A Method of Finding Source Function for Inverse Diffusion Problem with Time-Fractional Derivative

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Abstract

The Homotopy Perturbation Method is developed to find a source function for inverse diffusion problem with time-fractional derivative. The inverse problem is with variable coefficients and initial and boundary conditions. The analytical solutions to the inverse problems are obtained in the form of a finite convergent power series with easily obtainable components.

1. Introduction

In recent years, fractional partial differential equations have drawn much consideration. Many important phenomena in physics, engineering, mathematics, finance, transport dynamics, and hydrology are well characterized by partial differential equations of fractional order. Fractional partial differential equations play an important role in modelling the so-called anomalous transport phenomena and in the theory of complex systems. These fractional derivatives are made up more appropriately compared to the standard integer-order models. So, the fractional derivatives are regarded to be very dominating and useful tool. Fractional partial equations are formulated using fractional derivative operators to replace regular derivatives. Different forms of fractional partial equations have been widely researched. For example, fluid flow, diffusive transport, materials with memory and hereditary effects, electrical networks, signal processing, electromagnetic theory, and many other physical processes are different applications of fractional partial equations. For mathematical properties of fractional derivatives and integrals, one can consult [1–6].

A direct problem is the procedure of identification of the effects from causes. An inverse problem is the opposite of a direct problem. An inverse problem is the procedure of calculating from a set of remarks the causal factors that yield them: for example, calculating an image in computer tomography, calculating the density of the earth from measurements of its gravity field, and source reconstructing in acoustic.

It is called an inverse problem because it starts with the results and then calculates the causes. This is the inverse of a forward problem, which starts with the causes and then calculates the results.

Inverse problems are some of the most important mathematical problems because they inform us about parameters that we cannot directly remark. They have wide application in optics, radar, communication theory, acoustics, computer vision, medical imaging, signal processing, astronomy, oceanography, remote sensing, and many other areas.

The field of inverse problems was first found and showed by Ambartsumian [7]; while still a student, Ambartsumian thoroughly studied the theory of atomic structure, the formation of energy levels. Then, the field of inverse problems has enjoyed a remarkable growth in the past few decades. High speed computers have made numerical solutions to many large scale inverse problems possible.

Applications of inverse problems are extremely various. One may say that this is an area attracted almost exclusively by applications. Because of the complexity of the problems and variety of the applications, the mathematical methods that are involved in solving inverse problems are also various. In the past few years, fractional calculus appears as an important form to deal with heat transfer equations. To obtain analytic solutions to fractional partial equations, two methods have been mainly used: the first method is the application of both Laplace-Fourier transforms and the second method is the separation of variables technique. Recently, several semianalytic methods have been also utilized to present...
series solution to fractional partial equations such as Ado-
mian decomposition method [8, 9], Homotopy Perturbation
Method [10–14], and variational iteration method [15, 16].

Many researchers also regard the regularization meth-
ods for the solution to the inverse problem of the one-
dimensional linear time-fractional heat equation. Murio [17]
recommended a space marching regularizing scheme using
mollification techniques for the solution to the inverse time-
fractional heat equation. In [18], the author considers the
problem of identification at the diffusion time-fractional
coefficient and the other problem of the fractional derivative
for the one-dimensional time-fractional diffusion equation.
Kirane et al. [19] proposed two-dimensional inverse source
problem and, in Section 4.2, finding of unknown source function
depending on $f(x, t)$.

In Section 4.1, finding of unknown source function
depending on $x$ is as follows:

$$D^\alpha_t u (x, t) = h(x) u_{xx} (x, t) + f(x),$$

and, in Section 4.2, finding of unknown source function
depending on $t$ is as follows:

$$D^\alpha_t u (x, t) = h(x) u_{xx} (x, t) + f(t).$$

Two numerical examples were given in Section 5. Conclusion
took place in the last section.

