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Research Article

Finite Time Control for Fractional Order Nonlinear Hydroturbine Governing System via Frequency Distributed Model

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This paper studies the application of frequency distributed model for finite time control of a fractional order nonlinear hydroturbine governing system (HGS). Firstly, the mathematical model of HGS with external random disturbances is introduced. Secondly, a novel terminal sliding surface is proposed and its stability to origin is proved based on the frequency distributed model and Lyapunov stability theory. Furthermore, based on finite time stability and sliding mode control theory, a robust control law to ensure the occurrence of the sliding motion in a finite time is designed for stabilization of the fractional order HGS. Finally, simulation results show the effectiveness and robustness of the proposed scheme.

1. Introduction

Nowadays, fractional calculus has attracted numerous scientific researchers' attention in various fields. It has been widely used in mechanics [1], electrical engineering, [2] and some other fields [3, 4]. Since many practical models of engineering applications could be better described by fractional order calculus, like fractional order PMSM system [5, 6], chemical processing systems [7], and wind turbine generators [8], fractional calculus still has great potential especially for the description of hereditary and memory attributes of numerous processes and materials [9, 10].

The hydroturbine governing system plays a very important role in a hydroelectric station, and its running conditions directly affect the stable operation of hydroelectric stations and electrical systems, which has arisen many researchers' interests [11–13]. In recent years, many scholars try to establish the nonlinear model of HGS [14–16]. However, most of the models are on the basis of integer order calculus. As we all know, HGS is a highly coupling, nonlinear as well as nonminimum phase system. For this reason, integer calculus is not suitable for describing complex hydroturbine governing system. According to the history-dependent and memory character of hydraulic servo system, the fractional order hydroturbine governing system that is more in line with actual project is considered in this paper.

Many studies have indicated that the hydroturbine governing system exhibits nonlinear even chaotic vibration in nonrated operating conditions [17, 18]. So it is very important to design robust controller for suppressing nonlinear even chaotic vibration of HGS. Recently, fractional order nonlinear control has attracted increasing attention. Some control methods have been presented for stability control of fractional order nonlinear or chaotic systems, such as fuzzy control method, sliding mode control, pinning control, and predictive control [19–22]. It is clear that all of the above schemes are focused on the asymptotical stability, which needs infinite time theoretically in order to achieve the control objectives. From the perspective of optimizing the control time, finite time stability theory based control methods should be studied, which has good performance on improving the transition time, overshoot, and oscillation frequency [23–25]. Until now, some finite time control techniques such as terminal sliding mode (TSM) have been proposed [26–29].

Besides, as we all know, Lyapunov stability theorem is often used in the analysis of integer order system stability.

However, it has not yet received satisfactory results in fractional systems. Reference [30] proposes applying the frequency distributed model (FDM) to Lyapunov's method for some simple linear and nonlinear fractional differential equations. In [31, 32], by using the FDM, FDE initial conditions problem where converted into an equivalent ODE initialization problem for the first time. By applying FDM, stability analysis of sliding mode dynamics systems was studied in [33, 34]. Reference [35] introduces the FDM into the fractional order complex dynamic networks, and a robust nonfragile observer-based controller is designed. The main advantage using FDM is that the approach provides a reference for generalization of integer order system theory to fractional order ones, which is obvious a bridge between fractional order system and integer order system.

That is, both FDM in analyzing the stability of fractional order system and finite time control in improving control quality have potential advantages. Can finite time control of fractional order HGS be implemented via FDM? It is still an open problem. Research in this area should be meaningful and challenging.

In light of the above analysis, there are several advantages which make our study attractive. Firstly, a frequency distributed model is proposed by an auxiliary function and the properties of fractional calculus, which is easier to implement. Secondly, a novel fractional order TSM is firstly proposed and its stability to origin is guaranteed based on the proposed FDM and Lyapunov stability theorem. Then, a robust finite time control law to ensure the occurrence of the sliding motion in a finite time is proposed for stabilization of the fractional order HGS regardless the external disturbances. Lastly, simulation results have demonstrated the robustness and effectiveness of this new approach.

The rest of this paper is organized as follows. In Section 2, the fractional order HGS model is presented. Some definitions of fractional order calculus and relevant properties, the FDM, and controller design are given in Section 3. In Section 4, simulation results are provided. Some conclusions end this paper in Section 5.

2. Modeling of HGS

The physical model of penstock system is shown in Figure 1.

