Research Article

New Applications of a Kind of Infinitesimal-Operator Lie Algebra

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Applying some reduced Lie algebras of Lie symmetry operators of a Lie transformation group, we obtain an invariant of a second-order differential equation which can be generated by a Euler-Lagrange formalism. A corresponding discrete equation approximating it is given as well. Finally, we make use of the Lie algebras to generate some new integrable systems including $(1+1)$ and $(2+1)$ dimensions.

1. Introduction

Some research on difference equations admitting Lie-point transformations can be found in the literature [1, 2]. Specifically, some symmetry-preserving difference schemes for nonlinear differential equations were discovered in [3–5]. We first briefly recall some fundamental notations. Given L satisfying the Euler-Lagrange equation,

$$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} - D \left(\frac{\partial L}{\partial y'} \right) = 0, \quad (1)$$

where

$$D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots \quad (2)$$

is a Lie group operator, it is well-known that the functional

$$\tilde{L} = \int_{\Omega} L(x, y, y') dx, \quad \Omega \subset \mathbb{R}^1 \quad (3)$$

achieves its extremal values on [1].

Equation (1) is an ODE that can be rewritten as

$$y'' = f(x, y, y'). \quad (4)$$

Group G generated by the vector field

$$X = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \quad (5)$$

is a variational symmetry of the functional $\tilde{L}$ if and only if the Lagrangian satisfies

$$\text{pr } X(L) + LD(\xi) = 0. \quad (6)$$

If there exists a function $V(x, y)$ such that

$$\text{pr } X(L) + LD(\xi) = D(V), \quad (7)$$

then the group G is known as an infinitesimal divergence symmetry. By Noether's theorem, one infers that [5]

$$\begin{aligned} \text{pr } X(L) + LD(\xi) &= (\eta - \xi y') \frac{\delta L}{\delta y} \\ &+ D \left(\xi L + (\eta - \xi y') \frac{\partial L}{\partial y'} \right). \end{aligned} \quad (8)$$

Therefore, if the vector operator X is a divergence symmetry, we have a first integral

$$K = \xi L + (\eta - \xi y') \frac{\partial L}{\partial y'} - V, \quad (9)$$

which corresponds to the Euler-Lagrange equation.

In [4], the Lagrangian formalism for second-order difference equations was reviewed. Consider a finite difference functional

$$\hat{L} = \sum_{\Omega} \mathcal{L}(x, x_+, y, y_+) h_+ \quad (10)$$

defined on some one-dimensional lattice Ω with step spacing h_+ , which can be represented by

$$h_+ = \varphi(x, y, x_+, y_+). \quad (11)$$

For an arbitrary curve the stationary value of a differential functional is given by any solution of the quasiextremal equations [4, 5]:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} &= h_+ \frac{\partial \mathcal{L}}{\partial y} + h_- \frac{\partial \mathcal{L}^-}{\partial y} = 0, \\ \frac{\delta \mathcal{L}}{\delta x} &= h_+ \frac{\partial \mathcal{L}}{\partial x} + h_- \frac{\partial \mathcal{L}^-}{\partial x} + \mathcal{L}^- - \mathcal{L}. \end{aligned} \quad (12)$$

It can be verified that if the Lagrangian density $\mathcal{L}$ is divergence invariant under group G , it holds that

$$\text{pr } X(\mathcal{L}) + \mathcal{L} D_{+h}(\xi) = D_{+h}(V); \quad (13)$$

then each element X of the Lie algebra corresponding to G provides us with a first integral of (12):

$$K = h_- \eta \frac{\partial \mathcal{L}^-}{\partial y} + h_- \xi \frac{\partial \mathcal{L}^-}{\partial x} + \xi \mathcal{L}^- - V. \quad (14)$$

