Research Article

Existence of Positive Solutions for Two-Point Boundary Value Problems of Nonlinear Finite Discrete Fractional Differential Equations and Its Application

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This paper is concerned with the two-point boundary value problems of nonlinear finite discrete fractional differential equations. On one hand, we discuss some new properties of the Green function. On the other hand, by using the main properties of Green function and the Krasnoselskii fixed point theorem on cones, some sufficient conditions for the existence of at least one or two positive solutions for the boundary value problem are established.

1. Introduction

Since 1974, the development of fractional differential equations is driven by an extremely wide application background. In those thirty years, a lot of papers and monographs have been produced and several international conferences were held about fractional calculus and fractional differential equation theory and application. At the same time, due to its application in physics, it also caused the fierce debate to classical laws of physics (see [1–11] and the references therein). However, it was basically confined to the fractional order differential equations and there is less literature available on paper concerned with discrete fractional differential equations.

In this paper, we will be interested in the nonlinear finite discrete FBVP given by

$$\begin{aligned} -\Delta^\nu y(t) &= f(t + \nu - 1, y(t + \nu - 1)), \\ y(\nu - 2) &= 0 = y(\nu + b + 1), \end{aligned} \quad (1)$$

where $1 < \nu \leq 2$, $t \in [0, b + 1]_{\mathbb{N}_0}$, $f : [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $b \in \mathbb{N}_0$. In Section 2, we will deduce some new properties of the Green function. In Section 3, by means of using the properties of the Green function and Krasnoselskii fixed point theorem on cones, the existence of two positive solutions and one positive solution of problem

(1) is established. At last, an example is given to illustrate the main results of this paper.

The difference with other literatures is that we get some new results about Green's function firstly, and then with the help of the nature of Green's function we introduce a new function, thus establishing the new operator equation. By discussing the fixed point of the new operator equation, we gain some existence results of solution in (1).

2. Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory.

Definition 1. The ν th fractional sum of a function f defined on $\mathbb{N}_a := \{a, a + 1, \dots\}$, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} = \sum_{s=a}^{t-\nu} (t-s-1)^{\nu-1} f(s), \quad (2)$$

where $t \in \{a + \nu, a + \nu + 1, \dots\} := \mathbb{N}_{a+\nu}$. We also define the ν th fractional difference, where $\nu > 0$ and $0 \leq N - 1 < \nu \leq N$ with $N \in \mathbb{N}$ to be

$$\Delta^\nu f(t) = \Delta^N \Delta^{-(N-\nu)} f(t), \quad (3)$$

where $t \in \mathbb{N}_{a+N-\nu}$.

Definition 2. We define

$$t^\nu = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)} \quad (4)$$

for any t and ν for which the right-hand side is defined. One appeals to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^\nu = 0$.

Lemma 3. Let t and ν be any numbers for which t^ν and $t^{\nu-1}$ are defined. Then,

$$\Delta t^\nu = \nu t^{\nu-1}. \quad (5)$$

Lemma 4. Let f be a real-valued function, and let $\mu, \nu > 0$. Then, for all t such that $t = a + \mu + \nu$, we have

$$\Delta^{-\nu} [\Delta^{-\mu} f(t)] = \Delta^{-(\nu+\mu)} f(t) = \Delta^{-\mu} [\Delta^{-\nu} f(t)], \quad (6)$$

where $t \in \mathbb{N}_{a+\nu+\mu}$.

Lemma 5. Let $0 \leq N-1 < \nu \leq N$. Then,

$$\begin{aligned} \Delta^{-\nu} [\Delta^\nu f(t)] &= f(t) + C_1 t^{\nu-1} + C_2 t^{\nu-2} + \dots \\ &\quad + C_N t^{\nu-N}, \end{aligned} \quad (7)$$

for some $C_i \in \mathbb{R}$ with $1 \leq i \leq N$.

Definition 6. Let E be a real Banach space. A nonempty convex closed set P is called a cone provided that (1) $au \in P$, for all $u \in P$; $a \geq 0$ and (2) $u, -u \in P$ implies $u = 0$.

Lemma 7. Let E be a Banach space and let $P \subset E$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Assume, further, that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ is a completely continuous operator. If either

$$(H_1) \quad \|Tu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1 \text{ and } \|Tu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2 \text{ or}$$

$$(H_2) \quad \|Tu\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2 \text{ and } \|Tu\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1$$

then T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Moreover, there is one paper [12] in which the following two statements have been shown.