2. Definitions

**Definition 1.** The Riemann–Liouville fractional integral of $f \in C_\alpha$ of the order $\alpha \geq 0$ is defined as

$$\mathcal{J}^\alpha_t f (t) = \begin{cases} f(t), & \text{if } \alpha = 0, \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau, & \text{if } \alpha > 0, \end{cases}$$

where $\Gamma$ denotes gamma function: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$, $z \in C$.

**Definition 2.** The fractional derivative of $f \in C_\alpha$ of the order $\alpha \geq 0$, in Caputo sense, is defined as

$$D^\alpha_t f (t) = \mathcal{D}^\alpha_t f (t) + f(t),$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \, d\tau$$

for $n-1 < \alpha \leq n$, $n \in N$, $t > 0$, $f \in C^n_\alpha$, and $\alpha \geq -1$.

**Definition 3.** The Caputo-time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D^\alpha_t u (x, t) = \mathcal{J}_1^{\alpha-\alpha} u (x, t),$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u}{\partial \tau^n} d\tau.$$

**Lemma 4.** Let $n-1 < \alpha \leq n$, $n \in N$, and $f \in C^n_\alpha$, $\alpha \geq -1$; then

$$D^\alpha_t \mathcal{J}_1^{\alpha-\alpha} f (t) = f(t),$$

$$f^{(k)}(t) = f(t) - \sum_{k=0}^n \frac{t^k}{k!}$$

for $t > 0$.

**Lemma 5.** If $n-1 < \alpha \leq n$, $n \in N$, and $k \geq 0$, then one has

$$\mathcal{J}_1^{\alpha+k} \left( \frac{t^k}{\Gamma(k+1)} \right) = \frac{t^{\alpha+k}}{\Gamma(\alpha+k+1)}.$$
nonlinear operator. Hence, (8) can be written, following He [10], as follows:

\[ D_t^\alpha u (x, t) = L (u) + N (u) + f (x, t). \]  

(9)

For solving (8) by Homotopy Perturbation Method, we construct the following homotopy:

\[ H (V, p) = (1 - p) [D_t^\alpha V - u_0] + p [D_t^\alpha V - L (V) - N (V) - f (x, t)] = 0. \]  

(10)

And, equivalently,

\[ H (V, p) = D_t^\alpha V - u_0 \]

\[ + p [u_0 - L (V) - N (V) - f (x, t)] = 0, \]

where \( p \in [0, 1] \) is an embedding or homotopy parameter, \( H(x; t; p) : \Omega x[0, 1] \rightarrow \mathbb{R} \), and \( u_0 \) is the initial approximation for solution (9).

Clearly, the homotopy equations \( H(V, 0) = 0 \) and \( H(V, 1) = 1 \) are equivalent to the equations \( D_t^\alpha V = u_0 \) and \( D_t^\alpha V - L (V) - N (V) - f (x, t) = 0 \), respectively. Thus, a monotonous change of parameter \( p \) from 0 to 1 corresponds to a continuous change of the trivial problem \( D_t^\alpha V - u_0 = 0 \) to the original problem. Now, we assume that the solution to (9) can be written as a power series in embedding parameter \( p \), as follows:

\[ V = V_0 + pV_1, \]  

(12)

where \( V_0 \) and \( V_1 \) are functions which should be determined. Now, we can write (12) in the following form:

\[ D_t^\alpha V (x, t) = u_0 + p [-u_0 + L (V) + N (V) + f (x, t)]. \]  

(13)

Applying the inverse operator, \( f_t^\alpha \), which is the Riemann-Liouville fractional integral of order \( \alpha > 0 \), on both sides of (13), we have

\[ V (x, t) = V (x, 0) + f_t^\alpha u_0 \]

\[ + p f_t^\alpha [-u_0 + L (V) + N (V) + f (x, t)]. \]  

(14)

Suppose that the initial approximation of solution (9) is in the following form:

\[ u_0 = \sum_{k=0}^{\infty} a_k (x) t^{k\alpha} \frac{\Gamma (k\alpha + 1)}{\Gamma (k\alpha + 1)}, \]  

