Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2016, Article ID 7304659, 7 pages
http://dx.doi.org/10.1155/2016/7304659

Research Article

Similarity Solutions for Multiterm Time-Fractional Diffusion Equation

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Received 12 January 2016; Revised 19 March 2016; Accepted 29 March 2016

Academic Editor: Xiao-Jun Yang

Similarity method is employed to solve multiterm time-fractional diffusion equation. The orders of the fractional derivatives belong to the interval (0, 1] and are defined in the Caputo sense. We illustrate how the problem is reduced from a multiterm two-variable fractional partial differential equation to a multiterm ordinary fractional differential equation. Power series solution is obtained for the resulting ordinary problem and the convergence of the series solution is discussed. Based on the obtained results, we propose a definition for a multiterm error function with generalized coefficients.

1. Introduction

Fractional differential equations (FDEs) appear in modeling many problems in the fields of science and engineering [1–3]. These problems are modeled using different types of fractional derivative operators which include Riemann-Liouville definition [1], Caputo definition [4], Riesz definition [5], Riesz-Feller definition [6], and the modified Riemann-Liouville definition recently proposed by Jumarie [7].

To obtain analytic solutions to fractional partial differential equations (FPDEs), two methods have been basically used: the first method is the application of both Laplace and Fourier transforms and the second method is the separation of variables technique [1]. But recently several semianalytic methods have been also utilized to present series solution to FPDEs such as Adomian decomposition method [8, 9], homotopy analysis method [10, 11], homotopy perturbation method [12, 13], variational iteration method [14, 15], and fractional differential transformation method [16, 17].

Fractional diffusion models are formulated using fractional derivative operators to replace regular derivatives. Different forms of fractional diffusion equations have been widely researched. For example, a time-fractional diffusion equation has been explicitly introduced in physics by Nigmatullin [18] to describe diffusion in special types of porous media which exhibit a fractal geometry. Giona et al. [19] also presented a time-fractional diffusion equation that describes relaxation phenomena in complex viscoelastic materials. Wyss [20] considered the time-fractional diffusion and wave equations and obtained the solution in closed form in terms of Fox functions. Gorenflo et al. [21] used the similarity method and the Laplace transform method to obtain the scale invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. Recently, fractional diffusion equations have been modeled and studied via local fractional derivatives as in the work of Liu et al. [22] and Zhao et al. [23].

Fractional diffusion equations have been also utilized to model anomalous diffusion where a particle plume spreads faster than predicted by classical models and may exhibit significant asymmetry. The anomalous diffusion processes differ from regular diffusion in that the dispersion of particles proceeds faster (superdiffusion) or slower (subdiffision) than for the regular case. Anomalous diffusion is one of the most popular phenomena in the theory of random walks and transport processes [24–26]. Also anomalous diffusion is a vast and exciting topic, in particular for "crowded" biological systems, where the transport of molecules plays a central functional role [27]. Lenzi et al. investigate the solutions for a fractional diffusion equation subjected to boundary

conditions which can be connected to adsorption-desorption processes; the analytical solutions were obtained using the Green function approach [28]. Prehl et al. in [29] deal with diffusion equation with a focus on space-fractional diffusion equation.

The large number of applications that are modeled via fractional diffusion equations motivated mathematicians to develop and apply several techniques to obtain analytic and approximate solutions to this class of FPDEs. Some examples of this work include studying the existence and uniqueness of solution [30], presenting the fundamental solution to some types of fractional diffusion equations [31, 32], applying Green's function approach [33], obtaining approximate solution via pseudospectral scheme [34], solution via radial basis functions [35], stochastic solution via Monte Carlo simulation [36], approximate solution based on the shifted Legendre-tau technique [37], solution by integral transform method [38], numerical solution by finite difference scheme [39], approximate analytical solution by homotopy analysis method [40], approximate analytical solution by optimal homotopy analysis method [41], approximate analytical solution via Adomian method [42], approximate analytical solution by variational iteration method [43], and approximate analytical solution via generalized differential transform method [44].