Lemma 8 (see [13]). The unique solution of the FBVP (1) is given by

$$y(t) := \sum_{s=0}^{b+1} G(t, s) h(s + \nu - 1), \quad (8)$$

where $G(t, s)$ is Green's function for the problem

$$\begin{aligned} -\Delta^\nu y(t) &= h(t + \nu - 1), \\ y(\nu - 2) &= 0 = y(\nu + b + 1), \end{aligned} \quad (9)$$

where $1 < \nu \leq 2$, which is given by

$$G(t, s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{\nu-1}(\nu + b - s)^{\nu-1}}{(\nu + b + 1)^{\nu-1}} - (t - s - 1)^{\nu-1}, & 0 \leq s < t - \nu + 1 \leq b + 1, \\ \frac{t^{\nu-1}(\nu + b - s)^{\nu-1}}{(\nu + b + 1)^{\nu-1}}, & 0 \leq t - \nu + 1 < s \leq b + 1. \end{cases} \quad (10)$$

Lemma 9 (see [13]). Let $G(t, s)$ be given as in the statement of Lemma 8. Then, we find that

- (1) $G(t, s) > 0$ for each $(t, s) \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$;
- (2) $\max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} G(t, s) = G(s + \nu - 1, s)$, $s \in [0, b+1]_{\mathbb{N}_0}$;
- (3) there exists a number $\gamma \in (0, 1)$ such that

$$\begin{aligned} \min_{t \in [(b+\nu)/4, 3(b+\nu)/4]} G(t, s) &\geq \gamma \cdot \max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} G(t, s) \\ &= \gamma G(s + \nu - 1, s) \end{aligned} \quad (11)$$

for $s \in [0, b+1]_{\mathbb{N}_0}$.

Lemma 10. For given $s \in [0, b+1]_{\mathbb{N}_0}$, $G(t, s)$ is decreasing and $\Delta_t G(t, s)$ is increasing with respect to t for $0 \leq s < t - \nu + 1 \leq b + 1$, where $\Delta_t G(t, s) = dG(t, s)/dt$.

Proof. For $0 \leq s < t - \nu + 1 \leq b + 1$, we find that

$$\begin{aligned} \Delta_t G(t, s) &= \frac{\nu-1}{\Gamma(\nu)} \left[\frac{t^{\nu-2}(\nu + b - s)^{\nu-1}}{(\nu + b + 1)^{\nu-1}} \right. \\ &\quad \left. - (t - s - 1)^{\nu-2} \right] = \frac{(\nu-1)\Gamma(b+3)}{\Gamma(\nu)\Gamma(\nu+b+2)} \\ &\quad \cdot \left[\frac{\Gamma(t+1)\Gamma(\nu+b-s+1)}{\Gamma(t-\nu+3)\Gamma(b-s+2)} \right. \\ &\quad \left. - \frac{\Gamma(t-s)\Gamma(\nu+b+2)}{\Gamma(t-s-\nu+2)\Gamma(b+3)} \right]. \end{aligned} \quad (12)$$

Similar to the proof of [14], we have

$$\begin{aligned} \frac{\Gamma(t-s)\Gamma(\nu+b+2)\Gamma(t-\nu+3)\Gamma(b-s+2)}{\Gamma(t+1)\Gamma(\nu+b-s+1)\Gamma(t-s-\nu+2)\Gamma(b+3)} \\ > 1. \end{aligned} \quad (13)$$

So, it shows that

$$\Delta_t G(t, s) < 0, \quad 0 \leq s < t - \nu + 1 \leq b + 1. \quad (14)$$

Then, $G(t, s)$ is decreasing with respect to t for $0 \leq s < t - \nu + 1 \leq b + 1$.

This implies that

$$\begin{aligned} \Delta_{tt} G(t, s) &:= \Delta_t (\Delta_t G(t, s)) \\ &= \frac{(\nu - 1)(\nu - 2)}{\Gamma(\nu)} \left[\frac{t^{\nu-3} (\nu + b - s)^{\nu-1}}{(\nu + b + 1)^{\nu-1}} \right. \\ &\quad \left. - (t - s - 1)^{\nu-3} \right]. \end{aligned} \quad (15)$$

Similar to the proof of [14], we have

$$\begin{aligned} &\frac{\Gamma(t-s)\Gamma(\nu+b+2)\Gamma(t-\nu+4)\Gamma(b-s+2)}{\Gamma(t+1)\Gamma(\nu+b-s+1)\Gamma(t-s-\nu+3)\Gamma(b+3)} \\ &= \frac{t-\nu+3}{t-\nu-s+2} \\ &\cdot \frac{\Gamma(t-s)\Gamma(\nu+b+2)\Gamma(t-\nu+3)\Gamma(b-s+2)}{\Gamma(t+1)\Gamma(\nu+b-s+1)\Gamma(t-s-\nu+2)\Gamma(b+3)} \\ &> \frac{t-\nu+3}{t-\nu-s+2} \\ &\cdot \frac{\Gamma(t-s)\Gamma(\nu+b+2)\Gamma(t-\nu+3)\Gamma(b-s+2)}{\Gamma(t+1)\Gamma(\nu+b-s+1)\Gamma(t-s-\nu+2)\Gamma(b+3)} \\ &> 1. \end{aligned} \quad (16)$$

So, since $\nu - 2 \leq 0$, we have

$$\Delta_{tt} G(t, s) > 0, \quad 0 \leq s < t - \nu + 1 \leq b + 1. \quad (17)$$

Then, $\Delta_t G(t, s)$ is increasing with respect to t for $0 \leq s < t - \nu + 1 \leq b + 1$. $\square$

Lemma 11. For given $s \in [0, b + 1]_{\mathbb{N}_0}$, $G(t, s)/(b + \nu + 1 - t)$ is decreasing with respect to t for $0 \leq s < t - \nu + 1 \leq b + 1$.

