Research Article

Tight $K$-$g$-Frame and Its Novel Characterizations via Atomic Systems

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Abstract

Frame theory has been widely used in filter theory [3], image processing [4], numerical analysis, and other areas. We refer to [5–8] for an introduction to frame theory in Hilbert space and its application. With the deepening of research on frame theory, various generalizations of frames have been proposed; see [9–12]. Atomic systems for subspaces were first introduced by Feichtinger and Werther in [13] based on examples arising in sampling theory. In 2011, Gavrut¸a [14] introduced atomic systems for subspaces were first introduced by Duffin and Schaeffer [1] to deal with nonharmonic Fourier series and reintroduced in 1986 by Daubechies et al. [2]. Since then the frame theory began to be more wildly studied. Today, frame theory has been widely used in filter theory [3], image processing [4], numerical analysis, and other areas. We refer to [5–8] for an introduction to frame theory in Hilbert space and its application.

With the deepening of research on frame theory, various generalizations of frames have been proposed; see [9–12]. Atomic systems for subspaces were first introduced by Feichtinger and Werther in [13] based on examples arising in sampling theory. In 2011, Găvruţa [14] introduced $K$-frame in Hilbert spaces to study atomic decomposition systems and discussed some properties of them. In [15–18], some conclusions of $K$-frame were given. With the extensive research of $K$-frame and $g$-frame in Hilbert space, Zhu et al. [19, 20] began to study $K$-$g$-frame, which was limited to the range of a bounded linear operator in Hilbert space and had gained greater flexibility in practical application relative to $g$-frame. $K$-$g$-frame, as a more general frame than $g$-frame and $K$-frame, has become one of the most active fields in frame theory in recent years. In [20, 21], several properties and characterizations of $K$-$g$-frame were obtained. However, many problems of $K$-$g$-frame have not been studied. Based on these important results of $K$-$g$-frame, we extend tight $g$-frame to $K$-$g$-frame and put forward the concept of tight $K$-$g$-frame. In this paper, we give equivalent characterizations and necessary conditions of tight $K$-$g$-frame are given. Finally, by means of methods and techniques of frame theory, several properties of tight $K$-$g$-frame are given.

1. Introduction

Frame in Hilbert space was first introduced in 1952 by Duffin and Schaeffer [1] to deal with nonharmonic Fourier series and reintroduced in 1986 by Daubechies et al. [2]. Since then the frame theory began to be more wildly studied. Today, frame theory has been widely used in filter theory [3], image processing [4], numerical analysis, and other areas. We refer to [5–8] for an introduction to frame theory in Hilbert space and its application. Throughout this paper, $H$ is separable Hilbert space and $I$ is the identity operator. $L(H_1, H_2)$ is a collection of all bounded linear operators from $H_1$ to $H_2$, where $H_1$ and $H_2$ are two Hilbert spaces. In particular, $L(H)$ is a collection of all bounded linear operators from $H$ to $H$. For any $T \in L(H_1, H_2)$, $R(T)$ is the range of $T$ and $T^*$ is the adjoint operator of $T$. $\{H_j : j \in J\}$ is a sequence of closed subspaces of $H$, where $J$ is a subset of integers $\mathbb{Z}$. $F^2(\{H_j\}_{j \in J})$ is defined by

$$i^2(\{H_j\}_{j \in J}) = \left\{ \{a_j\}_{j \in J} : a_j \in H_j, \sum_{j \in J} \|a_j\|^2 < +\infty \right\}, \quad (1)$$

and characterizations of $K$-$g$-frame were obtained. However, many problems of $K$-$g$-frame have not been studied. Based on these important results of $K$-$g$-frame, we extend tight $g$-frame to $K$-$g$-frame and put forward the concept of tight $K$-$g$-frame. In this paper, we give equivalent characterizations and necessary conditions of tight $K$-$g$-frame for Hilbert space. We also obtain the necessary and sufficient condition of tight $K$-$g$-frame to be tight $g$-frame. Finally, we present several properties of tight $K$-$g$-frame for Hilbert space.
with the inner product given by
\[
\langle \{a_j\}_{j \in J}, \{b_j\}_{j \in J} \rangle = \sum_{j \in J} \langle a_j, b_j \rangle_{H_j}.
\] (2)

It is clear that \(\tilde{P}(\{H_j\}_{j \in J})\) is a complex Hilbert space.

