Research Article

Elastic Equilibrium of Porous Cosserat Media with Double Porosity

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The static equilibrium of porous elastic materials with double porosity is considered in the case of an elastic Cosserat medium. The corresponding three-dimensional system of differential equations is derived. Detailed consideration is given to the case of plane deformation. A two-dimensional system of equations of plane deformation is written in the complex form and its general solution is represented by means of three analytic functions of a complex variable and two solutions of Helmholtz equations. The constructed general solution enables one to solve analytically a sufficiently wide class of plane boundary value problems of the elastic equilibrium of porous Cosserat media with double porosity. A concrete boundary value problem for a concentric ring is solved.

1. Introduction

A model of elastic equilibrium of porous media with double porosity was constructed in the works [1–3]. The theory justified in these papers combines the previously proposed model of Barenblatt for media with double porosity [4] and that of Biot for media with ordinary porosity [5]. For a detailed account of the development of the theory of porous media and relevant references, see [6]. Various issues related to the elastic equilibrium of bodies with double porosities are treated in [7–15].

It should be noted that all the papers mentioned above dealt with a classical (symmetric) medium. But we do not know of any works where problems of double porous elasticity would have been considered for a nonsymmetric elastic Cosserat medium [16–34]. In our opinion, the investigation of such problems is interesting from both theoretical and practical standpoints. For this reason, we considered the elastic equilibrium of porous bodies with double porosity in the case of the nonsymmetric Cosserat theory.

In Section 2 we give the basic three-dimensional equations of the static equilibrium of porous elastic materials with double porosity in the case of an elastic Cosserat medium. In Section 3 we consider the case of a plane deformed state and write the corresponding two-dimensional system of equilibrium equations in the complex form. In Section 4 we construct the general solution of the abovementioned system of equations by means of analytic functions of a complex variable and solutions of Helmholtz equations. The obtained analogues of the Kolosov-Muskhelishvili formulas [35] make it possible to solve analytically plane boundary value problems of the elastic equilibrium of porous Cosserat media with double porosity. Finally, in Section 5 we solve a boundary value problem for a concentric circular ring.

2. Basic Three-Dimensional Relations

Let an elastic body with double porosity occupy the domain \( \Omega \subset R^3 \). Denote by \((x_1, x_2, x_3)\) a point of the domain \( \Omega \) in the Cartesian coordinate system. Let the domain \( \Omega \) be filled with an elastic Cosserat medium having double porosity. The considered solid body is characterized by the displacement vector \( \mathbf{u} = (u_1, u_2, u_3) \), rotation vector \( \mathbf{\omega} = (\omega_1, \omega_2, \omega_3) \), and also the fluid pressures \( p_1(x_1, x_2, x_3) \) and \( p_2(x_1, x_2, x_3) \) occurring, respectively, in the pores and fissures of the porous medium.
Then a homogeneous system of static equilibrium equations is written in the form [25]
\[ \partial_t \sigma_{ij} = 0, \]
\[ \partial_t \mu_j + \epsilon_{ijk} \sigma_{ik} = 0, \quad j = 1, 2, 3 \]  
(1)
in \( \Omega \),
where \( \sigma_{ij} \) are stress tensor components, \( \mu_j \) are moment stress tensor components, \( \epsilon_{ijk} \) is the Levi-Civita symbol, and \( \partial_t \equiv \partial / \partial t \); the summation over the recurring index \( i \) is assumed to be done from 1 to 3.

Formulas that interrelate the stress and moment stress components, the displacement and rotation vector components, and the pressures \( p_1, p_2 \) have the form [13, 25]
\[ \sigma_{ij} = (\lambda \text{ div } \mathbf{u} - \beta_1 p_1 - \beta_2 p_2) \delta_{ij} + (\mu + \alpha) \partial_t \mu_j \]
\[ + (\mu - \alpha) \partial_t \mu_i - 2\alpha \epsilon_{ijk} \omega_k, \]
\[ \mu_{ij} = \sigma \text{ div } \omega_{ij} + (\nu + \beta) \partial_t \omega_j + (\nu - \beta) \partial_t \omega_i, \]  
(2)
j = 1, 2, 3,
where \( \lambda, \mu \) are the Lamé parameters, \( \alpha, \beta, \nu, \sigma \) are the constants characterizing the microstructure of the considered elastic medium, \( \beta_1 \) and \( \beta_2 \) are the effective stress parameters, and \( \delta_{ij} \) is the Kronecker delta.