In the paper, we start from the infinitesimal symmetry operators of an eight-parameter projective transformation group in R^2 [6],

$$\begin{aligned} X_1 &= x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \\ X_2 &= xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}, \\ X_3 &= x \frac{\partial}{\partial x}, \\ X_4 &= y \frac{\partial}{\partial x}, \\ X_5 &= \frac{\partial}{\partial x}, \\ X_6 &= x \frac{\partial}{\partial y}, \\ X_7 &= y \frac{\partial}{\partial y}, \\ X_8 &= \frac{\partial}{\partial y}, \end{aligned} \quad (15)$$

to discuss some continuous symmetries of Lagrangians and general solutions of discrete equations. We generate new $(1+1)$ -dimensional and $(2+1)$ -dimensional integrable systems with three potential functions. Specifically, we obtain an integrable coupling of the standard Burgers equation and a $(2+1)$ -dimensional integrable coupling of the heat conduct equation. We also derive a $(2+1)$ -dimensional integrable coupling of the $(2+1)$ -dimensional hyperbolic equation

under the framework of the Tu scheme. Finally, we establish a vector Lie algebra to generate expanding integrable models of the $(1+1)$ -dimensional and $(2+1)$ -dimensional integrable hierarchies of evolution equations obtained in the paper.

2. A Few Lie Subalgebras of the Operator Algebra

The Lie algebra (15) has the following commutative operations [6]:

$$\begin{aligned} [X_1, X_2] &= 0, \\ [X_1, X_3] &= -X_1, \\ [X_1, X_4] &= -X_2, \\ [X_1, X_5] &= -2X_3 - X_7, \\ [X_1, X_6] &= [X_1, X_7] = 0, \\ [X_1, X_8] &= -X_6, \\ [X_2, X_3] &= [X_2, X_4] = 0, \\ [X_2, X_5] &= -X_4, \\ [X_2, X_6] &= -X_1, \\ [X_2, X_7] &= -X_2, \\ [X_2, X_8] &= -X_3 - 2X_7, \\ [X_3, X_4] &= -X_4, \\ [X_3, X_5] &= -X_5, \\ [X_3, X_6] &= X_6, \\ [X_3, X_7] &= [X_3, X_8] = [X_4, X_5] = 0, \\ [X_4, X_6] &= X_7 - X_3, \\ [X_4, X_7] &= -X_4, \\ [X_4, X_8] &= -X_5, \\ [X_5, X_6] &= X_8, \\ [X_5, X_7] &= [X_5, X_8] = 0, \\ [X_6, X_7] &= X_6, \\ [X_6, X_8] &= 0, \\ [X_7, X_8] &= -X_8. \end{aligned} \quad (16)$$

From this Lie algebra, some interesting Lie algebras given in [4] can be obtained. In fact, the Lie algebras in [4] denoted by

$D_{2,1}, D_{2,2}, D_{3,1}, D_{3,2}, D_{3,3}$ are actually linear combinations of the elements in (15):

$$\begin{aligned} D_{2,1} &= \{X_5, X_8\}, \\ D_{2,2} &= \{X_3 + X_7, X_8\}, \\ D_{3,1} &= \{X_5, X_8, X_3 + kX_7\}, \\ D_{3,2} &= \{X_5, 2X_3 + X_7, X_1\}, \\ D_{3,3} &= \{X_5, X_8, X_3 + X_6 + X_7\}. \end{aligned} \quad (17)$$

In [4], by using some subalgebras, some difference equations corresponding to Lagrangian invariants and some general solutions of the corresponding quasiextremal equations were obtained. Therefore, it would be important to further study the operator Lie algebra (15) for applications in generating new difference equations and new integrable systems. If we denote the Lie algebra (15) by

$$g = \text{span} \{X_1, X_2, \dots, X_8\}, \quad (18)$$

then it is easy to have

$$\begin{aligned} g &= g_1 + g_2, \\ g_1 &= \text{span} \{X_1, X_2, X_3, X_4\}, \\ g_2 &= \text{span} \{X_5, X_6, X_7, X_8\}, \end{aligned} \quad (19)$$

where g_1 and g_2 are Lie subalgebras of g and both are not semisimple. In addition, we have

$$\begin{aligned} [g_1, g_1] &\subset g_1, \\ [g_2, g_2] &\subset g_2, \\ [g_1, g_2] &\text{ not in } g_1, g_2. \end{aligned} \quad (20)$$