2. Preliminaries

In this section, some necessary definitions and lemmas are introduced.

**Definition 1** (see [9, Definition 1.1]). A sequence \(\{\Lambda_j\}_{j \in J} \subset L(H, H_j) : j \in J\) is called a g-frame for \(H\) with respect to \(\{H_j\}_{j \in J}\) if there exist two positive constants \(A\) and \(B\) such that, for all \(f \in H\),
\[
A \|f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B \|f\|^2.
\] (3)

The constants \(A\) and \(B\) are called the lower and upper bounds of \(g\)-frame, respectively. If the right inequality is satisfied, then \(\{\Lambda_j\}_{j \in J}\) is said to be a \(g\)-Bessel sequence for \(H\) with respect to \(\{H_j\}_{j \in J}\).

If \(A = B = 1\), we call this \(g\)-frame a tight \(g\)-frame, and if \(A = B = 1\), it is called a Parseval \(g\)-frame.

For a \(g\)-Bessel sequence \(\{\Lambda_j\}_{j \in J}\), \(T : \tilde{P}(\{H_j\}_{j \in J}) \to H\) defines a bounded linear operator, that is,
\[
T(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j, \quad \forall \{g_j\}_{j \in J} \in \tilde{P}(\{H_j\}_{j \in J}).
\] (4)

The adjoint operator \(T^* : H \to \tilde{P}(\{H_j\}_{j \in J})\) is given by
\[
T^* f = \{\Lambda_j f\}_{j \in J}, \quad \forall f \in H.
\] (5)

By composing \(T\) with its adjoint \(T^*\), we obtain the bounded linear operator
\[
S : H \to H, \quad S f = TT^* f = T \{\Lambda_j f\}_{j \in J} = \sum_{j \in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in H.
\] (6)

We call \(T, T^*\), and \(S\) the preframe operator, analysis operator, and frame operator of \(g\)-Bessel sequence, respectively.

**Definition 2** (see [22, Definition 2.6]). We say \(\{\Lambda_j\}_{j \in J} \subset L(H, H_j)_{j \in J}\) is \(g\)-orthonormal basis for \(H\) with respect to \(\{H_j\}_{j \in J}\), if it is \(g\)-biorthonormal with itself, that is, \(\langle \Lambda_j^* g_j, \Lambda_i^* g_i \rangle = \delta_{ji} \langle g_j, g_i \rangle, \forall j, i \in J, g_j \in H_j, g_i \in H_i\), and for any \(f \in H\) one has \(\sum_{j \in J} \|\Lambda_j f\|^2 = \|f\|^2\).

**Definition 3** (see [21, Theorem 2.5]). Let \(K \in L(H)\) and \(\Lambda_j \in L(H, H_j)\) for any \(j \in J\). A sequence \(\{\Lambda_j\}_{j \in J}\) is called a \(K\)-g-frame for \(H\) with respect to \(\{H_j\}_{j \in J}\) if there exist constants \(A, B > 0\) such that
\[
A \|K^* f\|^2 \leq \sum_{j \in J} \|\Lambda_j f\|^2 \leq B \|f\|^2, \quad \forall f \in H.
\] (7)

The constants \(A\) and \(B\) are called the lower and upper bounds of \(K\)-g-frame, respectively.

**Remark 4.** Every \(K\)-g-frame is a \(g\)-Bessel sequence for \(H\) with respect to \(\{H_j\}_{j \in J}\). If \(K = I\), then \(K\)-g-frame is just the ordinary \(g\)-frame.

Motivated by the definition of tight \(g\)-frame, we give the following definition of tight \(K\)-g-frame.