As is well known, for the internal energy to be positive it is necessary that the following conditions be fulfilled [28]:
\[ 3\lambda + 2\mu + \alpha \geq 0, \]
\[ 2\mu + \alpha \geq 0, \]
\[ \alpha \geq 0, \]  
(3)
\[ 3\sigma + 2\nu \geq 0, \]
\[ |\nu - \beta| \leq \nu + \beta. \]

In the stationary case, the values \( p_1 \) and \( p_2 \) satisfy the following system of [13]
\[ (k_1 \Delta - \gamma) p_1 + (k_{12} \Delta + \gamma) p_2 = 0, \]
\[ (k_{21} \Delta + \gamma) p_1 + (k_2 \Delta - \gamma) p_2 = 0 \]  
(4)
in \( \Omega \),
where \( k_1 = k_1/\mu', k_2 = k_2/\mu', k_{12} = k_{12}/\mu', \) and \( k_{21} = k_{21}/\mu' \), \( \mu' \) is fluid viscosity, \( k_1 \) and \( k_2 \) are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, \( k_{12} \) and \( k_{21} \) are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, \( \gamma > 0 \) is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, and \( \Delta \equiv \partial_{11} + \partial_{22} + \partial_{33} \) is the three-dimensional Laplace operator.

The three-dimensional system of (1), (2), and (4) describes the static equilibrium of a porous elastic Cosserat medium with double porosity. Substituting relations (2) into (1), we obtain equilibrium equations with respect to the components of the displacement and rotation vectors:
\[ (\mu + \alpha) \Delta u_j + (\lambda + \mu - \alpha) \partial_j (\partial_t \omega_k) - 2\alpha \epsilon_{ijk} \partial_t \omega_k \]
\[ - \partial_j (\beta_1 p_1 + \beta_2 p_2) = 0, \]
\[ (\nu + \beta) \Delta \omega_j + (\sigma + \nu - \beta) \partial_j (\partial_t \omega_k) + 2\alpha \epsilon_{ijk} \partial_t \omega_k \]  
(5)
in \( \Omega \).

If we add boundary conditions on the boundary \( \partial \Omega \) of the domain \( \Omega \), to the system of equilibrium equations, then we can consider various classical boundary value problems.

The following lemma is easy to prove.

**Lemma 1.** If \( \gamma > 0, k_1 k_2 - k_{12} k_{21} > 0 \), then the system of (4) is equivalent to two independent equations: to the Laplace equation
\[ \Delta \left[ (k_1 + k_{12}) p_1 + (k_2 + k_{21}) p_2 \right] = 0 \]  
(6)
with respect to the combination \( (k_1 + k_{12}) p_1 + (k_2 + k_{21}) p_2 \) and to the Helmholtz equation with respect to the difference \( p_1 - p_2 \)
\[ \Delta (p_1 - p_2) - \zeta^2 (p_1 - p_2) = 0, \]  
(7)
where \( \zeta^2 = \gamma (k_1 + k_2 + k_{12} + k_{21}) / (k_1 k_2 - k_{12} k_{21}) > 0. \)

**Proof.** Adding the first equation of system (4) to the second equation of this system, we immediately obtain (6).

Let us write system (4) in the matrix form:
\[ \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \end{pmatrix} - \gamma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]
(8)
By assumption, the determinant of the matrix \( \det \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} = k_1 k_2 - k_{12} k_{21} > 0 \) is positive. The left multiplication of all members of the latter equation by the matrix \( \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix}^{-1} \) gives the system
\[ \Delta p_1 - \gamma \frac{(k_2 + k_{12})}{k_1 k_2 - k_{12} k_{21}} (p_1 - p_2) = 0, \]
\[ \Delta p_2 + \gamma \frac{(k_1 + k_{21})}{k_1 k_2 - k_{12} k_{21}} (p_1 - p_2) = 0. \]
(9)
If from the first equation of the latter system we subtract the second equation of this system, then we obtain (7).

Since the transformation determinant is defined as
\[ \det \begin{pmatrix} k_1 + k_{21} & k_2 + k_{12} \\ 1 & -1 \end{pmatrix} = -(k_1 + k_2 + k_{12} + k_{21}) \]
(10)
< 0,
the validity of the lemma is proved.
Corollary 2. If on the boundary \( \partial \Omega \) of the domain \( \Omega \), \( p_1 = p_2 \), then \( p_1 = p_2 \) throughout the body \( \Omega \).

This corollary follows from the fact that the homogeneous Helmholtz equation (7) with zero boundary conditions has only the trivial solution.

3. The Plane Deformation Case

From the basic three-dimensional equations we obtain the basic equations for the case of plane deformation. Let \( \Omega \) be a sufficiently long cylindrical body with generatrix parallel to the \( x_3 \)-axis. Denote by \( V \) the cross section of this cylindrical body; thus \( V \subset \mathbb{R}^2 \). In the case of plane deformation \( \omega_1 = 0 \), \( \omega_2 = 0 \), while the functions \( u_1, u_2, \omega_3, p_1 \), and \( p_2 \) do not depend on the coordinate \( x_3 \) (see, e.g., [29]).