(15)

where \( a_k (x) \), for \( k = 1, 2 \), are functions which must be computed. Substituting (12) and (15) into (14), we get

\[ V_0 + pV_1 = V (x, 0) + f_t^\alpha \left( \sum_{k=0}^{\infty} a_k (x) t^{k\alpha} \frac{\Gamma (k\alpha + 1)}{\Gamma (k\alpha + 1)} \right) \]

\[ + p f_t^\alpha \left[ -\sum_{k=0}^{\infty} a_k (x) t^{k\alpha} \frac{\Gamma (k\alpha + 1)}{\Gamma (k\alpha + 1)} + L (V_0 + pV_1) \right] \]

\[ + N (V_0 + pV_1) + f (x, t) \].  

(16)

Synchronizing the coefficients of the same powers leads to

\[ p^0 : V_0 = V (x, 0) + f_t^\alpha \left( \sum_{k=0}^{\infty} a_k (x) t^{k\alpha} \frac{\Gamma (k\alpha + 1)}{\Gamma (k\alpha + 1)} \right) \]

\[ + N (V_0 + pV_1) + f (x, t) \]

\[ + L (V_0 + pV_1) \]. \]  

(17)

Now, we obtain the coefficients \( a_k (x) \), \( k = 1, 2 \), and therefore the exact solution can be obtained as the following:

\[ u (x, t) = V (x, t) = V (x, 0) + f_t^\alpha \left( \sum_{k=0}^{\infty} a_k (x) t^{k\alpha} \frac{\Gamma (k\alpha + 1)}{\Gamma (k\alpha + 1)} \right). \]  

(18)

Efficiency and reliability of the method are shown.

4. Finding Source Function for Inverse Problem with Time-Fractional Derivative

In this section, we construct a new Homotopy Perturbation Method to obtain the source function for inverse time-fractional one-dimensional diffusion equation with initial-boundary conditions. Model problems have been received from Özkum et al. [22]. To obtain the unknown source function, we have defined new methods through Homotopy Perturbation Method as in the following subsections.

4.1. Finding of Unknown Source Function Depending on \( x \) . Let us assume inverse time-fractional differential equation is as follows:

\[ D_t^\alpha u (x, t) = h (x) u_{xx} (x, t) + f (x), \quad x, t \in \Omega, \]  

(19)

with the following initial and boundary conditions:

\[ x > 0, \]

\[ t > 0, \]  

(20)

\[ 0 < \alpha \leq 1, \]

\[ u (x, 0) = f_1 (x), \]  

(21)

\[ u (0, t) = h_1 (t), \]  

(22)

\[ u_x (0, t) = h_2 (t), \]  

(23)

where \( h_1 (t) \) and \( h_2 (t) \) are functions which must be determined. The model problem (19) with initial-boundary conditions (20) is solvable if \( h (x) \in C^\infty [0, \infty) \) and \( f (x) \in C^\infty [0, \infty) \).

To find the source function for (19), we apply Homotopy Perturbation Method. So, we construct the following homotopy:

\[ H (V, p) = (1 - p) [D_t^\alpha V - u_0] + p [D_t^\alpha V - L (V) - N (V) - f (x, t)] = 0. \]  

(24)
And, equivalently,

\[
H(V, p) = D_\alpha^t V - u_0 + p [u_0 - L(V) - N(V) - f(x)] = 0.
\]  

(25)

So we can write (25) in the following form:

\[
D_\alpha^t u(x,t) = u_0(x,t)
\]

\[
-p [u_0(x,t) - h(x) u_{xx}(x,t) - f(x)],
\]

(26)

where \( p \in [0,1] \) is an embedding or homotopy parameter, \( H(x, t; p) : \Omega x [0, 1] \to R \), and \( u_0 \) is the initial approximation for solution (19).