Hence, the Lie algebra g is not a direct sum of the Lie subalgebras g_1 and g_2 . Because of relation (20), g is not a symmetric Lie algebra. Therefore, g itself cannot be utilized to generate integrable couplings. However, subalgebras of g have the potential and will be investigated in the following. In addition, some new Lie subalgebras can be obtained from (15). For example, we take $X_3 \in g_1$ and $X_5, X_6, X_7 \in g_2$ to make a linear combination:

$$\begin{aligned} \bar{X} &= X_3 + X_5 + kX_6 + X_7 \\ &= (1+x) \frac{\partial}{\partial x} + (kx+y) \frac{\partial}{\partial y}. \end{aligned} \quad (21)$$

It is easy to verify that

$$g_5 = \text{span} \{X_5, X_6, X_7, X_8, \bar{X}\} \quad (22)$$

is a Lie subalgebra of g , with the commutative relations

$$\begin{aligned} [X_5, X_6] &= X_8, \\ [X_5, X_7] &= [X_5, X_8] = 0, \\ [X_5, \bar{X}] &= X_5 + kX_8, \\ [X_6, X_8] &= 0, \\ [X_6, X_7] &= X_6, \\ [X_6, \bar{X}] &= -X_8, \\ [X_7, \bar{X}] &= -kX_6, \\ [X_7, X_8] &= -X_8, \\ [X_8, \bar{X}] &= X_7. \end{aligned} \quad (23)$$

If we let $k = -1$ and remove the element X_5 from the Lie algebra g_5 , we get a 4-dimensional Lie subalgebra

$$g_4 = \text{span} \{X_6, X_7, X_8, \bar{X}\}. \quad (24)$$

The corresponding commutative operations are given by

$$\begin{aligned} [X_6, X_7] &= X_6, \\ [X_6, X_8] &= 0, \\ [X_6, \bar{X}] &= -X_8, \\ [X_7, X_8] &= -X_8, \\ [X_7, \bar{X}] &= X_6, \\ [X_8, \bar{X}] &= X_7. \end{aligned} \quad (25)$$

In what follows, we shall make use of the Lie subalgebras (22) and (24) to generate second-order differential equations and difference equations that preserve Lie-point symmetries. We derive the general solutions of the corresponding quasiextremal equations. Furthermore, we employ the Lie subalgebras to generate some new $(1+1)$ -dimensional and $(2+1)$ -dimensional integrable hierarchies of evolution equations.

3. General Solutions of Some Difference Equations

A direct calculation yields a second-order ODE corresponding to the vector field $\bar{X}$ on the space (x, y, y', y'') :

$$y'' = \frac{1}{1-2k} [k + (1-k) y']^{(1-2k)/(1-k)}. \quad (26)$$

This equation can be obtained from the Lagrangian

$$L = y + \frac{1-2k}{k} [k + (1-k) y']^{1/(1-k)}. \quad (27)$$

Obviously, the Lagrangian admits the symmetries X_5 and X_8 :

$$\begin{aligned} \text{pr } X_5 (L) + LD (\xi_5) &= 0, \\ \text{pr } X_8 (L) + LD (\xi_8) &= 1 = D(x). \end{aligned} \quad (28)$$

With the help of Noether's theorem we obtain the following first integrals:

$$\begin{aligned} J_1 &= y + \frac{1-2k}{k} [k + (1-k) y']^{1/(1-k)} \\ &\quad - \frac{1-2k}{k} y' [k + (1-k) y']^{k/(1-k)} \equiv A_0, \end{aligned} \quad (29)$$

$$J_2 = \frac{1-2k}{k} [k + (1-k) y']^{k/(1-k)} - x \equiv B_0.$$

Solving the second equation for y' in (29) and substituting it into the first equation, we obtain

$$\begin{aligned} y(x) &= A_0 + \frac{2k-1}{k} \left[\frac{k}{1-2k} (x + B_0) \right]^{1/k} \\ &\quad + \frac{2k-1}{k(1-k)} \left[\frac{k}{1-k} (x + B_0) \right]^{((1-k)/k)^2} \\ &\quad \cdot \left[\left(\frac{k}{1-2k} (x + B_0) \right)^{(1-k)/k} - k \right], \end{aligned} \quad (30)$$

which is the general solution to (26).