Multiterm FPDEs provide more generalizations to fractional order models. Thus many authors have dealt with them either analytically or numerically. For example, the authors of [45] developed a direct solution technique for solving the linear multiterm FDEs with constant coefficients using a spectral tau method. In [46], an effective finite element method is presented for the multiterm time-space Riesz fractional advection-diffusion equations. A numerical method is proposed in [47] to reduce the multiterm FDEs to a system of algebraic equations based on Bernstein polynomials basis. Also in [48], the authors present a numerical method for solving the multiterm time-fractional wave-diffusion equations. A study on the nonhomogeneous generalized multiterm fractional heat propagation and fractional diffusion-convection equation in three-dimensional space is presented in [49].

Though symmetry methods have been applied to different types of linear and nonlinear partial differential equations, the research on using these methods for solving FPDEs is still in the initial stage. The work reported on the Riemann-Liouville fractional derivative includes [50] where scaling transformations are derived for the time-fractional heat equation to reduce it to an ordinary FDE but with Erdelyi-Kober fractional differential operator. Also, similarity solutions for the time-fractional nonlinear conduction equations are presented in [51] to reduce the considered problems to ordinary FDEs that are solved by analytic and numerical techniques. The fractional derivative proposed by Jumarie is studied in [52] where the Lie group method is applied to a space-time-fractional diffusion equation. Choksi and Timol [53] derived similarity solution for fractional diffusion-wave equation using Lie group transformation and Erdelyi-Kober fractional differential operator. Through the invariants of the group of scaling transformations, Duan et al. [54] derived the integrodifferential equation for the similarity variable; then, by virtue of Mellin transform, the fundamental solution of the FDE was expressed in terms of Fox functions. Finally, El Kinani and Ouhadan [55] used Lie symmetry analysis to reduce the number of independent variables of time-fractional partial differential equations; then they employed symmetry properties to construct some exact solutions.

In this work, we use similarity methods to solve multiterm time-fractional diffusion equation with fractional derivatives defined in Caputo sense. The FPDE is reduced to an ordinary FDE which is solved by the power series method. This paper is arranged as follows. The definition and properties of Caputo fractional derivative are listed in Section 2. In Section 3, we illustrate how the similarity method technique is employed to transform a multiterm FPDE into a multiterm FDE. In Section 4, power series method is applied to solve the obtained FDE. The conclusion of this work is summarized in Section 5.

2. Fractional Calculus

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t) \in C_\mu$, $\mu \geq -1$ is defined as

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0,\ t > 0, \\ J^0 f(t) &= f(t). \end{aligned} \tag{1}$$

Definition 3. The fractional derivative in Caputo sense of $f(t) \in C_{-1}^m$, $m \in \mathbb{N}$, $t > 0$ is defined as [1]

$$D_t^\beta f(t) = \begin{cases} J^{m-\beta} \dfrac{d^m}{dt^m} f(t), & m-1 < \beta < m, \\ \dfrac{d^m}{dt^m} f(t), & \beta = m. \end{cases} \tag{2}$$

One basic property of Caputo fractional derivative is

$$D_t^\beta t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\beta+1)} t^{\mu-\beta}, \quad \mu \geq \beta. \tag{3}$$

For more details on Caputo fractional derivative definition and properties see [1, 56, 57].

3. Similarity Method Solution

In this section, we illustrate the technique for using similarity methods in solving multiterm FPDEs with Caputo fractional derivative.

Problem 4. Consider the following problem:

$$\frac{\partial^2 u}{\partial x^2} = \sum_{i=1}^{r} C_i t^{\alpha_i+2} D_t^{\alpha_i} u, \tag{4}$$

where C_i are constants, with an initial condition $u(x,0) = 0$. Here $D_t^{\alpha_i} u = (\partial^{\alpha_i}/\partial t^{\alpha_i})u$ denotes the partial fractional derivative of order α_i of $u = u(x,t)$ with respect to the time variable t in the Caputo sense. To solve (4), first we perform its scaling transformation using similarity methods; see [51, 58]. Consider the new independent and dependent variables denoted by $\bar{t}$, $\bar{x}$, and $\bar{u}$ defined in the following way:

$$\begin{aligned} t &= \lambda^n \bar{t}, \\ x &= \lambda^p \bar{x}, \\ u &= \lambda^q \bar{u}, \end{aligned} \tag{5}$$

where λ is called the scaling parameter and p, q, and n are arbitrary constants to be determined such that (4) remains invariant under this transformation. From Caputo definition for $0 < \alpha_i \le 1$, one may easily verify that