Assume that the initial value of solution (19) is in the following form:

\[
u_0(x,t) = u(x,0) = u_0 = \sum_{k=0}^{\infty} a_k(x) t^{k\alpha} \Gamma(k\alpha + 1),\]

(27)

where \( a_k(x) \), for \( k = 1, 2 \), are functions which must be computed. Applying the inverse operator \( J_\alpha^t \) of \( D_\alpha^t \) to both sides of (26), we obtain

\[
u(x,t) = u(x,0) + J_\alpha^t u_0 + p J_\alpha^t [u_0(x,t) - h(x) u_{xx}(x,t) + f(x)].
\]  

(28)

Assume solution (28) has the following form:

\[
u(x,t) = u_0(x,t) + p u_1(x,t).
\]  

(29)

Substituting (29) into (28) collecting the same powers of \( p \) and equating each coefficient of \( p \) to zero yield

\[
u_0(x,t) + p u_1(x,t)
\]

\[
= u(x,0) + f_1(x) + p f_1^t [-u_0 + h(x) u_{xx}(x,t) + f(x)].
\]  

(30)

If initial conditions apply to (30), we obtain

\[
u_0(x,t) = f_1(x) + f_1^t u_0(x,t).
\]  

(31)

Then, \( u_0(x,t) \) can be written as follows:

\[
u_0(x,t) = f_1(x) \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots \right],
\]

(35)

In order to define the unknown source function, taking over the boundary conditions, we are taking \( h_1(t) \) and \( h_2(t) \) functions of Taylor series expansion for the space whose bases are

\[
\sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad 0 < \alpha \leq 1
\]  

(37)

\[
h_1(t) = h_1(0) + h_1'(0) t^\alpha \Gamma(\alpha+1) + \cdots
\]  

(38)

\[
h_2(t) = h_2(0) + h_2'(0) t^\alpha \Gamma(\alpha+1) + \cdots
\]  

(39)

Then, putting (30) in place of (35), we get the following:

\[
u(x,t) = \left[ h(x) f_1''(x) + f_1(x) \right]
\]

\[
+ \left[ h(x) f_1''(x) + f(x) \right] \sum_{k=1}^{\infty} \frac{a_k(x) t^{k\alpha}}{\Gamma(k\alpha + 1)}.
\]  

(36)
From (38) and (41), we can write

\[ h_1 (0) = f_1 (0) + h (0) f_1'' (0) \]
\[ h_1' (0) = [ h (0) + f_1'' (0) + f (0) ] \]
\[ h_1'' (0) = [ h (0) + f_1'' (0) + f (0) ] \]
\[ h_1''' (0) = [ h (0) + f_1'' (0) + f (0) ] \]

(42)

From (39) and (43), we can write

\[ h_2 (0) = [ f_1' (0) + h' (0) f_1'' (0) + h (0) f_1''' (0) ] \]
\[ h_2' (0) = [ h' (0) f_1'' (0) + h (0) f_1''' (0) + f_1 (0) ] \]
\[ h_2'' (0) = [ h' (0) f_1'' (0) + h (0) f_1''' (0) + f_1 (0) ] \]

(43)

Using the above data in the following Taylor series expansion of unknown function \( f(x) \), we get

\[ f (x) = f_1'' (0) + h_1' (0) - h (0) \]
\[ + [ f_1' (0) - h' (0) f_1'' (0) - h (0) f_1''' (0) ] x \]
\[ + 0. \]  

(45)

4.2. Finding of Unknown Source Function Depending on \( t \). Let us assume inverse time-fractional differential equation is as follows:

\[ D_t^a u (x, t) = h (x) u_{xx} (x, t) + f (t), \quad x, t \in \Omega, \]  

(46)

with the following initial and boundary conditions:

\[ x > 0, \]
\[ t > 0, \]  

(47)

\[ 0 < \alpha \leq 1 \]
\[ u (x, 0) = f_1 (x) \]  

(48)

\[ u (0, t) = h_1 (t) \]  

(49)

\[ u_x (0, t) = h_2 (t), \]  