**Definition 5.** Let \(K \in L(H)\) and \(\Lambda_j \in L(H, H_j)\) for any \(j \in J\). A sequence \(\{\Lambda_j\}_{j \in J}\) is called a tight \(K\)-g-frame for \(H\) with respect to \(\{H_j\}_{j \in J}\) if there exists constant \(A > 0\) such that
\[
A \|K^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \forall f \in H.
\] (8)

The constant \(A\) is called the bound of tight \(K\)-g-frame. If \(A = 1\), we call this tight \(K\)-g-frame a Parseval \(K\)-g-frame for \(H\) with respect to \(\{H_j\}_{j \in J}\).

**Remark 6.** If \(K = I\), then tight \(K\)-g-frame and Parseval \(K\)-frame are tight \(g\)-frame and Parseval \(g\)-frame, respectively.

**Definition 7.** Let \(K \in L(H)\). An operator \(K\) is said to be left-invertible if there exists an operator \(P \in L(H)\) such that \(PK = I\). The operator \(P\) is called a left-inverse of \(K\); that is, \(K_P = P\). Similarly, an operator \(K\) is said to be right-invertible if there exists an operator \(P \in L(H)\) such that \(KP = I\). The operator \(P\) is called a right-inverse of \(K\); that is, \(K_P = P\).

**Lemma 8** (see [23, Theorem 1]). Let \(T_1 \in L(H_1, H)\) and \(T_2 \in L(H, H_2, H)\). The following conditions are equivalent:

1. \(R(T_1) \subset R(T_2)\).
2. There exists \(\lambda > 0\) such that \(T_1^* T_2 \lambda \leq \lambda T_2^* T_1\).
3. There exists a bounded operator \(X \in L(H_1, H_2)\) so that \(T_1 = T_2 X\).

**Lemma 9** (see [21, Theorem 2.5]). Let \(K \in L(H)\). Then the following statements are equivalent:

1. \(\{\Lambda_j\}_{j \in J}\) is a \(K\)-g-frame for \(H\) with respect to \(\{H_j\}_{j \in J}\).
2. \(\{\Lambda_j\}_{j \in J}\) is a \(g\)-Bessel sequence for \(H\) with respect to \(\{H_j\}_{j \in J}\) and there exists a \(g\)-Bessel sequence for \(H\) with respect to \(\{H_j\}_{j \in J}\) such that
\[
Kf = \sum_{j \in J} \Lambda_j^* T_j f, \quad \forall f \in H.
\] (9)

3. Properties of Tight \(K\)-g-Frame for Hilbert Space

In this section, we first give characterizations of tight \(K\)-g-frame and then give several properties of tight \(K\)-g-frame.

**Theorem 10.** Let \(K \in L(H)\); \(\Lambda_j \in L(H, H_j)\), and let \(T^*\) be the preframe operator of \(\{\Lambda_j\}_{j \in J}\). Then the following statements are equivalent:

1. \(\{\Lambda_j\}_{j \in J}\) is a tight \(K\)-g-frame for \(H\) with respect to \(\{H_j\}_{j \in J}\) with bound \(A\).
(2) There exists constant \( A > 0 \) such that \( A\|K^* f\|^2 = \|T^* f\|^2 \) for any \( f \in H \).

(3) There exists constant \( A > 0 \) such that \( AK^* = TT^* \).

Proof. (1) \( \Rightarrow \) (2). Suppose that \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A \). By the definition of tight \( K^* \)-frame, we get

\[
A\|K^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \forall f \in H. \tag{10}
\]

Since \( T \) is the preframe operator of \( \{\Lambda_j\}_{j \in J} \), we have

\[
\sum_{j \in J} \|\Lambda_j f\|^2 = \left\langle \sum_{j \in J} \Lambda_j^* f, f \right\rangle = \langle TT^* f, f \rangle = \|T^* f\|^2, \quad \forall f \in H. \tag{11}
\]

This implies that \( A\|K^* f\|^2 = \|T^* f\|^2 \) for any \( f \in H \).