Proof. By Lemma 10, for any given $s \in [0, b + 1]_{\mathbb{N}_0}$, $s < t - \nu + 1 \leq b + 1$, we have

$$\begin{aligned} G(t, s) &\geq G(b + \nu + 1, s) = 0, \quad s < t - \nu + 1, \\ \Delta_t G(t, s) &\leq \Delta_t G(\theta, s) = 0, \quad t - \nu + 1 \leq \theta \leq b + 1. \end{aligned} \quad (18)$$

Because

$$\begin{aligned} \frac{G(t, s)}{t - b - \nu - 1} &= \frac{G(t, s) - G(b + \nu + 1, s)}{t - b - \nu - 1} \\ &= \Delta_t G(\theta, s), \quad t - \nu + 1 < \theta < b + 1, \end{aligned} \quad (19)$$

we get

$$\begin{aligned} &\Delta_t \left[\frac{G(t, s)}{b + \nu + 1 - t} \right] \\ &= \frac{(b + \nu + 1 - t) \Delta_t G(t, s) - G(t, s) \Delta_t (b + \nu + 1 - t)}{(b + \nu + 1 - t)(b + \nu + 1 - t)} \\ &= \frac{\Delta_t G(t, s) + G(t, s) / (b + \nu + 1 - t)}{(b + \nu + 1 - t)} \\ &= \frac{\Delta_t G(t, s) - (G(b + \nu + 1, s) - G(t, s)) / (b + \nu + 1 - t)}{(b + \nu + 1 - t)} \\ &= \frac{\Delta_t G(t, s) - \Delta_t G(\theta, s)}{(b + \nu + 1 - t)} \leq 0. \end{aligned} \quad (20)$$

Namely, for given $s \in [0, b + 1]_{\mathbb{N}_0}$, $G(t, s)/(b + \nu + 1 - t)$ is decreasing with respect to t for $0 \leq s < t - \nu + 1 \leq b + 1$. $\square$

Theorem 12. The Green function $G(t, s)$ defined by Lemma 8 has the following: $G(t, s) = G(\nu + b - s, \nu + b - t)$ and for each $(t, s) \in [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \times [0, b + 1]_{\mathbb{N}_0}$,

$$\begin{aligned} &\frac{t^{\nu-1} (\nu + b - s)^{\nu-1} (b + \nu + 1 - t) (\nu - 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1} (b + 2) (b + 3)} \leq G(t, s) \\ &\leq \frac{t^{\nu-1} (b + \nu - s)^{\nu-1} (b + \nu + 1 - t)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}. \end{aligned} \quad (21)$$

Proof. By the definition of $G(t, s)$, it was obvious that $G(t, s) = G(\nu + b - s, \nu + b - t)$ for $s \in [0, b + 1]_{\mathbb{N}_0}$. By Lemma 11, for $0 \leq s < t - \nu + 1 \leq b + 1$, we get

$$\begin{aligned} &\frac{G(t, s)}{b + \nu + 1 - t} \leq \frac{G(s + \nu - 1, s)}{b + 2 - s} \\ &\leq \frac{(s + \nu - 1)^{\nu-1} (b + \nu - s)^{\nu-1}}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} \\ &\leq \frac{t^{\nu-1} (b + \nu - s)^{\nu-1}}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}, \\ &\frac{G(t, s)}{b + \nu + 1 - t} \geq \lim_{t \rightarrow (b + \nu + 1)^-} \frac{G(t, s)}{b + \nu + 1 - t} \\ &= \lim_{t \rightarrow (b + \nu + 1)^-} (-\Delta_t G(t, s)) \\ &= \frac{\nu - 1}{\Gamma(\nu)} (\nu + b - s)^{\nu-1} \frac{b - s + 1}{b - s + 2} \\ &\geq \frac{(\nu - 1) (\nu + b - s)^{\nu-1} t^{\nu-1}}{\Gamma(\nu) (b + 2) (b + 3) (\nu + b + 1)^{\nu-1}}. \end{aligned} \quad (22)$$

For $0 \leq t - \nu + 1 < s \leq b + 1$, we get

$$\begin{aligned} G(t, s) &= \frac{t^{\nu-1} (\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \\ &\leq \frac{t^{\nu-1} (b + \nu - s)^{\nu-1} (b + \nu + 1 - t)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}, \\ G(t, s) &= \frac{t^{\nu-1} (\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \\ &\geq \frac{t^{\nu-1} (\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \frac{(b + \nu + 1 - t) (\nu - 1)}{(b + 2) (b + 3)}, \end{aligned} \quad (23)$$

so

$$\begin{aligned} \frac{t^{\nu-1} (\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \frac{(b + \nu + 1 - t) (\nu - 1)}{(b + 2) (b + 3)} &\leq G(t, s) \\ &\leq \frac{t^{\nu-1} (b + \nu - s)^{\nu-1} (b + \nu + 1 - t)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}. \end{aligned} \quad (24)$$

Hence, for each $(t, s) \in [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \times [0, b + 1]_{\mathbb{N}_0}$,

$$\begin{aligned} \frac{t^{\nu-1} (\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \frac{(b + \nu + 1 - t) (\nu - 1)}{(b + 2) (b + 3)} &\leq G(t, s) \\ &\leq \frac{t^{\nu-1} (b + \nu - s)^{\nu-1} (b + \nu + 1 - t)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}. \end{aligned} \quad (25)$$