(50)

where \( h_1 (t) \) and \( h_2 (t) \) \( \in C^{\alpha} [0, \infty) \) and \( h (x), f_1 (x) \) \( \in C^{\alpha} [0, \infty) \), and \( f(t) \in C^1_p [0, \infty) \), \( \mu \geq -1 \). As in the previous case, we apply Homotopy Perturbation Method to determine the unknown \( f \) function of \( t \):

\[ H (W, p) = (1 - p) [ D_t^a W - u_0 ] \]
\[ + p [ D^a W - L (W) - N (W) - f (t) ] \]

(51)

\[ = 0, \]

\[ H (W, p) = D_t^a W - u_0 \]
\[ + p [ u_0 - L (W) - N (W) - f (t) ] = 0, \]  

(52)

where \( p \in [0, 1] \) is an embedding or homotopy parameter, \( H(x, t; p) : \Omega \times [0, 1] \to \mathbb{R} \), and \( u_0 \) is the initial approximation for solution (46).

So, we can write (52) in the following form:

\[ D_t^a u (x, t) = u_0 (x, t) \]
\[ - p [ u_0 (x, t) - h (x) u_{xx} (x, t) - f (t) ] . \]  

(53)

Assume that the initial value of solution (46) is in the following form:

\[ u_0 (x, t) = u (x, 0) = u_0 = \sum_{k=0}^{\infty} a_k (x) \frac{t^{k\alpha}}{\Gamma (k\alpha + 1)}. \]  

(54)

Applying the inverse operator \( f_t^a \) of \( D_t^a \) to both sides of (53), we obtain

\[ u (x, t) = u (x, 0) + f_t^a u_0 \]
\[ + p [ u_0 (x, t) - h (x) u_{xx} (x, t) + f (t) ]. \]  

(55)

Suppose solution (55) has the following form:

\[ u (x, t) = u_0 (x, t) + p u_1 (x, t) . \]  

(56)

Substituting (56) into (55) collecting the same powers of \( p \) and equating each coefficients of \( p \) to zero yield

\[ u_0 (x, t) + p u_1 (x, t) \]
\[ = u (x, 0) + f_t^a u_0 (x, 0) \]
\[ - p f_t^a [ u_0 - h (x) u_{xx} (x, t) - f (t) ], \]  

(57)

\[ p^0 : u_0 (x, t) = f_1 (x) + \sum_{k=0}^{\infty} a_k (x) \frac{t^{k\alpha + \alpha}}{\Gamma (k\alpha + \alpha + 1)} \]
\[ p^1 : u_1 (x, t) = - f_t^a [ u_0 - h (x) u_{xx} (x, t) - f (t) ] \]
so

\[
\begin{aligned}
\ u_1(x,t) &= - \sum_{k=0}^{\infty} \frac{a_k(x) t^{ka+\alpha}}{\Gamma(ka+\alpha+1)} + h(x) \sum_{k=0}^{\infty} \frac{t^{ka}}{\Gamma(ka+1)} \\
&+ \int_0^t f_1(t) dt.
\end{aligned}
\]  

(58)

If we define \(I_t^\alpha f(t)\) as \(I_t^\alpha f(t) = \omega(t)(t^{ka}/\Gamma(ka + 1))\), then we can write for \(p = 1\)

\[
\begin{aligned}
\ u(x,t) &= f_1(x) + \sum_{k=0}^{\infty} \frac{a_k(x) t^{ka+\alpha}}{\Gamma(ka+\alpha+1)} \\
&- \sum_{k=0}^{\infty} \frac{a_k(x) t^{ka+\alpha}}{\Gamma(ka+\alpha+1)} \\
&+ [h(x) + \omega(t)] \sum_{k=0}^{\infty} \frac{t^{ka}}{\Gamma(ka+1)}
\end{aligned}
\]  

(59)

By using boundary conditions (49) and (50) into (59), we obtain the following coefficients:

\[
\begin{aligned}
\ a_0(x) &= f_1(x) \\
\ a_1(x) &= a_0(x) = f_1(x) \\
\ a_2(x) &= a_1(x) = a_0(x) = f_1(x)
\end{aligned}
\]  