(2) \( \Rightarrow \) (3). If there exists constant \( A > 0 \) such that \( A\|K^* f\|^2 = \|T^* f\|^2 \) for any \( f \in H \), then we obtain

\[
\left\langle AKK^* f, f \right\rangle = \left\langle K^* f, f \right\rangle = A\|K^* f\|^2 = \|T^* f\|^2 = \left\langle TT^* f, f \right\rangle, \quad \forall f \in H. \tag{12}
\]

Hence \( AKK^* = TT^* \).

(3) \( \Rightarrow \) (1). If there exists constant \( A > 0 \) such that \( AKK^* = TT^* \), then \( \langle AKK^* f, f \rangle = \langle TT^* f, f \rangle, \forall f \in H \). That is,

\[
A\|K^* f\|^2 = \|T^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \forall f \in H. \tag{13}
\]

Therefore, \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A \). The proof of Theorem 10 is completed.

\[\square\]

Corollary 11. Suppose that \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A \), \( T \) is the preframe operator of \( \{\Lambda_j\}_{j \in J} \), and \( S \) is the frame operator of \( \{\Lambda_j\}_{j \in J} \), then

(1) \( R(K) = R(T) \);

(2) \( S = AKK^* \);

(3) \( \|T\| = \sqrt{A}\|K\| \).

Proof. Theorem 10 together with Lemma 8 shows that (1) and (2) are satisfied. We only need to prove that (3) holds. Assume that \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A \). By Theorem 10, we have \( A\|K^* f\|^2 = \|T^* f\|^2 \), \( \forall f \in H \). Therefore,

\[
\|T\| = \|T^*\| = \sup_{\|f\|=1, f \in H} \|T^* f\| = \sup_{\|f\|=1, f \in H} \sqrt{A}\|K^* f\| \tag{14}
\]

\[
= \sqrt{A}\|K\|.
\]

The proof of Corollary 11 is completed.

\[\square\]

Lemma 9 gives an equivalent characterization of \( K^* \)-frame; does the tight \( K^* \)-frame have the similar characterization? Clearly, if \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \), then there exists a \( g \)-Bessel sequence \( \{\Gamma_j\}_{j \in J} \) for \( H \) with respect to \( \{H_j\}_{j \in J} \) such that \( Kf = \sum_{j \in J} \Lambda_j^* \Gamma_j f \), \( \forall f \in H \). The theorem below gives a necessary condition of tight \( K^* \)-frame.

Theorem 12. Let \( K \in L(H) \). Suppose that \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A \). Then there exists a \( g \)-Bessel sequence \( \{\Gamma_j\}_{j \in J} \) for \( H \) with respect to \( \{H_j\}_{j \in J} \) such that \( Kf = \sum_{j \in J} \Lambda_j^* \Gamma_j f \), \( \forall f \in H \), and \( AB \geq 1 \).

Proof. Let \( \{\Lambda_j\}_{j \in J} \) be a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \). By Lemma 9, there exists a \( g \)-Bessel sequence \( \{\Gamma_j\}_{j \in J} \) for \( H \) with respect to \( \{H_j\}_{j \in J} \) such that \( Kf = \sum_{j \in J} \Lambda_j^* \Gamma_j f \), \( \forall f \in H \). For any \( f, g \in H \), we have

\[
\left\langle K^* f, g \right\rangle = \langle f, K g \rangle = \left\langle f, \sum_{j \in J} \Lambda_j^* \Gamma_j g \right\rangle = \left\langle \sum_{j \in J} \Gamma_j^* \Lambda_j f, g \right\rangle. \tag{15}
\]

Via (15),

\[
K^* f = \sum_{j \in J} \Gamma_j^* \Lambda_j f, \quad \forall f \in H. \tag{16}
\]

Since \( \{\Lambda_j\}_{j \in J} \) is a tight \( K^* \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A \), we get

\[
A\|K^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \forall f \in H. \tag{17}
\]