As follows from formulas (2), in the case of plane deformation

\[
\begin{align*}
\sigma_{\alpha \beta} &= 0, \\
\sigma_{3 \alpha} &= 0, \\
\mu_{\alpha \beta} &= 0, \\
\mu_{33} &= 0, \\
\alpha &= 1, 2; \\
\beta &= 1, 2.
\end{align*}
\]

Therefore the system of equilibrium equations (1) takes the form

\[
\begin{align*}
\partial_1 \sigma_{11} + \partial_2 \sigma_{21} &= 0, \\
\partial_1 \sigma_{12} + \partial_2 \sigma_{22} &= 0, \\
\partial_1 \mu_{13} + \partial_2 \mu_{23} + (\sigma_{12} - \sigma_{21}) &= 0
\end{align*}
\]

in \( V \).

Equations (6) and (7) take the form

\[
\begin{align*}
\Delta_2 \left[ (k_1 + k_{31}) p_1 + (k_2 + k_{32}) p_2 \right] &= 0 \quad \text{in } V, \\
\Delta_2 (p_1 - p_2) - \zeta^2 (p_1 - p_2) &= 0 \quad \text{in } V,
\end{align*}
\]

where \( \Delta_2 := \partial_{11} + \partial_{22} \) is the Laplace operator in two dimensions.

If relations (13) are substituted into system (12), then we obtain the following system of equilibrium equations with respect to the functions \( u_1, u_2, \omega_3, \) and \( \omega_3 \):

\[
\begin{align*}
(\mu + \alpha) \Delta_2 u_1 + (\lambda + \mu - \alpha) \partial_1 \theta + 2 \alpha \partial_2 \omega_3 &= 0, \\
- \partial_1 (\beta_1 p_1 + \beta_2 p_2) &= 0, \\
(\mu + \alpha) \Delta_2 u_2 + (\lambda + \mu - \alpha) \partial_2 \theta - 2 \alpha \partial_1 \omega_3 &= 0, \\
- \partial_2 (\beta_1 p_1 + \beta_2 p_2) &= 0, \\
(\nu + \beta) \Delta_2 \omega_3 + 2 \alpha (\partial_1 u_2 - \partial_2 u_1) - 4 \alpha \omega_3 &= 0
\end{align*}
\]

in \( V \).

On the plane \( x_1, x_2 \), we introduce the complex variable \( z = x_1 + i x_2 = r e^{i \theta} \) (\( i^2 = -1 \)) and the operators \( \partial_z = 0.5(\partial_1 - i \partial_2) \), \( \partial_{\bar{z}} = 0.5(\partial_1 + i \partial_2) \), \( \bar{z} = x_1 - i x_2 \), and \( \Delta_2 = 4 \partial_z \partial_{\bar{z}} \).

To write system (12) in the complex form, the second equation of this system is multiplied by \( i \) and summed up with the first equation [35–37] (see also [38, 39]):

\[
\begin{align*}
\partial_z (\sigma_{11} - \sigma_{22} + i (\sigma_{12} + \sigma_{21})) + \partial_{\bar{z}} (\sigma_{11} + \sigma_{22} + i (\sigma_{12} - \sigma_{21})) &= 0, \\
\partial_z (\mu_{13} + i \mu_{32}) + \partial_{\bar{z}} (\mu_{13} - i \mu_{32}) + \sigma_{12} - \sigma_{21} &= 0
\end{align*}
\]

in \( V \),

where, by formulas (13),

\[
\begin{align*}
\sigma_{11} - \sigma_{22} + i (\sigma_{12} + \sigma_{21}) &= 4 \mu \partial_z \partial_{\bar{z}} u_1, \\
\sigma_{11} + \sigma_{22} + i (\sigma_{12} - \sigma_{21}) &= 2 (\lambda + \mu - \alpha) \theta + 4 \alpha \partial_z \partial_{\bar{z}} u_4 - 4 \alpha \omega_3 \\
&- 2 (\beta_1 p_1 + \beta_2 p_2), \\
\mu_{13} + i \mu_{32} &= 2 (\nu + \beta) \partial_z \omega_3, \\
\mu_{31} + i \mu_{23} &= 2 (\nu - \beta) \partial_{\bar{z}} \omega_3, \\
u_+ &= u_1 + i u_2, \\
\theta &= \partial_z u_+ + \partial_{\bar{z}} \partial_z u_+.
\end{align*}
\]