$$\begin{aligned} \frac{\partial^{\alpha_i} u}{\partial t^{\alpha_i}} &= \frac{1}{\Gamma(1-\alpha_i)} \int_0^t \frac{1}{(t-\tau)^{\alpha_i}} \frac{\partial u(x,\tau)}{\partial \tau} d\tau \\ &= \frac{1}{\Gamma(1-\alpha_i)} \int_0^{\bar{t}} \frac{\lambda^q}{(\lambda^n \bar{t} - \lambda^n \bar{\tau})^{\alpha_i}} \frac{\partial \bar{u}(\bar{x},\bar{\tau})}{\partial \bar{\tau}} \lambda \, d\bar{\tau} \\ &= \lambda^{q-n\alpha_i} \frac{\partial^{\alpha_i} \bar{u}}{\partial \bar{t}^{\alpha_i}}, \quad i = 1, 2, \ldots, r, \end{aligned} \tag{6}$$

where $\tau = \lambda^n \bar{\tau}$. Also, for $\partial^2 u/\partial x^2$, we have

$$\frac{\partial^2 u}{\partial x^2} = \lambda^{q-2p} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}. \tag{7}$$

Hence, by substituting (6) and (7) into (4), we get

$$\lambda^{q-2p} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \sum_{i=1}^{r} C_i \lambda^{n(\alpha_i+2)} \bar{t}^{(\alpha_i+2)} \lambda^{q-n\alpha_i} \frac{\partial^{\alpha_i} \bar{u}}{\partial \bar{t}^{\alpha_i}}. \tag{8}$$

From (8), it is clear that, by setting $n = 1$ and $p = -1$, then (4) is invariant under transformation (5). The characteristic equation associated with transformation (5) is given by

$$\frac{du}{qu} = \frac{dx}{px} = \frac{dt}{nt}. \tag{9}$$

At $q = 0$, this shows that $u(x,t)$ can be expressed as

$$u(x,t) = f(\zeta), \tag{10}$$

where $\zeta = tx$.

By using formula (10), we have

$$\begin{aligned} \frac{\partial^{\alpha_i} u}{\partial t^{\alpha_i}} &= \frac{1}{\Gamma(1-\alpha_i)} \int_0^t \frac{1}{(t-\tau)^{\alpha_i}} \frac{\partial u(x,\tau)}{\partial \tau} d\tau \\ &= \frac{1}{\Gamma(1-\alpha_i)} \int_0^{\zeta} \frac{f'(\zeta)\, x}{(\zeta/x - y/x)^{\alpha_i}} x^{-1} dy \\ &= \frac{1}{\Gamma(1-\alpha_i)} \int_0^{\zeta} \frac{f'(\zeta)}{x^{-\alpha_i} (\zeta - y)^{\alpha_i}} dy = x^{\alpha_i} \frac{d^{\alpha_i} f}{d\zeta^{\alpha_i}}, \\ &\qquad i = 1, 2, \ldots, r, \end{aligned} \tag{11}$$

where $y = \tau x$.

Again, by using formula (10), we have

$$\frac{\partial^2 u}{\partial x^2} = \zeta^2 x^{-2} \frac{d^2 f}{d\zeta^2}. \tag{12}$$

Substituting (11) and (12) into (4), the resulting FDE is given by

$$\frac{d^2 f}{d\zeta^2} = \sum_{i=1}^{r} C_i \zeta^{\alpha_i} \frac{d^{\alpha_i} f}{d\zeta^{\alpha_i}}, \tag{13}$$

with the initial condition $f(0) = 0$.

4. Power Series Solution

In this section, we develop the series solution of (13). Let

$$f(\zeta) = \sum_{\ell=0}^{\infty} a_\ell \zeta^\ell. \tag{14}$$

Using (3), we have

$$\begin{aligned} \sum_{i=1}^{r} \frac{d^{\alpha_i} f}{d\zeta^{\alpha_i}} &= \sum_{i=1}^{r} a_1 \frac{\Gamma(2)}{\Gamma(2-\alpha_i)} \zeta^{1-\alpha_i} + a_2 \frac{\Gamma(3)}{\Gamma(3-\alpha_i)} \zeta^{2-\alpha_i} \\ &\quad + a_3 \frac{\Gamma(4)}{\Gamma(4-\alpha_i)} \zeta^{3-\alpha_i} + \cdots, \end{aligned} \tag{15}$$

$$\frac{d^2 f}{d\zeta^2} = \sum_{\ell=2}^{\infty} \ell(\ell-1) a_\ell \zeta^{\ell-2}. \tag{16}$$