Furthermore,

\[
\sum_{j \in J} \|\Lambda_j f\|^2 = A\|K^* f\|^2 = A\left\| \sum_{j \in J} \Gamma_j^* \Lambda_j f \right\|^2 \\
= A \sup_{\|g\|=1, g \in H} \left\| \sum_{j \in J} \left( \frac{\langle \Lambda_j f, \Gamma_j g \rangle}{\|\Gamma_j\|^2} \right) g \right\|^2 \\
= A \sup_{\|g\|=1, g \in H} \left\| \sum_{j \in J} \left( \frac{\langle \Lambda_j f, \Gamma_j g \rangle}{\|\Gamma_j\|^2} \right) g \right\|^2 \\
\leq A \sup_{\|g\|=1, g \in H} \left\| \sum_{j \in J} \|\Lambda_j f\|^2 \right\| \sum_{j \in J} \|\Gamma_j g\|^2 \\
\leq A \sup_{\|g\|=1, g \in H} \left\| \sum_{j \in J} \|\Lambda_j f\|^2 \right\| B \|g\|^2 \\
= AB \sum_{j \in J} \|\Lambda_j f\|^2.
\]

Therefore, \( AB \geq 1 \). The proof of Theorem 12 is completed.

\[\square\]
Note that when \( K = I \), tight \( K-g \)-frame is tight \( g \)-frame. One may wonder whether \( K = I \) when tight \( K-g \)-frame is tight \( g \)-frame as well. In fact, the answer is negative. The following example demonstrates this.

**Example 13.** Suppose that \( H = \mathbb{R}^2; J = \{1, 2, 3\} \). Let \( \{e_j\}_{j \in J} \) be an orthonormal basis of \( H \), and let \( H_j = \text{span}(e_j) \). Now define \( \{\Lambda_j\}_{j \in J} \) and \( \{\Gamma_j\}_{j \in J} \) as follows:

\[
\Lambda_j : H \rightarrow H,
\Lambda_j f = (f, e_1) e_1,
\Lambda_2 f = (f, e_2) e_2,
\Lambda_3 f = (f, e_3) e_3;
\Gamma_j : H \rightarrow H_j,
\Gamma_1 f = \frac{1}{\sqrt{2}} ((f, e_1) + (f, e_2)) e_1,
\Gamma_2 f = \frac{1}{\sqrt{2}} ((f, e_1) - (f, e_2)) e_2,
\Gamma_3 f = (f, e_3) e_3.
\]

By a simple calculation, we have

\[
\Lambda_1^*: H_1 \rightarrow H,
\Lambda_1^* (a_1 e_1) = a_1 e_1,
\Lambda_2^*: H_2 \rightarrow H,
\Lambda_2^* (a_2 e_2) = a_2 e_2,
\Lambda_3^*: H_3 \rightarrow H,
\Lambda_3^* (a_3 e_3) = a_3 e_3;
\Gamma_1^*: H_1 \rightarrow H,
\Gamma_1^* (a_1 e_1) = \frac{1}{\sqrt{2}} a_1 e_1 + \frac{1}{\sqrt{2}} a_2 e_2,
\Gamma_2^*: H_2 \rightarrow H,
\Gamma_2^* (a_2 e_2) = \frac{1}{\sqrt{2}} a_2 e_1 - \frac{1}{\sqrt{2}} a_2 e_2,
\Gamma_3^*: H_3 \rightarrow H,
\Gamma_3^* (a_3 e_3) = a_3 e_3.
\]

For any \( f = (c_1, c_2, c_3) \in H \), we have

\[
\sum_{j \in J} \|\Lambda_j f\|^2 = c_1^2 + c_2^2 + c_3^2 = \|f\|^2;
\sum_{j \in J} \|\Gamma_j f\|^2 = \frac{1}{2} (c_1 + c_2)^2 + \frac{1}{2} (c_1 - c_2)^2 + c_3^2 = \|f\|^2.
\]

Obviously, \( \{\Lambda_j\}_{j \in J} \) and \( \{\Gamma_j\}_{j \in J} \) are Parseval \( g \)-frames for \( H \) with respect to \( \{H_j\}_{j \in J} \).