(60)

and so on. Then, we can write

\[
\begin{aligned}
\ u(x,t) &= f_1(x) + \left[ h(x) + \omega(t) \right] \sum_{k=1}^{\infty} \frac{t^{ka}}{\Gamma(ka+1)} \\
&= f(x) + \left[ h(x) + \omega(t) \right] \sum_{k=1}^{\infty} \frac{t^{ka}}{\Gamma(ka+1)}.
\end{aligned}
\]  

(61)

Since \(D_t^\alpha D_t^\alpha f(t) = f(t)\), we find source function \(f(t)\) as follows:

\[
\begin{aligned}
\ f(t) &= D_t^\alpha \left\{ u(x,t) - f_1(x) - h(x) \right\} \\
&= \left[ e^x + \sin x \right] + \left[ 2e^x - 2\sin x + 2x \right] \\
&\cdot \left[ 1 + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots \right].
\end{aligned}
\]  

(62)

5. Numerical Examples

5.1. Example 1. We consider problem as follows:

\[
D_t^\alpha u(x,t) = 2u_{xx}(x,t) + f(x),
\]  

(63)

with the following initial and boundary conditions:

\[
\begin{aligned}
\ x &> 0, \quad t > 0, \quad 0 < \alpha \leq 1 \\
\ u(x,0) &= f_1(x) = e^x + \sin x \\
\ u(0,t) &= h_1(t) = e^{2t} \\
\ u_x(0,t) &= h_2(t) = e^{2t+1},
\end{aligned}
\]  

(64-70)

Figures 1 and 2 show the Homotopy Perturbation Method solutions \(u(x,t)\) for source function \(f(x)\) at different \(\alpha\) values.

5.2. Example 2. We consider problem as follows:

\[
D_t^\alpha u(x,t) = \frac{1}{2} x^2 u_{xx}(x,t) + f(t),
\]  

(71)
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with the following initial and boundary conditions:

\[ x > 0, \]
\[ t > 0, \]
\[ 0 < \alpha \leq 1 \]
\[ u(x, 0) = f_1(x) = x^2 + \frac{1}{2} \]  
(72)
\[ u(0, t) = h_1(t) = \frac{1}{2} e^{2t} \]
\[ u_x(0, t) = h_2(t) = 0, \]

where \( h_1(t) \) and \( h_2(t) \) are functions of space whose bases are \( \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \). Therefore, we obtain

\[ \omega(t) = e^{2t}. \]  
(74)

In order to define the unknown source function taking over the boundary conditions, we are taking \( h_1(t) \) and \( h_2(t) \) functions of Taylor series expansion for the space whose bases are \( \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \). Therefore, we obtain

\[ \omega(t) = e^{2t}. \]

In the definition \( D_t^\alpha u(t) = D_t^\alpha f(t) \), finally we obtain the source function

\[ f(t) = t^{-\alpha} E_{1,1-\alpha}(2t), \]  
(75)

where \( E_{1,1-\alpha}(2t) \) is Mittag-Leffler function with two parameters given as [23].

Figure 3 shows the Homotopy Perturbation Method solutions \( u(x, t) \) for source function \( f(t) \) at \( \alpha = 1/2 \) value.

6. Conclusion

Being effortless and also simple to apply, we can say that the new Homotopy Perturbation Method is an effective method

and has appropriate technique to find the analytic solution to inverse problems and many complex problems. Matlab has been used for presenting graph of solution in the present paper.

Nomenclature

\[ p: \] Homotopy parameter
\[ H(V, p): \] Homotopy function
\[ \Gamma: \] Gamma function
\[ u(x, t): \] Diffusion
\[ D_t^\alpha u(x, t): \] Diffusion with Caputo-time-fractional derivative
\[ J_t^\alpha u(x, t): \] Diffusion with Riemann-Liouville fractional integral
\[ f(x, t): \] Source function of \( x, t \)
\[ f(x): \] Source function of \( x \)
\[ f(t): \] Source function of \( t \)
\[ E_{\alpha}(z): \] Mittag-Leffler function.

Competing Interests

The author declares no competing interests.

References