Substituting (15) and (16) into (13) and comparing the coefficients of identical powers on both sides, we get

$$\begin{aligned} a_{2k} &= 0, \quad k = 0, 1, 2, \ldots \\ a_3 &= \frac{1}{3!} \sum_{i=1}^{r} C_i \frac{\Gamma(2)}{\Gamma(2-\alpha_i)} a_1, \\ a_5 &= \frac{1}{5!} \left(\sum_{i=1}^{r} C_i \frac{\Gamma(4)}{\Gamma(4-\alpha_i)} \right) \left(\sum_{i=1}^{r} C_i \frac{\Gamma(2)}{\Gamma(2-\alpha_i)} \right) a_1, \\ &\vdots \end{aligned} \tag{17}$$

For the initial condition $f(0) = 0$, we get $a_0 = 0$. Hence, $f(\zeta)$ takes the form

$$f(\zeta) = \zeta + \sum_{\ell=1}^{\infty} \frac{1}{(2\ell+1)!} \zeta^{2\ell+1} \prod_{i=1}^{\ell} \sum_{k=1}^{r} C_k \frac{\Gamma(2i)}{\Gamma(2i-\alpha_k)}. \tag{18}$$

Based on these results, we propose a definition for multiterm error function with generalized coefficients and discuss its convergence.

Definition 5. A multiterm error function with generalized coefficients is defined in the form

$$\operatorname{erf}(\zeta;\tau;\alpha_1,\alpha_2,\ldots,\alpha_r) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} a_{2n+1}\zeta^{2n+1}, \tag{19}$$

where

$$a_1 = 1,$$
$$a_{2n+1} = \frac{\tau^n}{(2n+1)!}\prod_{i=1}^{n}\sum_{k=1}^{r} C_k \frac{\Gamma(2i)}{\Gamma(2i-\alpha_k)}, \quad n = 1,2,\ldots. \tag{20}$$

By this definition, the solution of the ordinary FDE obtained in (13) is given by

$$f(\zeta) = c_1 + c_2 \operatorname{erf}(\zeta;1;\alpha_1,\alpha_2,\ldots,\alpha_r), \tag{21}$$

where c_1 and c_2 are arbitrary constants.

To check the radius of convergence of the multiterm error function with generalized coefficients (19) and (20), we evaluate the following limit:

$$\begin{aligned}
\lim_{\ell\to\infty}\left|\frac{a_{2\ell+3}}{a_{2\ell+1}}\right| &= \lim_{\ell\to\infty}\left|\frac{\zeta^{2\ell+3}}{\zeta^{2\ell+1}}\right| \\
&\cdot\left|\frac{\tau^{\ell+1}(2\ell+1)!\prod_{i=1}^{\ell+1}\sum_{k=1}^{r}C_k\left(\Gamma(2i)/\Gamma(2i-\alpha_k)\right)}{\tau^{\ell}(2\ell+3)!\prod_{i=1}^{\ell}\sum_{k=1}^{r}C_k\left(\Gamma(2i)/\Gamma(2i-\alpha_k)\right)}\right| \\
&= \lim_{\ell\to\infty}\left|\zeta^2\right| \\
&\cdot\left|\frac{\tau(2\ell+1)!\Gamma(2\ell+2)}{(2\ell+3)!}\sum_{k=1}^{r}C_k\frac{1}{\Gamma(2\ell+2-\alpha_k)}\right| \\
&= \lim_{\ell\to\infty}\left|\zeta^2\right| \\
&\cdot\left|\frac{\tau\Gamma(2\ell+2)}{(2\ell+3)(2\ell+2)}\sum_{k=1}^{r}C_k\frac{1}{\Gamma(2\ell+2-\alpha_k)}\right| = 0.
\end{aligned} \tag{22}$$

Hence, the series solution converges for all ζ.

We consider the case $r = 3$ in (18). Figures 1, 2, and 3 illustrate the effect of changing the orders of fractional derivatives α_1, α_2, and α_3 on the behavior of Taylor series representation of the solution function $f(\zeta)$ at different values of the constants C_i, $i = 1,2,3$.