The dynamic characteristic of synchronous generator can be represented as

$$\begin{aligned} \dot{\delta} &= \omega_0 \omega \\ \dot{\omega} &= \frac{1}{T_{ab}} \left[m_t - m_e - D\omega \right], \end{aligned} \tag{1}$$

where δ is the rotor angle, D is the damping factor of the generator, ω is the variation of the speed of the generator, m_t is the output torque of hydroturbine and T_a, T_b denote the inertia time constant of generator and load, respectively, $T_{ab} = T_a + T_b$.

Here,

$$m_e = P_e. \tag{2}$$

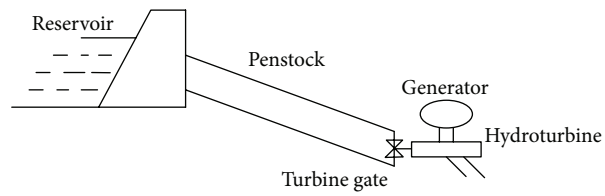


Figure 1: The physical model of penstock system.

The electromagnetic power of the generator can be expressed as

$$P_e = \frac{E_q' V_s}{x_{d\Sigma}'} \sin\delta + \frac{V_s^2}{2} \frac{x_{d\Sigma}' - x_{q\Sigma}}{x_{d\Sigma}' x_{q\Sigma}} \sin 2\delta, \tag{3}$$

where E_q is the transient internal voltage of the armature, V_s is the bus voltage at infinity, $x_{d\Sigma}'$ is the direct axis transient reactance, $x_{q\Sigma}$ is the quadrature axis reactance.

The dynamic characteristics of a hydraulic servo system can be got as

$$T_y \frac{dy}{dt} + y = 0, \tag{4}$$

where y is the incremental deviation of the guide vane opening.

The hydraulic servo system has significant historical reliance. Since it is a powerful advantage for fractional calculus to describe the function which has significant historical reliance, the fractional order hydraulic servo system is adopted.

According to fractional calculus, the fractional order hydraulic servo system is described as [36]

$$D^\alpha y = -\frac{1}{T_y} y, \tag{5}$$

where T_y is the major relay connector response time.

The output torque of turbine governing system is obtained as

$$\dot{m}_t = \frac{1}{e_{qh} T_w} \left[-m_t + e_y y + \frac{e e_y T_w}{T_y} y \right], \tag{6}$$

where e_{qh} is the transfer coefficient of turbine flow on the head, e_y is the transfer coefficient of turbine torque on the main servomotor stroke, $e = e_{qh} e_h / e_y - e_{qh}$, and e_h is the transfer coefficient of turbine torque on the water head.

According to formulae (1) to (6), the mathematical model of HGS can be described as

$$\begin{aligned} \dot{\delta} &= \omega_0 \omega, \\ \dot{\omega} &= \frac{1}{T_{ab}} \left(m_t - D\omega - \frac{E_q' V_s}{x_{d\Sigma}'} \sin\delta \right. \\ &\quad \left. - \frac{V_s^2}{2} \frac{x_{d\Sigma}' - x_{q\Sigma}}{x_{d\Sigma}' x_{q\Sigma}} \sin 2\delta \right), \end{aligned}$$

$$\dot{m}_t = \frac{1}{e_{qh}T_w}\left[-m_t + e_y y + \frac{ee_yT_w}{T_y}y\right],$$
$$D^\alpha y = -\frac{1}{T_y}y. \tag{7}$$

Here, the parameters of system (7) are, respectively, $\omega_0 = 300$, $T_{ab} = 19.0$, $D = 2.0$, $E'_q = 1.35$, $T_w = 0.8$, $T_y = 0.1$, $x'_{d\Sigma} = 1.25$, $x_{q\Sigma} = 1.474$, $V_s = 1.0$, $e = 0.7$, $e_{qh} = 0.5$, $e_y = 1.0$, and $\alpha = 0.98$. For convenience, we use x_1, x_2, x_3, x_4 to replace δ, ω, m_t, y, and the random disturbances are considered. The fractional order HGS (8) can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= 300x_2 + 0.1\,\mathrm{rand}(1),\\
\dot{x}_2 &= -\frac{2}{19}x_2 + \frac{1}{19}x_3\\
&\quad - \frac{1}{19}\left(1.08\sin x_1 + 0.061\sin 2x_1\right)\\
&\quad + 0.1\,\mathrm{rand}(1),\\
\dot{x}_3 &= -2.5x_3 + 6.6x_4 + 0.1\,\mathrm{rand}(1),\\
D^\alpha x_4 &= -10x_4 + 0.1\,\mathrm{rand}(1).
\end{aligned} \tag{8}$$

The state trajectories of system (8) are illustrated in Figure 2. It is clear that the system exhibits nonlinear irregular oscillations. Therefore, it is necessary to design controller for suppressing the complex even chaotic vibration of HGS.