We write (14) and (15) as

\[
\begin{align*}
\partial_z \partial_{\bar{z}} \left[ (k_1 + k_{31}) p_1 + (k_2 + k_{32}) p_2 \right] &= 0 \quad \text{in } V, \\
4 \partial_z \partial_{\bar{z}} (p_1 - p_2) - \zeta^2 (p_1 - p_2) &= 0 \quad \text{in } V.
\end{align*}
\]
If relations (18) are substituted into system (17), then system (16) is written in the complex form:

\[ 2(μ + α) \partial_2 \partial_z u_+ + (λ + μ − α) \partial_2 \theta - 2αi\partial_2 ω_3 \]
\[ - \partial_2 (β_1 p_1 + β_2 p_2) = 0, \]
\[ 2(ν + β) \partial_2 \partial_z ω_3 + αi(θ - 2\partial_z u_+) - 2αω_3 = 0 \]  

(20)
in V.

4. The General Solution of System (19)-(20)

In this section, we construct the analogues of the Kolosov-Muskhelishvili formulas [35] (see also [36–39]) for system (19)-(20).

Equations (19) imply that

\[ (k_1 + k_{21})p_1 + (k_2 + k_{12})p_2 = k_0 \left[ f'(z) + f'(z) \right], \]
\[ p_1 - p_2 = k_0 η(z, \overline{z}), \]  

(21)
where \( k_0 = k_1 + k_2 + k_{12} + k_{21}, \) \( f(z) \) is an arbitrary analytic function of a complex variable \( z \) in the domain \( V, \) and \( η(z, \overline{z}) \) is an arbitrary solution of the Helmholtz equation

\[ 4\partial_2 \partial_2 η - z^2 η = 0. \]  

(22)

From system (21) we easily obtain the expressions for the pressures \( p_1 \) and \( p_2: \)

\[ p_1 = f'(z) + \overline{f'(z)} + (k_2 + k_{12}) η(z, \overline{z}), \]
\[ p_2 = f'(z) + \overline{f'(z)} - (k_1 + k_{21}) η(z, \overline{z}). \]  

(23)

**Theorem 3.** The general solution of the system of (20) is represented as follows:

\[ 2μu_+ = ϵp(z) - zq'(z) - ψ'(z) + 2i\partial_2 χ(z, \overline{z}) \]
\[ + \frac{μ(β_1 + β_2)}{λ + 2μ} \left( f(z) + zf'(z) \right) \]
\[ + δ\partial_2 η(z, \overline{z}), \]
\[ 2ω_3 = \frac{2μ}{ν + β} χ(z, \overline{z}) - \frac{κ + 1}{2i} \left( q'(z) - ψ'(z) \right), \]  

(24)
where \( κ = (λ + 3μ)/(λ + μ), \) \( δ = 4μ(k_2 + k_{12})β_1 - (k_1 + k_{21})β_2)/(λ + 2μ)^2, \) \( q(z) \) and \( ψ(z) \) are arbitrary analytic functions of a complex variable \( z \) in the domain \( V, \) and \( χ(z, \overline{z}) \) is an arbitrary solution of the Helmholtz equation

\[ 4\partial_2 \partial_2 χ - χ^2 = 0, \]  

(26)
where

\[ ξ^2 = \frac{4μκ}{(ν + β)(μ + α)} > 0. \]  

(27)

**Proof.** We take the operator \( \partial_2 \) out of the brackets in the left-hand part of the first equation of system (20):

\[ \partial_2 \left[ 2(μ + α) \partial_2 u_+ + (λ + μ − α) \theta - 2αiω_3 \right] \]
\[ - (β_1 p_1 + β_2 p_2) = 0. \]  

(28)

Since (28) is a system of Cauchy-Riemann equations, we have

\[ 2(μ + α) \partial_2 u_+ + (λ + μ − α) \theta - 2αiω_3 \]
\[ = (κ + 1) \varphi'(z) + β_1 p_1 + β_2 p_2, \]  

(29)
where \( \varphi(z) \) is an arbitrary analytic function of \( z. \)

A conjugate equation to (29) has the form

\[ 2(μ + α) \partial_2 \overline{u_+} + (λ + μ − α) \theta - 2αiω_3 \]
\[ = (κ + 1) \varphi'(z) + β_1 p_1 + β_2 p_2. \]  

(30)
Summing up (29) and (30) and taking into account that

\[ \theta = \partial_2 u_+ + \partial_2 \overline{u_+}, \]  

(31)
we obtain

\[ \theta = \frac{1}{λ + μ} \left( \varphi'(z) + \varphi'(\overline{z}) \right) + \frac{1}{λ + 2μ} \left( β_1 p_1 + β_2 p_2 \right). \]  