Define the bounded linear operator \( K \) as follows:

\[
K : H \rightarrow H,
Kf = \sum_{j \in J} \Lambda_j^* \Gamma_j f, \quad \forall f \in H.
\]

Now we prove that \( \{\Lambda_j\}_{j \in J} \) is a tight \( K-g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \). For any \( f, g \in H \), we have

\[
\langle K^* f, g \rangle = \langle f, Kg \rangle = \left\langle f, \sum_{j \in J} \Lambda_j^* \Gamma_j g \right\rangle = \left\langle \sum_{j \in J} \Gamma_j^* \Lambda_j f, g \right\rangle.
\]

Hence, \( K^* f = \sum_{j \in J} \Gamma_j^* \Lambda_j f \). It follows that

\[
\|K^* f\|^2 = \left\| \sum_{j \in J} \Gamma_j^* \Lambda_j f \right\|^2
= \left\| \Gamma_1^* c_1 e_1 + \Gamma_2^* c_2 e_2 + \Gamma_3^* c_3 e_3 \right\|^2
= \frac{1}{\sqrt{2}} \left[ \left| c_1 + c_2 \right|^2 + \left| c_1 - c_2 \right|^2 + c_3^2 \right]
= \|K^* f\|^2, \quad \forall f \in H.
\]

Therefore, for any \( f \in H \), we have \( \sum_{j \in J} \|\Lambda_j f\|^2 = \|f\|^2 = \|K^* f\|^2 \). Via the definition of tight \( K-g \)-frame, \( \{\Lambda_j\}_{j \in J} \) is a tight \( K-g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \). For any \( f = (c_1, c_2, c_3) \in H \), we have \( Kf = \sum_{j \in J} \Lambda_j^* \Gamma_j f = ((1/\sqrt{2})(c_1 + c_2), (1/\sqrt{2})(c_1 - c_2), c_3) \neq f \).

Example 13 shows that if a tight \( K-g \)-frame is tight \( K \)-frame, then \( K \) cannot be \( I \). In the following theorem, we state a necessary and sufficient condition for a tight \( K-g \)-frame being a tight \( g \)-frame.

**Theorem 14.** Let \( K \in L(H) \) and \( A_1, A_2 > 0 \). Suppose that \( \{\Lambda_j\}_{j \in J} \) is a tight \( K-g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A_1 \). Then \( \{\Lambda_j\}_{j \in J} \) is a tight \( g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A_2 \) if and only if \( K \) is right-invertible and the right-invertible operator is \( K^{-1} = (A_1/A_2)K^* \).

**Proof.** First, we prove the sufficient condition. Since \( \{\Lambda_j\}_{j \in J} \) is a tight \( K-g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A_1 \), we have

\[
A_1 \|K^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \forall f \in H.
\]

Assume that \( \{\Lambda_j\}_{j \in J} \) is a tight \( g \)-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( A_2 \). Then, for any \( f \in H \), we get

\[
\sum_{j \in J} \|\Lambda_j f\|^2 = A_2 \|f\|^2.
\]
By (25) and (26), $A_1\|K^*f\|^2 = A_2\|f\|^2$ for any $f \in H$, implying that $\|K^*f\|^2 = (A_2/A_1)\|f\|^2$. Then, for any $f \in H$, we have $(KK^*f, f) = (A_2/A_1, f, f)$. This implies that $K((A_1/A_2)K^*) = I$. So $K$ is right-invertible and the right-invertible operator is $K_r^{-1} = (A_1/A_2)K^*$.