3. Finite Time Controller Design for Fractional Order HGS Based on FDM

3.1. Preliminaries. In this section, some basic definitions and properties would be used related to fractional calculus. The two most usually used definitions of fractional derivative are Riemann-Liouville and Caputo definitions.

Definition 1 (see [37]). The αth fractional order Riemann-Liouville integration of function $f(t)$ is defined by

$${}_{t_0}I_t^\alpha f(t) = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \tag{9}$$

where $\alpha \in R^+$ and $\Gamma(\cdot)$ is the Gamma function.

It can be known that when α approaches to zero, fractional integral (9) would change into the identity operator in the weak sense. In this paper, 0th fractional integral is considered to be the identity operator which is defined as

$$I^0 f(t) = f(t). \tag{10}$$

Remark 2. $\Gamma(\cdot)$ is the well-known Euler's gamma function which is defined as

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \quad (\mathrm{Re}(z) > 0) \tag{11}$$

and the following identity holds:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (0 < \mathrm{Re}(z) < 1). \tag{12}$$

Definition 3 (see [37]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f(t)$ is defined as the nth derivative of fractional integral (9) of order $n-\alpha$:

$$\begin{aligned}
{}^{\mathrm{RL}}_{t_0}D_t^\alpha f(t) &= \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t)\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,
\end{aligned} \tag{13}$$

where n is the smallest integer larger than or equal to α and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 4 (see [37]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f(t)$ at time instant $t \geq 0$ is defined as the fractional integral (9) of order $n-\alpha$ of the nth derivative of $f(t)$:

$$\begin{aligned}
{}^{C}_{t_0}D_t^\alpha f(t) &= I^{n-\alpha}\left(\frac{d}{dt}\right)^n f(t)\\
&= \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,
\end{aligned} \tag{14}$$

where n is the smallest integer number larger than or equal to α and $\Gamma(\cdot)$ denotes the Gamma function.

The next are some useful properties of fractional differential and integral operators which will be used for the controller design [38].

Property 1. The fractional integral meets the semigroup property. Let $\alpha > 0$ and $\beta > 0$; then

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t). \tag{15}$$

Property 2. For the Caputo fractional derivative, the following equality holds:

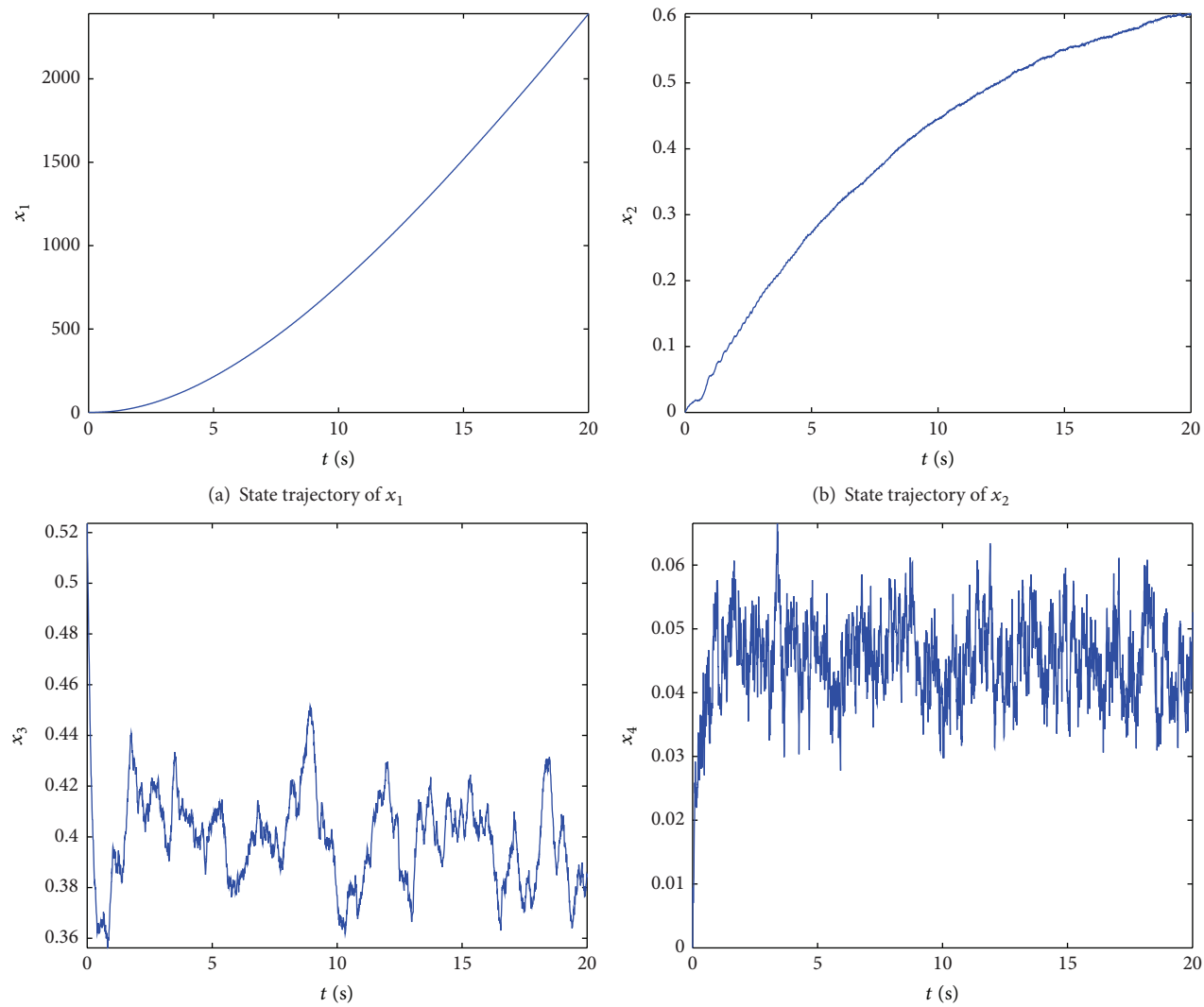
$$I^\alpha {}^{C}D^\alpha f(t) = f(t) - f(0). \tag{16}$$