□

Theorem 13. The function $G^*(t, s) = (1/t^{\nu-1})G(t, s)$ has the following properties:

$$\begin{aligned} \frac{(\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \frac{(b + \nu + 1 - t) (\nu - 1)}{(b + 2) (b + 3)} &\leq G^*(t, s) \\ &\leq \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}. \end{aligned} \quad (26)$$

Proof. By Theorem 12, we get

$$\begin{aligned} \frac{t^{\nu-1} (\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \frac{(b + \nu + 1 - t) (\nu - 1)}{(b + 2) (b + 3)} &\leq G(t, s) \\ &= G(\nu + b - s, \nu + b - t) \\ &\leq \frac{(b + \nu - s)^{\nu-1} t^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}. \end{aligned} \quad (27)$$

Let $G^*(t, s) = (1/t^{\nu-1})G(t, s)$. It is clear that

$$\begin{aligned} \frac{(\nu + b - s)^{\nu-1}}{\Gamma(\nu) (\nu + b + 1)^{\nu-1}} \frac{(b + \nu + 1 - t) (\nu - 1)}{(b + 2) (b + 3)} &\leq G^*(t, s) \\ &\leq \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}}. \end{aligned} \quad (28)$$

It is worth noting that two inequalities of Theorem 13 are very useful in the research of positive solutions to Dirichlet-type

boundary value problems of discrete fractional differential equation (see the proofs of Theorems 15 and 17). Theorems 12 and 13 extend the results of integer-order Dirichlet boundary value problems. □

3. Main Results and Proofs

In this section, we will give the existence results of positive solution to the boundary value problem (1) on the basis of Theorem 13 and Lemma 7 and make the following assumptions:

(H_0) There exist $g \in C([0, +\infty) \times [0, +\infty))$, $q_1, q_2 \in C([\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}} \times (0, +\infty))$ such that

$$\begin{aligned} q_1(t) g(y) &\leq f\left(t, \frac{y}{t^{\nu-1}}\right) \leq q_2(t) g(y), \\ t &\in [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}}, \quad y \in [0, +\infty); \end{aligned} \quad (29)$$

here

$$\sum_{s=0}^{b+1} q_i(s + \nu - 1) < +\infty, \quad i = 1, 2. \quad (30)$$

For convenience, we introduce the following notations:

$$\begin{aligned} g_\infty &= \limsup_{y \rightarrow \infty} \frac{g(y)}{y}, \\ g_0 &= \liminf_{y \rightarrow 0} \frac{g(y)}{y}. \end{aligned} \quad (31)$$

(H_1) $g_0 = \infty, g_\infty = \infty$.

(H_2) $g_0 = 0, g_\infty = 0$.

(H_3) There exists $p > 0$ such that $g(y) < M_1 p$ for $0 \leq y \leq p$, where $M_1 = [(1/\Gamma(\nu)(b + \nu + 1)^{\nu-1}) \sum_{s=0}^{b+1} (b + \nu - s)^{\nu-1} (s + 1) q_2(s + \nu - 1)]^{-1}$.

(H_4) There exists $p > 0$ such that $g(y) > M_2 p$ for $p(\alpha - 1)/16 \leq y \leq p$, where $M_2 = [\sum_{s=(b+\nu)/4}^{3(b+\nu)/4} G^*((b + \nu)/2, s) q_1(s + \nu - 1)]^{-1}$.

We note that y is a solution of the boundary value problem (1) if and only if y is a fixed point of the operator

$$Ty := \sum_{s=0}^{b+1} G^*(t, s) f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right), \quad (32)$$

where G^* is Green's function derived in this paper and $T : B \rightarrow B$ and B is the Banach space

$$\begin{aligned} B &:= \left\{ y : [\nu - 2, \nu + b + 1]_{\mathbb{N}_{\nu-2}} \longrightarrow \mathbb{R} \mid y(\nu - 2) = 0 \right. \\ &\quad \left. = y(\nu + b + 1) \right\} \end{aligned} \quad (33)$$

equipped with the norm

$$\|y(t)\| := \max_{t \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}} |y(t)|. \quad (34)$$

Define the cone $K \subset B$ by

$$K = \left\{ y \in B : y(t) \geq 0, \quad y(t) \geq \frac{(\nu-1)(b+\nu+1-t)}{(b+2)^2(b+3)} \|y\| \right\}. \quad (35)$$

Lemma 14. Assume $T : K \rightarrow B$ is the operator defined by

$$Ty := \sum_{s=0}^{b+1} G^*(t, s) f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right). \quad (36)$$

Then, the mapping $T : K \rightarrow K$ is completely continuous.