Next, we prove the necessary condition. Suppose that $K$ is right-invertible and the right-invertible operator is $K_r^{-1} = (A_1/A_2)K^*$. Then $KK_r^{-1} = K((A_1/A_2)K^*) = I$; that is, $KK^* = (A_2/A_1)I$. So

$$\langle KK^*f, f \rangle = \left( \frac{A_2}{A_1} f, f \right), \quad \forall f \in H. \quad (27)$$

That is,

$$\|K^*f\|^2 = \frac{A_2}{A_1} \|f\|^2. \quad (28)$$

Since $\{\Lambda_j\}_{j \in J}$ is a tight $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A_1$, we have

$$\sum_{j \in J} \|\Lambda_j f\|^2 = A_1 \|K^*f\|^2 = A_2 \|f\|^2. \quad (29)$$

This implies that $\{\Lambda_j\}_{j \in J}$ is a tight $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A_2$. The proof of Theorem 14 is completed.

In the following, we will verify whether the $K_r^{-1}$ in Example 13 is equal to $K^*$. For any $f = (c_1, c_2, c_3) \in H$, we have

$$KK^* f = \sum_{j \in J} \Gamma_j K^* f = \sum_{j \in J} \Gamma_j \left( \sum_{j \in J} \Gamma_j^* \Lambda_j f \right)$$

$$= \sum_{j \in J} \Gamma_j \left( \frac{1}{\sqrt{2}} (c_1 + c_2), \frac{1}{\sqrt{2}} (c_1 - c_2), c_3 \right)$$

$$= \Gamma_1^* \left( \frac{1}{2} (c_1 + c_2) + \frac{1}{2} (c_1 - c_2) \right) e_1$$

$$+ \Gamma_2^* \left( \frac{1}{2} (c_1 + c_2) - \frac{1}{2} (c_1 - c_2) \right) e_2$$

$$+ \Gamma_3^* (c_3 e_3)$$

$$= \Gamma_1^* (c_1 e_1) + \Gamma_2^* (c_2 e_2) + \Gamma_3^* (c_3 e_3) = f. \quad (30)$$

It follows that $KK^* = I$. This implies that $K$ is right-invertible and the right-invertible operator is $K_r^{-1} = K^*$.

**Theorem 16.** Let $K \in L(H)$. If $\{\Lambda_j\}_{j \in J}$ is a tight $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$, then $\{\Lambda_j K^*\}_{j \in J}$ is a tight $K$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$.

**Proof.** Since $\{\Lambda_j\}_{j \in J}$ is a tight $g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$, we have

$$\sum_{j \in J} \|\Lambda_j f\|^2 = A \|f\|^2, \quad \forall f \in H. \quad (31)$$

Again, for any $f \in H$, we have $K^* f \in H$; then

$$\sum_{j \in J} \|\Lambda_j K^* f\|^2 = A \|K^* f\|^2, \quad \forall f \in H. \quad (32)$$

Therefore, $\{\Lambda_j K^*\}_{j \in J}$ is a tight $K$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$. The proof of Theorem 16 is completed.

**Corollary 17.** Let $K \in L(H)$. If $\{\Lambda_j\}_{j \in J}$ is a $g$-orthonormal basis for $H$ with respect to $\{H_j\}_{j \in J}$, then $\{\Lambda_j K^*\}_{j \in J}$ is a tight $K$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$.

**Theorem 18.** Let $T, K \in L(H)$. If $\{\Lambda_j\}_{j \in J}$ is a tight $K$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$, then $\{\Lambda_j T^*\}_{j \in J}$ is a tight $TK$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$.

**Proof.** Since $\{\Lambda_j\}_{j \in J}$ is a tight $K$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$, we have

$$A \|K^* f\|^2 = \sum_{j \in J} \|\Lambda_j f\|^2, \quad \forall f \in H. \quad (33)$$

And, for any $f \in H$, we have $T^* f \in H$; then

$$\sum_{j \in J} \|\Lambda_j T^* f\|^2 = A \|K^* T^* f\|^2 = A \|(TK)^* f\|^2, \quad (34) \forall f \in H.$$  

So $\{\Lambda_j T^*\}_{j \in J}$ is a tight $TK$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$ with bound $A$. The proof of Theorem 18 is completed.