Property 3. The following equality for the Caputo derivative and the Riemann-Liouville derivative are established:

$${}^{\mathrm{RL},C}D^\alpha\left({}^{\mathrm{RL},C}D^{-\beta} f(t)\right) = {}^{\mathrm{RL},C}D^{\alpha-\beta} f(t), \tag{17}$$

where $\alpha \geq \beta \geq 0$.

Remark 5. Compared with Riemann-Liouville fractional derivative, the Laplace transform of the Caputo definition allows utilization of initial conditions of classical integer order derivatives with clear physical interpretations. And the Caputo fractional derivative has the widespread application in the actual modeling process. Therefore in this paper, the Caputo definition of fractional derivative and integral is selected. To simplify the notation, we denote the Caputo fractional derivative of order α as D^α instead of ${}^{C}_{t_0}D_t^\alpha$.



(a) State trajectory of x_1

(b) State trajectory of x_2

(c) State trajectory of x_3

(d) State trajectory of x_4

Figure 2: State trajectories of fractional order HGS (8).

3.2. Frequency Distributed Model Transformation. For the convenience of mathematical analysis, the n-dimensional fractional order system is equally written as

$$D^{\alpha} X(t) = F(X), \tag{18}$$

where α is the order of the system, $X(t) \in R^n$ is the system state vector, and $F(X)$ is the nonlinear term.

Then, an auxiliary time and frequency domain function is defined as

$$\phi(\omega, t) = \int_0^t e^{-\omega^2 (t-\tau)} F(\tau)\, d(\tau). \tag{19}$$

Theorem 6. *It follows from (19) that the fractional order system (18) can be equivalently written as*

$$\frac{\partial \phi(\omega, t)}{\partial t} = -w^2 \phi(\omega, t) + F(X, t),$$

$$X(t) = \int_0^{\infty} u(\omega)\, \phi(\omega, t)\, d\omega, \tag{20}$$

where $u(\omega) = (2\sin(\pi\alpha)/\pi)\omega^{1-2\alpha}$, $\alpha \in (0, 1)$.

Proof. The process of proving is divided into two steps.

Step 1. Equation (19) can be transformed into the form as follows:

$$\phi(\omega, t) = e^{-\omega^2 t} \int_0^t e^{\omega^2 \tau} F(\tau)\, d(\tau). \tag{21}$$

Take the derivative of (21) with respect to time, and one can get

$$\frac{\partial \phi(\omega, t)}{\partial t} = -\omega^2 e^{-\omega^2 t} \int_{t_0}^t e^{\omega^2 \tau} F(\tau)\, d(\tau) + e^{-\omega^2 t} e^{\omega^2 t} F(t)$$

$$= -\omega^2 \int_{t_0}^{t} e^{-\omega^2(t-\tau)} F(\tau)\, d(\tau) + F(t)$$
$$= -\omega^2 \phi(\omega, t) + F(t). \tag{22}$$