Proof ($K \subset B$). Hence, $T : K \rightarrow K$. On one hand,

$$\begin{aligned} (Ty)(t) &= \sum_{s=0}^{b+1} G^*(t, s) f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right) \\ &\geq \frac{(b + \nu + 1 - t)(\nu - 1)}{(b + 2)(b + 3)} \\ &\quad \cdot \sum_{s=0}^{b+1} \frac{(\nu + b - s)^{\nu-1}}{\Gamma(\nu)(\nu + b + 1)^{\nu-1}} f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right). \end{aligned} \quad (37)$$

On the other hand,

$$\begin{aligned} \|Ty\| &= \max_{t \in [\nu-1, \nu+b+1]_{\mathbb{N}_{\nu-1}}} |Ty(t)| \\ &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1}(s + 1)}{\Gamma(\nu)(b + \nu + 1)^{\nu-1}} f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right) \leq (b + 2) \\ &\quad \cdot \sum_{s=0}^{b+1} \frac{(\nu + b - s)^{\nu-1}}{\Gamma(\nu)(\nu + b + 1)^{\nu-1}} f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right). \end{aligned} \quad (38)$$

Then, we get

$$(Ty)(t) \geq \frac{(\nu-1)(b+\nu+1-t)}{(b+2)^2(b+3)} \|Ty\|. \quad (39)$$

And by means of the expression of $G^*(t, s)$, there are clearly $(Ty)(t) \geq 0$ whenever $y \in K$, whence $Ty \in K$.

Next, we show that T is uniformly bounded.

Let $D \subset K$ be bounded; that is, there exists a positive real number $L > 0$ such that $\|y\| \leq L$ for all $y \in D$.

Let $M = \max_{0 \leq y \leq L} |g(y)| + 1$; then for $y \in D$, by Theorem 13 and (H_0) , we have

$$\begin{aligned} |Ty(t)| &\leq \sum_{s=0}^{b+1} \left| G^*(t, s) f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right) \right| \\ &\leq \sum_{s=0}^{b+1} |G^*(t, s) q_2(s + \nu - 1) g(y(s + \nu - 1))| \quad (40) \\ &\leq M \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1}(s + 1)}{\Gamma(\nu)(b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1). \end{aligned}$$

Thus, $T(D)$ is bounded.

Finally, we prove that T is equicontinuous.

Note that $G^*(t, s)$ is uniformly continuous on $(t, s) \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times [0, b+1]_{\mathbb{N}_0}$; then, for $\varepsilon > 0$, each $u \in D$, $t_1, t_2 \in [\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}}$, $t_1 < t_2$, there exists $\eta > 0$ such that $|G^*(t_2, s) - G^*(t_1, s)| < \varepsilon/M \sum_{s=0}^{b+1} q_2(s + \nu - 1)$ for $t_2 - t_1 < \eta$. Then, we get

$$\begin{aligned} |Ty(t_2) - Ty(t_1)| &= \left| \sum_{s=0}^{b+1} G^*(t_2, s) f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right) - \sum_{s=0}^{b+1} G^*(t_1, s) f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right) \right| \\ &\leq \sum_{s=0}^{b+1} |G^*(t_2, s) - G^*(t_1, s)| \\ &\quad \cdot f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right) \\ &\leq \sum_{s=0}^{b+1} |G^*(t_2, s) - G^*(t_1, s)| \\ &\quad \cdot |q_2(s + \nu - 1) g(y(s + \nu - 1))| < \varepsilon. \end{aligned} \quad (41)$$

Hence, we get

$$|Ty(t_2) - Ty(t_1)| < \varepsilon. \quad (42)$$

In view of the Arzela-Ascoli theorem; it is easy to see that $T : K \rightarrow K$ is completely continuous. $\square$

Theorem 15. Suppose that (H_0) , (H_1) , and (H_3) are satisfied. Then, problem (1) has two positive solutions.

Proof. Assume that (H_1) holds. Since $g_0 = \infty$, there exists $r \in (0, p)$ such that $g(y) \geq Ny$ for $0 < y < r$, where $N > 0$ satisfies

$$\begin{aligned} &\sum_{s=(b+\nu)/4-\nu+1}^{3(b+\nu)/4-\nu+1} G^*\left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s\right) q_1(s + \nu - 1) N \\ &\quad \cdot \frac{(\nu-1)\nu}{(b+2)^2(b+3)} \geq 1. \end{aligned} \quad (43)$$

Let $\Omega_r = \{y \in B \mid \|y\| < r\}$ for $y \in K \cap \partial\Omega_r$; we note that $\{[(b+1)/2] + \nu\} \subset [(b+\nu)/4, 3(b+\nu)/4]$, where $[\cdot]$ is the usual so-called floor function. So we get

$$\begin{aligned}
\|Ty\| &\geq Ty \left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu \right) \\
&= \sum_{s=0}^{b+1} G^* \left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) \\
&\quad \cdot f \left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}} \right) \\
&\geq \sum_{s=0}^{b+1} G^* \left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) q_1(s + \nu - 1) \\
&\quad \cdot g(y(s + \nu - 1)) \geq \sum_{s=0}^{b+1} G^* \left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) \\
&\quad \cdot q_1(s + \nu - 1) Ny(s + \nu - 1) \\
&\geq \sum_{s=(b+\nu)/4-\nu+1}^{3(b+\nu)/4-\nu+1} G^* \left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) \\
&\quad \cdot q_1(s + \nu - 1) N \frac{(\nu-1)(b + \nu + 1 - t)}{(b+2)^2(b+3)} \|y\| \\
&\geq \sum_{s=(b+\nu)/4-\nu+1}^{3(b+\nu)/4-\nu+1} G^* \left(\left\lfloor \frac{b+1}{2} \right\rfloor + \nu, s \right) \\
&\quad \cdot q_1(s + \nu - 1) N \frac{(\nu-1)\nu}{(b+2)^2(b+3)} \|y\| \geq \|y\|.
\end{aligned} \tag{44}$$

Then, this implies $\|Ty\| \geq \|y\|$ for $y \in K \cap \partial\Omega_r$.

