2 \\
 &\quad + K_\epsilon (\text{mes}(\Omega_n))^{1/4} \left(\sum_{i=1}^n \omega_i t_i^2 \right)^{3/4},
 \end{aligned} \tag{121}$$

where

$$\omega_i = \int_{\mathbb{R}^N} \psi_i^2 dx > 0, \quad i = 1, \dots, n. \tag{122}$$

Choosing

$$\epsilon = \frac{\min \{-\alpha_i \mid 1 \leq i \leq n\}}{4 \max \{\omega_i \mid 1 \leq i \leq n\}}, \tag{123}$$

by (118) and (121), we get that

$$\Phi(u) \leq \frac{1}{4} \sum_{i=1}^n \alpha_i t_i^2 + K_\epsilon (\text{mes}(\Omega_n))^{1/4} \left(\sum_{i=1}^n \omega_i t_i^2 \right)^{3/4}. \tag{124}$$

Since $\alpha_i < 0$, for every i , (124) implies

$$\sup_{V_n} \Phi < \infty. \tag{125}$$

Therefore, Φ satisfies the condition (A_4) of Theorem 9.

Because Φ satisfies the conditions (A_1) – (A_4) of Theorem 9, by Theorem 9, it has infinitely many critical points. It follows that (6) has infinitely many solutions. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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