Step 2. According to definition (11) of Euler function and definition (9) of fractional calculus, there is

$$\begin{aligned} {}_{t_0}I_t^{\alpha} F(t) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{F(\tau)}{(t-\tau)^{1-\alpha}} d\tau \\ &\cdot \int_0^{\infty} e^{-z} z^{-\alpha} dz = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ &\cdot \int_0^{\infty} \int_{t_0}^{t} \frac{F(\tau)}{(t-\tau)^{1-\alpha}} e^{-z} z^{-\alpha} d\tau\, dz = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ &\cdot \int_0^{\infty} \int_{t_0}^{t} F(\tau) \left(\frac{z}{t-\tau}\right)^{-\alpha} \frac{1}{(t-\tau)} e^{-z} d\tau\, dz. \end{aligned} \tag{23}$$

Define the variable

$$z = \omega^2 (t-\tau) \tag{24}$$

with $dz = 2\omega(t-\tau)d\omega$.

According to (23) and (24), one gets

$$\begin{aligned} &{}_{t_0}I_t^{\alpha} F(t) \\ &= \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} \int_{t_0}^{t} F(\tau)\, \omega^{1-2\alpha} e^{-\omega^2(t-\tau)} d\tau\, d\omega \\ &= \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} \omega^{1-2\alpha} \int_{t_0}^{t} F(\tau)\, e^{-\omega^2(t-\tau)} d\tau\, d\omega. \end{aligned} \tag{25}$$

Introducing the auxiliary function (19), one has

$${}_{t_0}I_t^{\alpha} F(t) = \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} \omega^{1-2\alpha} \phi(\omega, t)\, d\omega. \tag{26}$$

Note

$$\mu(\omega) = \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \omega^{1-2\alpha}. \tag{27}$$

Based on (12), one can get

$$\mu(\omega) = \frac{2\sin \alpha\pi}{\pi} \omega^{1-2\alpha}. \tag{28}$$

Then (26) can be written as

$${}_{t_0}I_t^{\alpha} F(t) = \int_0^{\infty} \mu(\omega)\, \phi(\omega, t)\, d\omega. \tag{29}$$

Based on Properties 2 and 3 and (29), one gets

$$X(t) = D^{-\alpha} F(t) = \int_0^{\infty} \mu(\omega)\, \phi(\omega, t)\, d\omega. \tag{30}$$

This completes the proof. □

3.3. Controller Design

Lemma 7 (see [39]). *Consider the n-dimensional fractional order system (18); assume that there exists a positive constant $T = T(X(0))$, such that*

$$\lim_{t \to T} \|X(t)\| = 0, \tag{31}$$

and $\|X(t)\| \equiv 0$; if $t \geq T$, then the fractional order nonlinear system (18) will be stable in the finite time T.

In general, the design process of sliding mode control can be divided into two steps. Firstly, one can select an appropriate sliding surface which represents the required system dynamic characteristics. In this paper, a novel fractional order FTSM is defined as follows:

$$s(t) = D^{\alpha-1} x + D^{-1}\left(k_1 x + k_2 \ln^u (|x| + 1)\, \mathrm{sat}(x)\right), \tag{32}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ are the system states and k_1, k_2, u are the given sliding surface parameters, with $k_1 > 0$, $k_2 > 0$, $0 < u < 1$. The saturation function $\mathrm{sat}(\cdot)$ is presented as

$$\mathrm{sat}(\rho) = \begin{cases} \mathrm{sign}\left(\dfrac{\rho}{k}\right), & |\rho| > k \\ \dfrac{\rho}{k}, & |\rho| \leq k. \end{cases} \tag{33}$$

When the system reaches the sliding mode surface

$$s(t) = 0. \tag{34}$$

According to (32) and (34), one can obtain

$$\begin{aligned} s(t) &= D^{\alpha-1} x + D^{-1}\left(k_1 x + k_2 \ln^u (|x| + 1)\, \mathrm{sat}(x)\right) \\ &= 0. \end{aligned} \tag{35}$$

Then

$$D^{\alpha-1} x = -D^{-1}\left(k_1 x + k_2 \ln^u (|x| + 1)\, \mathrm{sat}(x)\right). \tag{36}$$

Based on Properties 2 and 3, there is

$$D^{\alpha} x = -\left(k_1 x + k_2 \ln^u (|x| + 1)\, \mathrm{sat}(x)\right). \tag{37}$$

Theorem 8. *If the terminal sliding mode is selected in the form of (32), the sliding mode dynamics system (37) is stable and its state trajectories will converge to zero.*

Proof. According to Theorem 6, the sliding mode dynamical system (37) can be described as

$$\begin{aligned} \frac{\partial \phi(\omega, t)}{\partial t} &= -w^2 \phi(\omega, t) \\ &\quad - \left(k_1 x + k_2 \ln^u (|x| + 1)\, \mathrm{sat}(x)\right), \\ x_i(t) &= \int_0^{\infty} u(\omega)\, \phi(\omega, t)\, d\omega. \end{aligned} \tag{38}$$