Now, let $\Omega_p = \{y \in B \mid \|y\| < p\}$ for $y \in K \cap \partial\Omega_p$. It follows from (H_3) that we get

$$\begin{aligned}
\|Ty\| &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1}(s+1)}{\Gamma(\nu)(b + \nu + 1)^{\nu-1}} f \left(s + \nu \right. \\
&\quad \left. - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}} \right) \leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1}(s+1)}{\Gamma(\nu)(b + \nu + 1)^{\nu-1}} q_2(s \\
&\quad + \nu - 1) g(y(s + \nu - 1)) \\
&< \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1}(s+1)}{\Gamma(\nu)(b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) M_1 p = p \\
&= \|y\|.
\end{aligned} \tag{45}$$

Then, it implies that $\|Ty\| < \|y\|$ for $y \in K \cap \partial\Omega_p$.

Therefore, by Lemma 7, it follows that T has a fixed point in $K \cap (\partial\Omega_p \setminus \Omega_p)$, such that $r \leq \|y_1\| \leq p$.

Further, since $g_\infty = \infty$, there exists $M^* > 0$ such that $g(y) \geq \rho y$ for $y > M^*$, where $\rho > 0$ is chosen so that $(\rho(\nu-1)((b+\nu)/4+1)/(b+2)^2(b+3))G^*((b+\nu)/2, s)q_1(s+\nu-1) \geq 1$.

Let $R = p + (4(b+2)^2/(\nu-1)(b+\nu))M^*$ and $\Omega_R = \{y \in B \mid \|y\| < R\}$, for $y \in K \cap \partial\Omega_R$; we get

$$\begin{aligned}
y(t) &\geq \frac{(\nu-1)((b+\nu)/4+1)}{(b+2)^2(b+3)} \|y\| > M^*, \\
t &\in \left[\frac{b+\nu}{4}, \frac{3(b+\nu)}{4} \right],
\end{aligned} \tag{46}$$

and so

$$\begin{aligned}
\|Ty\| &\geq Ty \left(\frac{b+\nu}{2} \right) = \sum_{s=0}^{b+1} G^* \left(\frac{b+\nu}{2}, s \right) \\
&\quad \cdot f \left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}} \right) \geq \sum_{s=0}^{b+1} G^* \left(\frac{b+\nu}{2}, s \right) \\
&\quad \cdot q_1(s + \nu - 1) g(y(s + \nu - 1)) \\
&\geq \sum_{s=(b+\nu)/4-\nu+1}^{3(b+\nu)/4-\nu+1} G^* \left(\frac{b+\nu}{2}, s \right) q_1(s + \nu - 1) \\
&\quad \cdot g(y(s + \nu - 1)) \geq \sum_{s=(b+\nu)/4-\nu+1}^{3(b+\nu)/4-\nu+1} G^* \left(\frac{b+\nu}{2}, s \right) \\
&\quad \cdot q_1(s + \nu - 1) \rho y(s + \nu - 1) \\
&\geq \frac{\rho(\nu-1)((b+\nu)/4+1)}{(b+2)^2(b+3)} G^* \left(\frac{b+\nu}{2}, s \right) q_1(s + \nu - 1) \|y\| \geq \|y\|.
\end{aligned} \tag{47}$$

Hence, for $y \in K \cap \partial\Omega_R$, $\|Ty\| \geq \|y\|$.

Thus, by Lemma 7, it follows that T has a fixed point y_2 in $K \cap (\partial\Omega_R \setminus \Omega_p)$ such that $p \leq \|y_2\| \leq R$.

Then, T has a fixed point y_1 in $K \cap (\partial\Omega_p \setminus \Omega_r)$ and a fixed point y_2 in $K \cap (\partial\Omega_R \setminus \Omega_p)$ such that $r \leq \|y_1\| \leq p \leq \|y_2\| \leq R$, and satisfy

$$\begin{aligned}
y_i(t) &= \sum_{s=0}^{b+1} G^*(t, s) f \left(s + \nu - 1, \frac{y_i(s + \nu - 1)}{s^{\nu-1}} \right), \\
t &\in [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}}, \quad i = 1, 2,
\end{aligned} \tag{48}$$

and $\|y_i\| \leq R$. It is clear that $u_i(t) = (1/t^{\nu-1})y_i(t)$ are two solutions of (1); that is,

$$\begin{aligned}
u_i(t) &= \sum_{s=0}^{b+1} G(t, s) f(s + \nu - 1, u_i(s + \nu - 1)), \\
t &\in [\nu - 1, \nu + b]_{\mathbb{N}_{\nu-1}}, \quad i = 1, 2.
\end{aligned} \tag{49}$$