(32)
If from (29) we subtract (30) and write the expression \( i(\partial_2 u_+ - \partial_2 \overline{u_+}) \), then we have

\[ i(\partial_2 u_+ - \partial_2 \overline{u_+}) = \frac{κ + 1}{2μ + α} i \left( \varphi'(z) - \varphi'(\overline{z}) \right) \]
\[ - \frac{2α}{μ + α} ω_3. \]  

(33)
The second equation of system (20) is written as

\[ 4\partial_2 \partial_2 ω_3 - \frac{2α}{ν + β} i(\partial_2 u_+ - \partial_2 \overline{u_+}) - \frac{4α}{ν + β} ω_3 = 0. \]  

(34)
Substituting formula (33) into formula (34) we obtain the equation

\[ Δ_2 ω_3 - ξ^2 ω_3 = \frac{κ + 1}{(ν + β)(μ + α)} i \left( \varphi'(z) - \varphi'(\overline{z}) \right). \]  

(35)
The general solution of (35) is written in the form

\[ 2μω_3 = \frac{2μ}{ν + β} χ(z, \overline{z}) - \frac{κ + 1}{2i} i \left( \varphi'(z) - \varphi'(\overline{z}) \right), \]  

(36)
where \( χ(z, \overline{z}) \) is a general solution of the Helmholtz equation

\[ Δ_2 χ - ξ^2 χ = 0. \]  

(37)
The multiplier \( 2μ/(ν + β) \) has been introduced for convenience in writing our subsequent formulas.
Substituting formulas (32) and (36) into (29) and taking into account that \( \chi(z, \bar{z}) \) is a solution of (37), we obtain

\[
2\mu \partial_z u_+ = \kappa \varphi'(z) - \varphi'(z) + 2i\partial_z \partial \bar{\chi}(z, \bar{z}) + \frac{\mu}{\lambda + 2\mu} \left( \beta_1 p_1 + \beta_2 p_2 \right).
\]

(38)

From formulas (23) we find the following expression for the combination \( \beta_1 p_1 + \beta_2 p_2 \):

\[
\frac{\mu}{\lambda + 2\mu} \left( \beta_1 p_1 + \beta_2 p_2 \right) = \frac{\mu (\beta_1 + \beta_2) }{\lambda + 2\mu} \left( f'(z) + \overline{f'(z)} \right) + \overline{\delta \partial \bar{\eta}}(z, \bar{z}).
\]

(39)

Substituting the latter formula into (38), integrating over \( z \), and using

\[
\eta = \frac{4}{\zeta^2} \partial_z \bar{\eta},
\]

we obtain formula (24) which we are proving:

\[
2\mu u_+ = \kappa \varphi(z) - \frac{2\varphi'(z) - \psi(z) + 2i\partial_z \partial \bar{\chi}(z, \bar{z})}{\lambda + 2\mu} \left( f'(z) + \overline{f'(z)} \right) + \delta \partial \bar{\eta}(z, \bar{z}).
\]

(40)

Thus, if the solution of system (20) is sufficiently smooth, then it is represented in the form of (24) and (25). Conversely, if expressions (24) and (25) are substituted into (20), then this system will be satisfied.

Substituting expressions (24) and (25) into formulas (18), for combinations of stress tensor components we obtain the following formulas:

\[
\sigma_{11} + \sigma_{22} + i \left( \sigma_{12} - \sigma_{21} \right) = 2 \left[ \varphi'(z) + \frac{\varphi'(z)}{\zeta} \right]
\]

\[
- 2i\partial_z \partial \bar{\chi} - \frac{\mu (\beta_1 + \beta_2) }{\lambda + 2\mu} \left( f'(z) + \overline{f'(z)} \right)
\]

\[
- \delta \partial \bar{\eta},
\]

\[
\sigma_{11} - \sigma_{22} + i \left( \sigma_{12} + \sigma_{21} \right) = 2 \left[ -2\varphi''(z) - \frac{\varphi'(z)}{\zeta} \right]
\]

\[
+ 2i\partial_z \partial \bar{\chi} + \frac{2\mu (\beta_1 + \beta_2) }{\lambda + 2\mu} \left( f'(z) + \overline{f'(z)} \right) + 2\delta \partial \bar{\eta}.,
\]

\[
\sigma_{33} = \frac{\lambda}{\lambda + \mu} \left[ \varphi'(z) + \frac{\varphi'(z)}{\zeta} \right]
\]

\[- \frac{2\mu}{\lambda + 2\mu} \left( (\beta_1 + \beta_2) \left( f'(z) + \overline{f'(z)} \right) + \delta \chi(z, \bar{z}) \right),
\]

\[
\mu_{13} + i \mu_{23} = \frac{(\kappa + 1) \left( \psi(z) + 2\varphi''(z) \right)}{2\mu},
\]

\[
\mu_{31} + i \mu_{32} = \frac{(\kappa + 1) \left( \psi(z) + \frac{2\varphi'(z)}{\zeta} \right)}{2\mu}.
\]

(42)

Thus, the general solution of a two-dimensional system of differential equations that describes the static equilibrium of a porous elastic medium with double porosity is represented by means of three analytic functions of a complex variable and two solutions of the Helmholtz equation. By an appropriate choice of these functions we can satisfy five independent classical boundary conditions.