In the case where $\sum_{i=1}^{r} C_i = -1/2$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_r = \alpha$, (13) is reduced to

$$\frac{d^2 f}{d\zeta^2} = \frac{-1}{2}\zeta^{\alpha}\frac{d^{\alpha} f}{d\zeta^{\alpha}}, \tag{23}$$

which has a solution of the form

$$f(\zeta) = k_1 + k_2 \operatorname{erf}\left(\zeta;\frac{-1}{2};\alpha\right), \tag{24}$$

where k_1 and k_2 are arbitrary constants and $\operatorname{erf}(\zeta;\tau;\alpha)$ is the error function with generalized coefficients defined in the form

$$\operatorname{erf}(\zeta;\tau;\alpha) = \frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty} a_{2m+1}\zeta^{2m+1}, \tag{25}$$

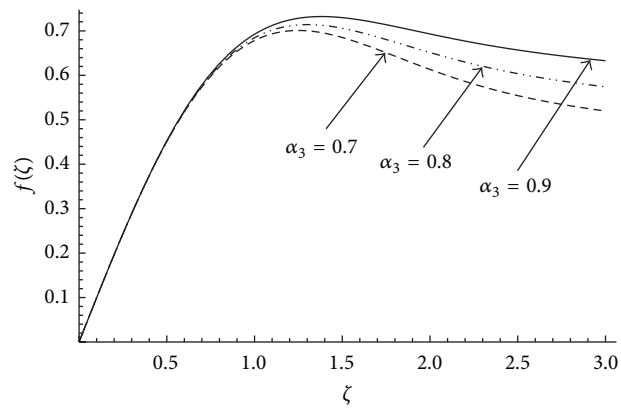

Figure 1: The effect of changing α_3 on the solution $f(\zeta)$ at $\alpha_1 = 0.9$, $\alpha_2 = 0.8$, $C_1 = -0.5$, $C_2 = -0.8$, and $C_3 = -1$.

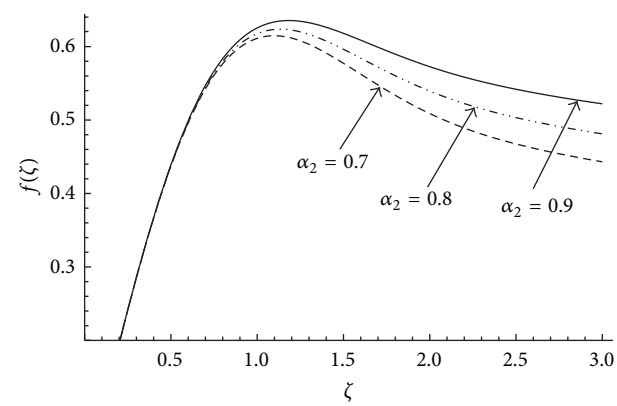

Figure 2: The effect of changing α_2 on the solution $f(\zeta)$ at $\alpha_1 = 0.9$, $\alpha_3 = 0.8$, $C_1 = -0.5$, $C_2 = -1$, and $C_3 = -1.5$.

where

$$a_1 = 1,$$
$$a_{2m+1} = \frac{\tau^m}{(2m+1)!}\prod_{i=1}^{m}\frac{\Gamma(2i)}{\Gamma(2i-\alpha)}, \quad m = 1,2,\ldots. \tag{26}$$

Hence for the condition $f(0) = 0$, k_1 equal zero.

Evidently, when we substitute $\alpha = 1$ into (24), the obtained series is the Taylor series of the classical integer-order error function $\operatorname{erf}(\zeta/2)$. This coincides with the fact that in this case (23) becomes the ordinary differential equation of the form $d^2 f/d\zeta^2 = (-1/2)\zeta(df/d\zeta)$ which has the general solution $f(\zeta) = k_1 + k_2\operatorname{erf}(\zeta/2)$, with $k_1 = 0$ to satisfy the initial condition $f(0) = 0$.