Select Lyapunov function as

$$V_1(t) = \frac{1}{2}\int_0^\infty \mu(\omega)\phi^2(\omega,t)\,d\omega. \tag{39}$$

Taking its time derivative, one gets

$$\begin{aligned}\frac{dV_1}{dt} &= \int_0^\infty \mu(\omega)\phi(\omega,t)\frac{\partial\phi(\omega,t)}{\partial t}d\omega = \int_0^\infty \mu(\omega)\\ &\cdot \phi(\omega,t)\left(-w^2\phi(\omega,t)\right.\\ &\left.-\left(k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)\right)d\omega\\ &= -\int_0^\infty \mu(\omega)\omega^2\phi^2(\omega,t)\,d\omega - \int_0^\infty \mu(\omega)\\ &\cdot \phi(\omega,t)\left(k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)d\omega\\ &= -\int_0^\infty \mu(\omega)\omega^2\phi^2(\omega,t)\,d\omega - \left(k_1x\right.\\ &\left.+ k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)\int_0^\infty \mu(\omega)\phi(\omega,t)\,d\omega\\ &= -\int_0^\infty \mu(\omega)\omega^2\phi^2(\omega,t)\,d\omega - k_1x^2\\ &- k_2\ln^u(|x|+1)\,x\operatorname{sat}(x).\end{aligned} \tag{40}$$

According to the definition of saturation function sat(·), there is the following.

Case 1 ($|x| > k$). In this case, one has

$$x\operatorname{sat}(x) = x\operatorname{sign}\left(\frac{x}{k}\right). \tag{41}$$

Because of $x\operatorname{sign}(x) = |x|$ and k is a given positive constant, one has

$$x\operatorname{sat}(x) = |x| > 0. \tag{42}$$

One can easily get

$$\frac{dV_1}{dt} \le 0. \tag{43}$$

Case 2 ($|x| \le k$). In this case, one has

$$x\operatorname{sat}(x) = x\operatorname{sign}\left(\frac{x}{k}\right) = \frac{x^2}{k} > 0. \tag{44}$$

It is clear that

$$\frac{dV_1}{dt} \le 0. \tag{45}$$

Considering both Cases 1 and 2, there is

$$\frac{dV_1}{dt} \le 0. \tag{46}$$

According to Lyapunov stability theory, the state trajectories of the sliding mode dynamics system (37) will converge to zero asymptotically. This completes the proof. □

As for the fractional order HGS (8), its controlled form can be briefly represented as

$$D^\alpha(x) = f(x) + d(t) + u(t), \tag{47}$$

where $x = [x_1, x_2, x_3, x_4]^T$ are the state variables, $d(t) = [d_1, d_2, d_3, d_4]^T$ are the external random disturbances, $u = [u_1, u_2, u_3, u_4]^T$ are the control inputs, and $\alpha = [1, 1, 1, 0.98]$ are the fractional orders.

Theorem 9. *Consider fractional order HGS (47) and the sliding surface in (32). If the system is controlled by the control law (48), then the states trajectories of the system will converge to the sliding surface $s(t) = 0$ in a finite time*

$$\begin{aligned}u(t) = &-\left(f(x) + k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x) + \eta s\right.\\ &\left.+ \left(\xi + L|s|^r\right)\operatorname{sign}(s)\right),\end{aligned} \tag{48}$$

where $\xi = [\xi_1, \xi_2, \xi_3, \xi_4] = [0.1, 0.1, 0.1, 0.1]$ present bounded values of the external disturbances, η, L, r are given positive constants with $\eta, L > 0$, $0 < r < 1$.

Proof. Select Lyapunov function $V_2(t) = |s|$, and one gets

$$\dot V_2(t) = \operatorname{sign}(s)\,\dot s. \tag{49}$$

Substituting $\dot s(t) = D^\alpha x + (k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x))$ into (49), there is

$$\begin{aligned}&\dot V_2(t)\\ &= \operatorname{sign}(s)\left(D^\alpha x + \left(k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)\right).\end{aligned} \tag{50}$$

Considering $D^\alpha(x) = f(x) + d(t) + u(t)$, one has

$$\begin{aligned}\dot V_2(t) &= \operatorname{sign}(s)\left(f(x) + d(t) + u(t)\right.\\ &\left.+ \left(k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)\right) \le \operatorname{sign}(s)\left(f(x)\right.\\ &\left.+ u(t) + \left(k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)\right) + |\xi|.\end{aligned} \tag{51}$$