Let mutually perpendicular unit vectors \( \mathbf{l} \) and \( \mathbf{s} \) be such that

\[
\mathbf{l} \times \mathbf{s} = \mathbf{e}_3,
\]

where \( \mathbf{e}_3 \) is the unit vector directed along the \( x_3 \) -axis. The vector \( \mathbf{l} \) forms the angle \( \theta \) with the positive direction of the \( x_1 \) -axis. Then the displacement components \( u_\theta = \mathbf{u} \cdot \mathbf{l} \), \( u_\phi = \mathbf{u} \cdot \mathbf{s} \), as well as the stress and moment stress components acting on an area of arbitrary orientation, are expressed by the formulas

\[
\sigma_\theta = \begin{cases} \sigma_{11} + \sigma_{22} - 2i \sigma_{12} - 2i \sigma_{21}, & \text{if } r = R_1, \\ \sigma_{11} + \sigma_{22} + i \sigma_{12} - i \sigma_{21}, & \text{if } r = R_2, \end{cases}
\]

\[
\sigma_\phi = \begin{cases} 0, & \text{if } r = R_1, \\ 0, & \text{if } r = R_2, \end{cases}
\]

\[
\mu_{13} = \begin{cases} 0, & \text{if } r = R_1, \\ 0, & \text{if } r = R_2, \end{cases}
\]

\[
\mu_{23} = \begin{cases} 0, & \text{if } r = R_1, \\ 0, & \text{if } r = R_2, \end{cases}
\]

\[
(k_1 + k_{21}) p_1 + (k_2 + k_{12}) p_2 = \begin{cases} p^{(1)}, & \text{if } r = R_1, \\ p^{(2)}, & \text{if } r = R_2, \end{cases}
\]

(43)

(44)

(45)

(46)

5. A Problem for a Concentric Circular Ring

In this section, we solve a concrete boundary value problem for a concentric circular ring. On the boundary of the considered domain which is free from stresses and moment stresses, the values of pressures \( p_1 \) and \( p_2 \) are given.

Let a porous elastic body with double porosity occupy the domain \( V \) which is bounded by the concentric circumferences \( L_1 \) and \( L_2 \) with radii \( R_1 \) and \( R_2 \), respectively (\( R_1 < R_2 \)) (Figure 1).

We consider the following problem:

\[
\sigma_{rr} - i \sigma_{\theta r} = \begin{cases} 0, & \text{if } r = R_1, \\ 0, & \text{if } r = R_2, \end{cases}
\]

\[
\mu_{13} = \begin{cases} 0, & \text{if } r = R_1, \\ 0, & \text{if } r = R_2, \end{cases}
\]

\[
(k_1 + k_{21}) p_1 + (k_2 + k_{12}) p_2 = \begin{cases} p^{(1)}, & \text{if } r = R_1, \\ p^{(2)}, & \text{if } r = R_2, \end{cases}
\]
The function \( \chi(z, \bar{z}) \) is sought as a series
\[
\chi(z, \bar{z}) = \sum_{n=-\infty}^{\infty} (\gamma_n I_n(\kappa r) + \delta_n K_n(\kappa r)) e^{i \alpha n},
\]
whence
\[
\alpha_1 = -\frac{p K_1(\kappa R_1)}{I_1(\kappa R_1) K_1(\kappa R_1) - I_1(\kappa R_2) K_1(\kappa R_2)},
\]
\[
\beta_1 = \frac{p I_1(\kappa R_1)}{I_1(\kappa R_1) K_1(\kappa R_1) - I_1(\kappa R_2) K_1(\kappa R_2)}.
\]

Thus
\[
\eta(z, \bar{z}) = \frac{2}{k_0} (\alpha_1 I_1(\kappa r) + \beta_1 K_1(\kappa r)) \cos \alpha,
\]
where the coefficients \( \alpha_1 \) and \( \beta_1 \) are calculated by formulas (53).