**Corollary 19.** Let $K \in L(H)$. If $\{\Lambda_j\}_{j \in J}$ is a tight $K$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$, then $\{\Lambda_j (K^*)^N\}_{j \in J}$ is a tight $K^{N+1}$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$, where $N$ is a given positive integer.

**Theorem 20.** Let $K_1, K_2 \in L(H)$, and let $gF_K(H)$ be the collection of all tight $K$-$g$-frames for $H$ with respect to $\{H_j\}_{j \in J}$. Then $gF_K(H) \subset gF_{K_2}(H)$ if and only if there exists $A > 0$ such that $K_1 K_2^* = AK_2 K_2^*.$

**Proof.** If $\{\Lambda_j\}_{j \in J}$ is a $g$-orthonormal basis for $H$ with respect to $\{H_j\}_{j \in J}$, by Corollary 17, we get that $\{\Lambda_j K_1^*\}_{j \in J}$ is a tight $K_1$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$. Since $gF_K(H) \subset gF_{K_2}(H)$, we have that $\{\Lambda_j K_2^*\}_{j \in J}$ is a tight $K_2$-$g$-frame for $H$ with respect to $\{H_j\}_{j \in J}$. Assume that the bound
of tight $K_2\cdot g$-frame $\{\Lambda_j K_1^*\}_j \in \mathcal{J}$ is $A$. By the definition of tight $K\cdot g$-frame, we obtain
\begin{equation}
\sum_{j \in \mathcal{J}} \|\Lambda_j K_1^* f\|^2 = A \|K_1^* f\|^2, \quad \forall f \in \mathcal{H}.
\end{equation}
Since $\{\Lambda_j\}_j \in \mathcal{J}$ is a $g$-orthonormal basis for $\mathcal{H}$ with respect to $\{H_j\}_j \in \mathcal{J}$, we have $K_1^* f \in \mathcal{H}$ for any $f \in \mathcal{H}$. By the definition of $g$-orthonormal basis, we get
\begin{equation}
\sum_{j \in \mathcal{J}} \|\Lambda_j K_1^* f\|^2 = \|K_1^* f\|^2, \quad \forall f \in \mathcal{H}.
\end{equation}
By (35) and (36), we get $\|K_1^* f\|^2 = A \|K_1^* f\|^2$ for any $f \in \mathcal{H}$. So $K_1 K_1^* = AK_1^*$. Suppose that $\{\Lambda_j\}_j \in \mathcal{J}$ is a tight $K_1\cdot g$-frame for $\mathcal{H}$ with respect to $\{H_j\}_j \in \mathcal{J}$ with bound $A_1$; then
\begin{equation}
\sum_{j \in \mathcal{J}} \|\Lambda_j f\|^2 = A_1 \|K_1^* f\|^2, \quad \forall f \in \mathcal{H}.
\end{equation}
Under this assumption, there exists $A > 0$ such that $K_1 K_1^* = AK_1^*$. So for any $f \in \mathcal{H}$, we have $\|K_1^* f\|^2 = A \|K_1^* f\|^2$. Hence
\begin{equation}
A_1 A \|K_1^* f\|^2 = A_1 \|K_1^* f\|^2 = \sum_{j \in \mathcal{J}} \|\Lambda_j f\|^2, \quad \forall f \in \mathcal{H}.
\end{equation}
Therefore, $\{\Lambda_j\}_j \in \mathcal{J}$ is a tight $K_2\cdot g$-frame for $\mathcal{H}$ with respect to $\{H_j\}_j \in \mathcal{J}$ with bound $A_1A$. The proof of Theorem 20 is completed.

Corollary 21. Let $K_1, K_2 \in L(\mathcal{H})$ and let $PgF_{K_2}(\mathcal{H})$ be the collection of all Parseval $K_2\cdot g$-frames for $\mathcal{H}$ with respect to $\{H_j\}_j \in \mathcal{J}$. Then $PgF_{K_2}(\mathcal{H}) = PgF_{K_1}(\mathcal{H})$ if and only if $K_1 K_1^* = K_2 K_2^*$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

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