Figures 4 and 5 illustrate that, as α approaches one, the graph of the solution coincides with the graph of the classical error function $\operatorname{erf}(\zeta/2)$. Also, when α approaches zero in (23), in this case the obtained series is the Taylor series of the classical trigonometric sine function $\sin((1/\sqrt{2})\zeta)$. This coincides with the fact that in this case (23) becomes the ordinary differential equation $d^2 f/d\zeta^2 = (-1/2)f$ which has

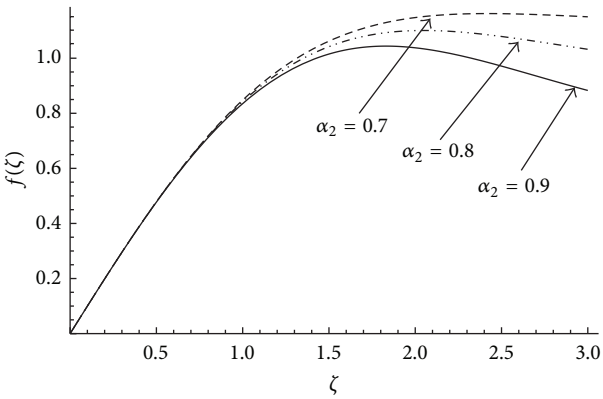

Figure 3: The effect of changing α_2 on the solution $f(\zeta)$ at $\alpha_1 = 0.9$, $\alpha_3 = 0.8$, $C_1 = -0.5$, $C_2 = 1$, and $C_3 = -1.5$.

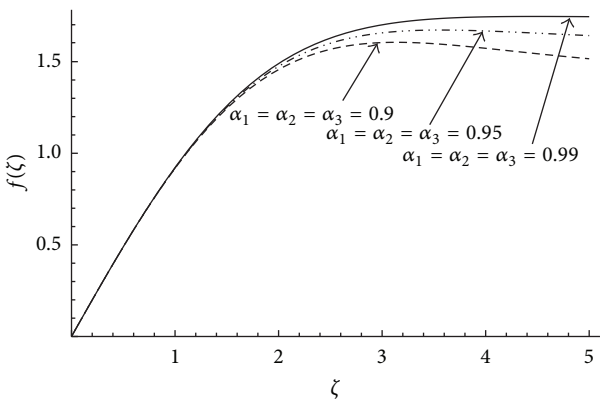

Figure 4: The solution $f(\zeta)$ in the case $\sum_{i=1}^{3} C_i = -1/2$ at different values of α.

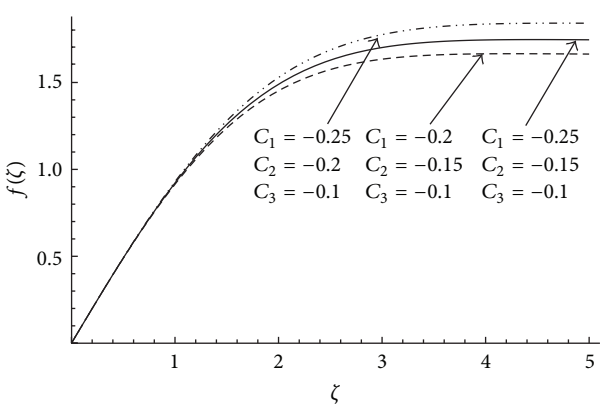

Figure 5: The solution $f(\zeta)$ at $\alpha_1 = \alpha_2 = \alpha_3 = 0.99$ and different values of C_i, $i = 1, 2, 3$.

the general solution of the form $f(\zeta) = k_1 \sin((1/\sqrt{2})\zeta) + k_2 \cos((1/\sqrt{2})\zeta)$, with $k_2 = 0$ to satisfy the initial condition $f(0) = 0$.

5. Conclusion

The similarity method is used to solve multiterm FPDEs where the fractional derivatives are given in Caputo sense. We considered time-fractional diffusion equation and illustrated that the similarity method successfully transforms this equation with two independent variables into an ordinary FDE in the same fractional derivative sense. Also it is noticed that the obtained FDE inherits the multiterm nature from the original FPDE.

The ordinary FDE is solved using power series expansion. We proved that the obtained series solution has an infinite radius of convergence. Thus, not only the solution obtained satisfies the equation, but also it can provide the exact solution at any point. So, any other semianalytic solution to this problem should be compared to this one.

Based on this series solution, a new definition is proposed for what we call the multiterm error function with generalized coefficients. This function is considered as a generalization of the classical error function and it coincides with it when the orders α_i of the fractional derivatives approach one and the sum of the coefficients C_i of the multiterm derivatives equals $-1/2$, whereas as the orders α_i approach zero, the function undergoes oscillation that tend to the trigonometric sine function.

The parameters involved in the definition of multiterm error function with generalized coefficients enrich this function. The figures presented in the considered case study illustrate that as the values of these parameters vary, the function exhibits different types of behaviors that can describe a wider range of applied models.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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