Introducing control law (48) to (51), one can get

$$\begin{aligned}\dot V_2(t) &\le \operatorname{sign}(s)\left(f(x) - \left(f(x) + k_1x\right.\right.\\ &\left.+ k_2\ln^u(|x|+1)\operatorname{sat}(x) + \eta s + \left(\xi + L|s|^r\right)\operatorname{sign}(s)\right)\\ &+ \left(k_1x + k_2\ln^u(|x|+1)\operatorname{sat}(x)\right)\Big) + \xi = \operatorname{sign}(s)\\ &\cdot\left(-\eta s - \left(\xi + L|s|^r\right)\operatorname{sign}(s)\right) + \xi = -\eta|s| - \left(\xi\right.\\ &\left.+ L|s|^r\right) + \xi = -\eta|s| - L|s|^r \le 0.\end{aligned} \tag{52}$$

According to $dV_2/dt \le -\eta|s| - L|s|^r$, one gets

$$\begin{aligned}dt &\le -\frac{d|s|}{\eta|s| + L|s|^r} = -\frac{|s|^{-r}\,d|s|}{\eta|s|^{1-r} + L}\\ &= -\frac{1}{1-r}\times\frac{d|s|^{1-r}}{\eta|s|^{1-r} + L}.\end{aligned} \tag{53}$$

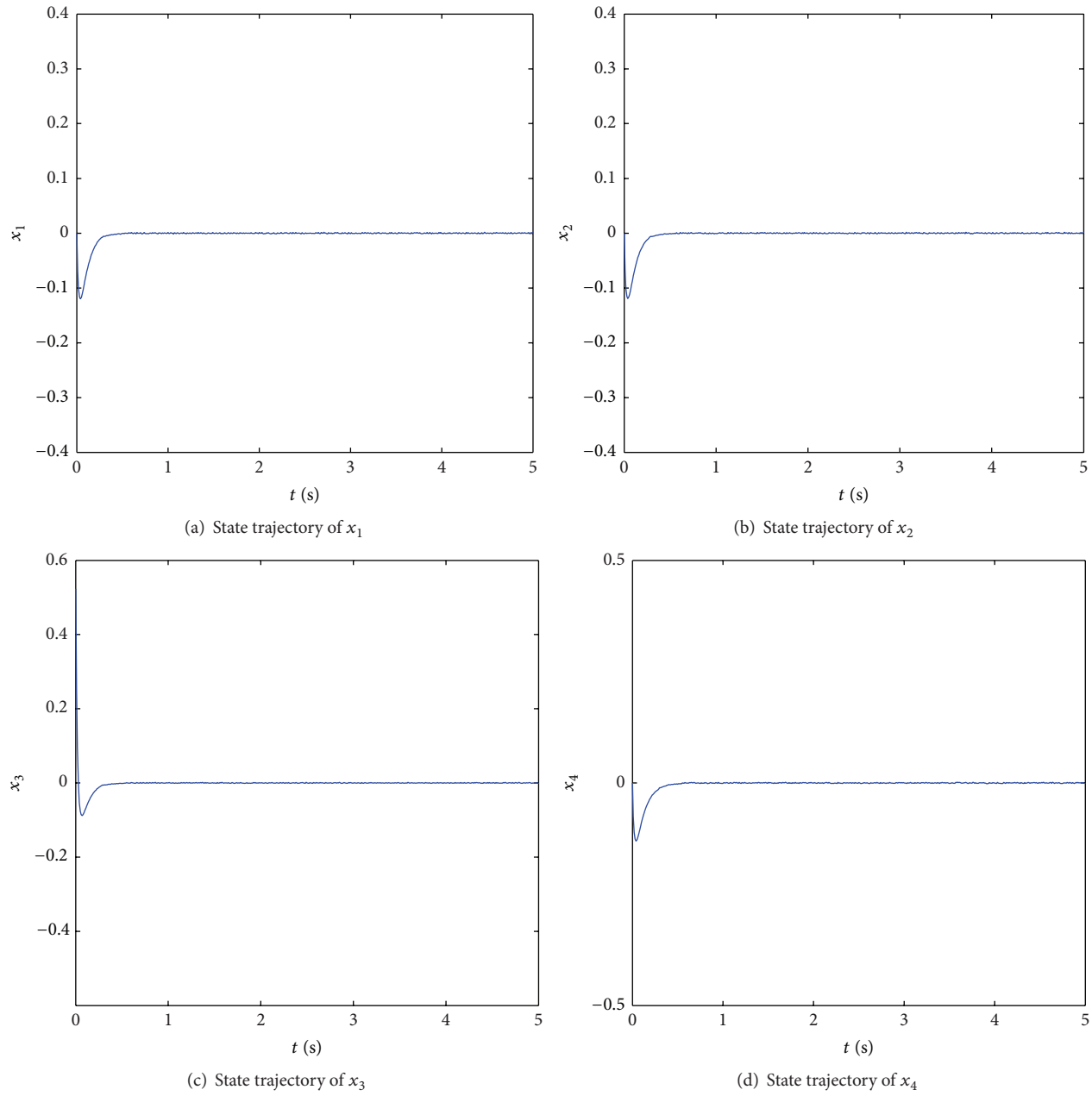


Figure 3: State trajectories of controlled fractional order HGS (8).