Next, we will prove $u_i(0) = 0$ ($i = 1, 2$). By means of (H_0) , we get

$$\begin{aligned}
 & \lim_{t \rightarrow 0^-} u_i(t) \\
 &= \lim_{t \rightarrow 0} \inf_{s=0}^{b+1} G(t, s) f(s + \nu - 1, u_i(s + \nu - 1)) \\
 &= \lim_{t \rightarrow 0} \inf_{s=0}^{b+1} G(t, s) f\left(s + \nu - 1, \frac{y_i(s + \nu - 1)}{s^{\nu-1}}\right) \\
 &\leq \lim_{t \rightarrow 0} \inf_{s=0}^{b+1} G(t, s) q_2(s + \nu - 1) g(y_i(s + \nu - 1)) \\
 &\leq \lim_{t \rightarrow 0} \inf_{s=0}^{b+1} G(t, s) q_2(s + \nu - 1) \\
 &\cdot \max_{\|y_i\| \leq R} g(y_i(s + \nu - 1)) = 0, \quad i = 1, 2.
 \end{aligned} \tag{50}$$

Therefore $u_i(0) = 0$, $i = 1, 2$, and $u_i(t) = (1/t^{\nu-1})y_i(t)$ is a solution of problem (1). $\square$

Theorem 16. Suppose that (H_0) , (H_2) , and (H_4) are satisfied. Then, problem (1) has two positive solutions.

Proof. Assume that (H_2) holds. Since $g_0 = 0$, there exists $r \in (0, p)$ such that $g(y) \leq \varepsilon y$ for $0 < y < r$, where ε satisfy

$$\sum_{s=0}^{b+1} \frac{\varepsilon (b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \leq 1. \tag{51}$$

Let $\Omega_r = \{y \in B \mid \|y\| < r\}$ for $y \in K \cap \partial\Omega_r$; we get

$$\begin{aligned}
 Ty(t) &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} f\left(s + \nu - 1, \frac{y_i(s + \nu - 1)}{s^{\nu-1}}\right) \\
 &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \varepsilon y(s + \nu - 1) \\
 &\leq \sum_{s=0}^{b+1} \frac{\varepsilon (b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \|y\| \\
 &\leq \|y\|.
 \end{aligned} \tag{52}$$

Then, this implies that $\|Ty\| \leq \|y\|$ for $y \in K \cap \partial\Omega_r$.

Now, let $\Omega_p = \{y \in B \mid \|y\| < p\}$ for $y \in K \cap \partial\Omega_p$; then, $y(t) \geq ((\alpha - 1)/16)p$, $t \in [(b + \nu)/4, 3(b + \nu)/4]$; by (H_4) , we get

$$\begin{aligned}
 \|Ty\| &\geq Ty\left(\frac{b + \nu}{2}\right) \geq \sum_{s=(b+\nu)/4}^{3(b+\nu)/4} G^*\left(\frac{b + \nu}{2}, s\right) \\
 &\cdot f\left(s + \nu - 1, \frac{y(s + \nu - 1)}{s^{\nu-1}}\right)
 \end{aligned}$$

$$\begin{aligned}
 &\geq \sum_{s=(b+\nu)/4}^{3(b+\nu)/4} G^*\left(\frac{b + \nu}{2}, s\right) q_1(s + \nu - 1) \\
 &\cdot g(y(s + \nu - 1)) > \sum_{s=(b+\nu)/4}^{3(b+\nu)/4} G^*\left(\frac{b + \nu}{2}, s\right) \\
 &\cdot q_1(s + \nu - 1) M_2 p = p = \|y\|.
 \end{aligned} \tag{53}$$

Then, we see that $\|Ty\| > \|y\|$ for $y \in K \cap \partial\Omega_p$.

As a result, by Lemma 7, we obtain that T has a fixed point in $K \cap (\partial\Omega_p \setminus \Omega_r)$, such that $r \leq \|y_1\| < p$.

Furthermore, because $g_\infty = 0$, there exists $M > 0$ such that $g(y) \leq \varepsilon y$ for $y > M$, where $\varepsilon > 0$ is chosen so that $\varepsilon \sum_{s=0}^{b+1} ((b + \nu - s)^{\nu-1} (s + 1) / \Gamma(\nu) (b + \nu + 1)^{\nu-1}) q_2(s + \nu - 1) < 1$.

Let $R = \max\{\max_{0 \leq y(s) \leq M} \{g(y)\} \sum_{s=0}^{b+1} ((b + \nu - s)^{\nu-1} (s + 1) / \Gamma(\nu) (b + \nu + 1)^{\nu-1}) q_2(s + \nu - 1) / (1 - \varepsilon \sum_{s=0}^{b+1} ((b + \nu - s)^{\nu-1} (s + 1) / \Gamma(\nu) (b + \nu + 1)^{\nu-1}) q_2(s + \nu - 1)), p + M\}$ and $\Omega_R = \{y \in B \mid \|y\| < R\}$, for $y \in K \cap \partial\Omega_R$; we get