Let us now satisfy the boundary conditions (45). Due to the general representations (42) and (44) obtained above we have
\[
\sigma_{\alpha\beta} - i\sigma_m
\]

\[
= \phi'(z) + \phi'(z) e^{2i\alpha} (\bar{z} \phi''(z) + \psi'(z))
\]

\[
- \frac{\kappa^2 i}{2} \left( \chi(z, \bar{z}) - \frac{4}{k^2} \partial_\kappa \partial_{\kappa} (\chi(z, \bar{z}) e^{2i\alpha}) \right)
\]

\[
- \frac{\delta^2}{4} \left( \eta(z, \bar{z}) - \frac{4}{k^2} \partial_\kappa \partial_{\kappa} (\eta(z, \bar{z}) e^{2i\alpha}) \right)
\]

\[
- \frac{\mu (\beta_1 + \beta_2)}{\lambda + 2\mu} \left( f'(z) + f'(z) e^{2i\alpha} - \bar{z} f''(z) e^{2i\alpha} \right)
\]

\[
= 0, \quad r = R_1,
\]

\[
= 0, \quad r = R_2;
\]

\[
\mu_3
\]

\[
= \frac{(\kappa + 1) (\nu + \beta) i}{4\mu} \left( \phi''(z) e^{-2i\alpha} - \phi''(z) e^{2i\alpha} \right)
\]

\[
+ \partial_\kappa \partial_{\kappa} (\chi(z, \bar{z}) e^{-2i\alpha} + \partial_\kappa \partial_{\kappa} (\eta(z, \bar{z}) e^{2i\alpha})
\]

\[
= 0, \quad r = R_1,
\]

\[
= 0, \quad r = R_2.
\]

The analytic functions \( \phi'(z) \) and \( \psi'(z) \) are represented as series
\[
\phi'(z) = A \ln z + \sum_{n=-\infty}^{\infty} a_n r^n e^{i\beta_n},
\]

\[
\psi'(z) = \sum_{n=-\infty}^{\infty} b_n r^n e^{i\beta_n}.
\]

From the first condition of displacement uniqueness it follows that \( A = 0 \). It is also assumed that \( a_0 \) is a real value; that is, \( a_0 = \overline{a_0} \) (see [35]).

The metaharmonic function \( \chi(z, \bar{z}) \) is sought as a series
\[
\chi(z, \bar{z}) = \sum_{n=-\infty}^{\infty} (\gamma_n I_n(\kappa r) + \delta_n K_n(\kappa r)) e^{i\alpha n}.
\]
If representations (56), (57), (54), and (49) are substituted into the boundary conditions (55), then we obtain
\[
\sum_{-\infty}^{+\infty} \left( 1 - n \right) a_n r^n e^{i\alpha x} + \sum_{-\infty}^{+\infty} \sum_{+\infty}^{-\infty} a_n r^{n-1} e^{-i\alpha x} - \sum_{-\infty}^{+\infty} b_{n-2} r^{n-2} e^{-2i\alpha x} - \frac{\xi_i}{r} \left( \sum_{-\infty}^{+\infty} \left( 1 - n \right) (y_n I_{n-1}(\xi r) - a_n K_{n-1}(\xi r)) e^{i\alpha x} \right)
\]
\[
= A_\gamma, \quad \gamma = 1, 2,
\]
Equating the free members on both sides of equalities (58), we obtain
\[
\sum_{-\infty}^{+\infty} \left( 1 - n \right) (y_n I_{n+1}(\xi r) - a_n K_{n+1}(\xi r)) e^{i\alpha x} = 0,
\]
Equating the free members on both sides of equalities (58) and (59), we obtain the system
\[
2a_0 - b_2 R_1^{-2} - \frac{\xi_i}{r} (y_0 I_1(\xi R_1) - a_0 K(\xi R_1)) = B_1,
\]
\[
2a_0 - b_2 R_2^{-2} - \frac{\xi_i}{r} (y_0 I_2(\xi R_2) - a_0 K(\xi R_2)) = B_2,
\]
where
\[
B_1 = \frac{\mu (\beta_1 + \beta_2)}{2 (\alpha + 2\mu) k_0} \left( p^{(2)} - p^{(1)} \right) (2 \ln R_1 - 1) + 2p^{(1)} \ln R_2 - 2p^{(2)} \ln R_1 - \frac{2\xi_i}{R_1} (y_0 I_1(\xi R_1) - a_0 K(\xi R_1)),
\]
\[
B_2 = \frac{\mu (\beta_1 + \beta_2)}{2 (\alpha + 2\mu) k_0} \left( p^{(2)} - p^{(1)} \right) (2 \ln R_2 - 1) + 2p^{(1)} \ln R_1 - 2p^{(2)} \ln R_1 - \frac{2\xi_i}{R_2} (y_0 I_2(\xi R_2) - a_0 K(\xi R_2)) = 0,
\]
From system (61) we define the constants \(a_0, b_2, y_0,\) and \(\delta_0:\)
\[
y_0 = \delta_0 = 0,
\]
\[
a_0 = \frac{B_2 R_1^2 - B_2 R_2^2}{2 (R_1^2 - R_2^2)}, \quad b_2 = \frac{(B_1 - B_2) R_1^2 R_2^2}{R_1^2 - R_2^2}.
\]
Equating the coefficients of \(e^{i\alpha x}\) in (58) and (59) and, using the second displacement uniqueness condition
\[
\alpha_{-1} - b_{-1} = 0,
\]
we have
\[
a_{-1} = 0,
\]
\[
b_{-1} = 0.
\]
Equating the coefficients of \(e^{-i\alpha x}\) in (58) and (59) and using (66), we obtain the following system of equations:
\[
\sum_{-\infty}^{+\infty} \left( 1 - n \right) (y_n I_{n+1}(\xi r) + a_n K_{n+1}(\xi r)) e^{-i\alpha x} = 0,
\]
where
\[
C_y := \frac{\delta_0}{k_0 R_1} \left[ a_1 I_1(c R_1) - \beta_1 K_1(c R_1) \right];
\]
\[
I'_y(\xi R_1) = 0.5 (I_0(\xi R_1) + I_2(\xi R_1)),
\]
\[
K'_y(\xi R_1) = -0.5 (K_0(\xi R_1) + K_2(\xi R_1)),
\]
\[
\gamma = 1, 2.
\]
Excluding \(b_3\) from the first two equations of system (67), for defining the coefficients \(a_{\gamma}, y_{-1},\) and \(\delta_{-1}\) we obtain the following system:
\[
\left( R_1^4 - R_2^4 \right) a_1 - 2\xi i \gamma_1 + 2\xi i K' \delta_{-1} = C_2 R_1^3 - C_1 R_2^3,
\]
\[
(\kappa + 1) (\gamma - \beta) a_1 - 4\mu \xi i K' \gamma_{-1} = 0,
\]
\[
\gamma = 1, 2.
\]