Taking integral of both sides of (53) from 0 to t_{r1}

$$\int_0^{t_{r1}} dt \le \int_{s(0)}^{s(t_{r1})} -\frac{1}{1-r} \times \frac{d\,|s|^{1-r}}{\eta\,|s|^{1-r} + L}. \tag{54}$$

There is

$$t_{r1} \le -\frac{1}{\eta\,(1-r)} \times \ln\left(\eta\,|s|^{1-r} + L\right)\Big|_{s(0)}^{s(t_{r1})}. \tag{55}$$

And let $s(t_{r1}) = 0$; after calculation we can get the finite time

$$t_{r1} \le \frac{1}{\eta\,(1-r)} \ln \frac{\left(\eta\,|s\,(0)|^{1-r} + L\right)}{L}. \tag{56}$$

According to Lyapunov stability theory, the state trajectories of the fractional order HGS (47) will converge to $s(t) = 0$ asymptotically. And we can easily get the reaching time $T \le (1/\eta(1-r))\ln((\eta|s(0)|^{1-r} + L)/L)$. This completes the proof. □

4. Numerical Simulations

For the fractional order HGS (8), according to the proposed method in Section 3, select the corresponding parameters as follows:

$$k_1 = 50,$$
$$k_2 = 1,$$

$$u = 0.2,$$
$$\eta = 10,$$
$$L = 0.01,$$
$$r = 0.2. \tag{57}$$

According to the sliding surface (32), one can get

$$s_i(t) = D^{0.98-1}x_i + D^{-1}\left(50x_i + \ln^{0.2}\left(|x_i| + 1\right)\operatorname{sat}(x_i)\right), \quad i = 1, 2, 3, 4. \tag{58}$$

Based on the control law (48), there is

$$\begin{aligned}
u_1(t) &= -\left(f(x_1) + 50x_1 + \ln^{0.2}\left(|x_1| + 1\right)\operatorname{sat}(x_1) + 10s_1 + \left(0.1 + 0.01\left|s_1\right|^{0.2}\right)\operatorname{sign}(s_1)\right),\\
u_2(t) &= -\left(f(x_2) + 50x_2 + \ln^{0.2}\left(|x_2| + 1\right)\operatorname{sat}(x_2) + 10s_2 + \left(0.1 + 0.01\left|s_2\right|^{0.2}\right)\operatorname{sign}(s_2)\right),\\
u_3(t) &= -\left(f(x_3) + 50x_3 + \ln^{0.2}\left(|x_3| + 1\right)\operatorname{sat}(x_3) + 10s_3 + \left(0.1 + 0.01\left|s_3\right|^{0.2}\right)\operatorname{sign}(s_3)\right),\\
u_4(t) &= -\left(f(x_4) + 50x_4 + \ln^{0.2}\left(|x_4| + 1\right)\operatorname{sat}(x_4) + 10s_4 + \left(0.1 + 0.01\left|s_4\right|^{0.2}\right)\operatorname{sign}(s_4)\right).
\end{aligned} \tag{59}$$

Figure 3 shows the control results of fractional order HGS (8) with initial condition $[0, 0, \pi/6, 0]$. From Figure 3, it is obvious that when the proposed controller (59) is put into system (8), the sliding mode is guaranteed and the state trajectories converge to zero immediately, which implies that the nonlinear vibration of the fractional order HGS (8) is efficiently suppressed in a finite time, regarding the system with external random disturbances. Simulation results have demonstrated the robustness and effectiveness of the proposed method.

5. Conclusions

A new robust finite time terminal sliding mode control scheme was designed to stabilize a nonlinear fractional order HGS in this paper. An auxiliary time and frequency domain function was introduced to transform the fractional order nonlinear systems into FDM. Then, a novel TSM is proposed and its stability to the origin was guaranteed based on the FDM and Lyapunov stability theory. Furthermore, a robust finite time control law to ensure the occurrence of the sliding motion in a finite time was proposed for stabilization of the fractional order HGS regardless the external disturbances. Numerical simulations were employed to demonstrate the effectiveness and robustness with the theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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