$$\begin{aligned}
 Ty(t) &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} f\left(s + \nu - 1, \frac{y_i(s + \nu - 1)}{s^{\nu-1}}\right) \\
 &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \\
 &\cdot g(y(s + \nu - 1)) \\
 &\leq \left(\sum_{0 \leq y(s) \leq M} + \sum_{M \leq y(s) \leq R} \right) \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} \\
 &\cdot q_2(s + \nu - 1) g(y(s + \nu - 1)) \\
 &\leq \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \\
 &\cdot \max_{0 \leq y(s) \leq M} g(y(s + \nu - 1)) \\
 &+ \sum_{M \leq y(s) \leq R} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \varepsilon y(s + \nu - 1) \\
 &\leq \left(\max_{0 \leq y(s) \leq M} \{g(y)\} + \varepsilon \|y\| \right) \\
 &\cdot \sum_{s=0}^{b+1} \frac{(b + \nu - s)^{\nu-1} (s + 1)}{\Gamma(\nu) (b + \nu + 1)^{\nu-1}} q_2(s + \nu - 1) \leq R = \|y\|.
 \end{aligned} \tag{54}$$

Therefore, for $y \in K \cap \partial\Omega_R$ we get $\|Ty\| \leq \|y\|$.

So it suffices to obtain that T has a fixed point in $K \cap (\partial\Omega_p \setminus \Omega_R)$ such that $p \leq \|y_1\| < R$ in view of Lemma 7.

Then, T has fixed points y_1 in $K \cap (\partial\Omega_p \setminus \Omega_r)$ and y_2 in $K \cap (\partial\Omega_R \setminus \Omega_p)$ such that $r \leq \|y_1\| < p \leq \|y_2\| < R$.

Therefore, problem (1) gets two positive solutions $u_1 = (1/t^{\nu-1})y_1(t)$ and $u_2 = (1/t^{\nu-1})y_2(t)$. $\square$

Theorem 17. Suppose that (H_0) is satisfied. Furthermore, either

$$\begin{aligned} g_0 &= 0, \\ g_\infty &= \infty \end{aligned} \quad (55)$$

(suplinear)

or

$$\begin{aligned} g_0 &= \infty, \\ g_\infty &= 0 \end{aligned} \quad (56)$$

(sublinear).

Then, problem (1) has one positive solution.

The proof of Theorem 17 is similar to the proof of Theorems 15 and 16.

4. Example

Consider the following boundary value problem:

$$\begin{aligned} -\Delta^\nu y(t) \\ = \mu \left(a \left(t^{\nu-1} y(t+\nu-1) \right)^a + b \left(t^{\nu-1} y(t+\nu-1) \right)^b \right), \end{aligned} \quad (57)$$

$$0 < a < 1 < b < \frac{1}{2-\nu}, \quad 1 < \nu < 2,$$

$$y(\nu-2) = 0 = y(\nu+b+1).$$

Proof. Let $f(t+\nu-1, y(t+\nu-1)) = \mu((t^{\nu-1}y(t+\nu-1))^a + (t^{\nu-1}y(t+\nu-1))^b)$; it is clear that we have $f(t+\nu-1, y(t+\nu-1)/t^{\nu-1}) = \mu(ay^a(t+\nu-1) + by^b(t+\nu-1))$.

Let $q_1(t) = a$, $q_2(t) = b$ and $g(y) = \mu(y^a + y^b)$; it follows that we get $q_1 g(y) \leq f(t+\nu-1, y(t+\nu-1)/t^{\nu-1}) \leq q_2 g(y)$ and $g \in C([0, +\infty) \times [0, +\infty))$, $q_1, q_2 \in C([\nu-1, \nu+b]_{\mathbb{N}_{\nu-1}} \times (0, +\infty))$. Hence, this implies (H_0) and (H_1) in Theorem 15 hold.

Because

$$\begin{aligned} & \frac{1}{\Gamma(\nu)(b+\nu+1)^{\nu-1}} \sum_{s=0}^{b+1} (b+\nu-s)^{\nu-1} (s+1) q_2(s) \\ &= \frac{1}{\Gamma(\nu)(b+\nu+1)^{\nu-1}} \sum_{s=0}^{b+1} (b+\nu-s)^{\nu-1} (s+1) b \quad (58) \\ &= \frac{b}{\Gamma(\nu)(b+\nu+1)^{\nu-1}} \sum_{s=0}^{b+1} (b+\nu-s)^{\nu-1} (s+1), \end{aligned}$$

$$M_1 = [(b/\Gamma(\nu)(b+\nu+1)^{\nu-1}) \sum_{s=0}^{b+1} (b+\nu-s)^{\nu-1} (s+1)]^{-1}.$$

Let $\mu^* = F(p) = \sup_{y>0} \{F(y) : F(y) = M_1 y/(y^a + y^b)\} = F(((1-a)/(b-1))^{1/(b-a)})$, where $p = ((1-a)/(b-1))^{1/(b-a)}$.

So, for $\mu < \mu^*$, we get $g(y) = \mu(y^a + y^b) \leq \mu(p^a + p^b) < M_1 p$, $0 \leq y \leq p$. Therefore, (H_3) holds. According to Theorem 15, problem (57) has two positive solutions. $\square$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All four authors read and approved the final paper.

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