(\kappa + 1)(\gamma - \beta)\overline{a}_1 - 4\mu \xi l^1_1(\xi R_2) \gamma_1
- 4\mu \xi l^1_1(\xi R_2) \delta_1 = 0,

(69)

where

\[ I^* = \left( I_2(\xi R_2) R_2^2 - I_2(\xi R_1) R_1^2 \right), \]

\[ K^* = \left( K_2(\xi R_2) R_2^2 - K_2(\xi R_1) R_1^2 \right). \]

After defining the coefficients \( a_1, \gamma_1, \) and \( \delta_1 \) from (69), the coefficient \( b_3 \) is defined from the first or the second equation of system (67). Since \( \chi(z, \overline{z}) \) is a real function, we have

\[ \gamma_1 = \overline{\gamma_1}, \]

\[ \delta_1 = \overline{\delta_1}. \]

(71)

All other coefficients are equal to zero. Thus the sought functions \( \phi'(z), \psi'(z), \) and \( \chi(z, \overline{z}) \) will have the form

\[ \phi'(z) = a_0 + a_1 z, \]

\[ \psi'(z) = \frac{b_2}{z^2} + \frac{b_3}{z^3}, \]

\[ \chi(z, \overline{z}) = 2 \text{Re} \left[ \left( \eta_1 I_1(\xi r) + \delta_1 K_1(\xi \overline{r}) \right) \overline{e}^a \right]. \]

(72)

By substituting the obtained values of these functions and functions \( f'(z) \) and \( \eta(z, \overline{z}) \) into formulas (42) we find the values of all stress and moment stress components. Though the domain boundary was free from moment stresses, due to fluid pressures in the pores the considered disturbance promoted their appearance inside the domain.

The procedure of solving a boundary value problem remains the same when stresses, moment stresses, and pressures on the domain boundary are given arbitrarily, but the condition that the principal vector and the principal moment of external forces are equal to zero is fulfilled.

6. Conclusion

We consider the static equilibrium of porous elastic materials with double porosity for a nonsymmetric elastic Cosserat medium. For the case of plane deformation, a general solution of the corresponding system of differential equations is constructed by means of three analytic functions of a complex variable and two solutions of the Helmholtz equations. The constructed general solution can be applied for solving analytically quite a wide class of boundary value problems. An explicit solution is obtained for a boundary value problem for a concentric circular ring.

In our opinion, problems of double porous elasticity for a nonsymmetric elastic medium may be of interest from theoretical and practical standpoints.

Disclosure

Any idea in this paper is possessed by the author and may not represent the opinion of Shota Rustaveli National Science Foundation itself.

Competing Interests

The author declares that there is no conflict of interests.

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References


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