Let us take the difference Lagrangians

$$\mathcal{L} = \frac{y + y_+}{2} + \frac{1-2k}{k} \left[k + (1-k) D_{+h}(y) \right]^{1/(1-k)}, \quad (31)$$

which satisfies

$$\text{pr } X_5 (\mathcal{L}) + \mathcal{L} D (\xi_5) = 0, \quad (32)$$

$$\text{pr } X_8 (\mathcal{L}) + \mathcal{L} D (\xi_8) = D_{+h}(x), \quad (33)$$

where

$$D_{+h}(y) \equiv y_x = \frac{1}{h_+} \left(S_{+h} - 1 \right) (y) = \frac{y_+ - y}{x_+ - x} = \frac{y_+ - y}{h_+}. \quad (34)$$

Then the variations of $\mathcal{L}$ yield the quasiextremal equations

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} : \frac{h_+ + h_-}{2} + \frac{2k-1}{k} [k + (1-k) y_x]^{k/(1-k)} \\ + \frac{1-2k}{k} [k + (1-k) y_{\bar{x}}]^{k/(1-k)} &= 0, \\ \frac{\delta \mathcal{L}}{\delta x} : \frac{y_+ - y_-}{2} + \frac{1-2k}{k} y_x [k + (1-k) y_x]^{k/(1-k)} \\ + \frac{2k-1}{k} y_{\bar{x}} [k + (1-k) y_{\bar{x}}]^{k/(1-k)} \\ + \frac{1-2k}{k} [k + (1-k) y_{\bar{x}}]^{1/(1-k)} \\ + \frac{2k-1}{k} [k + (1-k) y_x]^{1/(1-k)} &= 0, \end{aligned} \quad (35)$$

where

$$D_{-h}(y) \equiv y_{\bar{x}} = -\frac{1}{h_-} \left(S_{-h} - 1 \right) (y) = \frac{y - y_-}{x - x_-} = \frac{y - y_-}{h_-} \quad (36)$$

and h_+, h_- are spatial spacing steps. The case $h_+ = h_-$ implies that the grid mesh is uniform. By using formula (14), the first integrals are obtained:

$$\begin{aligned} I_1 &= \frac{2k-1}{k} y_x [k + (1-k) y_x]^{k/(1-k)} + \frac{y + y_+}{2} \\ &\quad + \frac{1-2k}{k} [k + (1-k) y_x]^{1/(1-k)} \equiv A, \\ I_2 &= x_+ + \frac{1-2k}{k} [k + (1-k) y_x]^{k/(1-k)} \equiv B, \\ I_3 &= \frac{x_+^2}{2} + \frac{1-2k}{k} x + [k + (1-k) y_x]^{k/(1-k)} \\ &\quad - \frac{(1-2k)^2}{k(1-k)} [k + (1-k) y_x] \equiv C. \end{aligned} \quad (37)$$

Solving y_x from the second equation in (37) and then substituting the solution into the third one yields the lattice equation:

$$\begin{aligned} Bx_+ - \frac{1}{2} x_+^2 - \frac{(1-2k)^2}{k(1-k)} \left[\frac{k}{1-2k} (B - x_+) \right]^{(1-k)/k} \\ \equiv C. \end{aligned} \quad (38)$$

Inserting y_x into the first equation in (37) gives rise to the general solution to the quasiextremal equation (35):

$$\begin{aligned} y_+ &= y + \frac{2k^{k-1} A^k}{(1-k)(1-2k)^{k-1}} \left[k \right. \\ &\quad \left. - \left(\frac{k}{1-2k} (B - x_+) \right)^{(1-k)/k} \right], \end{aligned} \quad (39)$$

which is defined on the lattice determined by the lattice equation (38).

We now consider the invariant model for (26). It is easy to see that (26) admits a Lie transformation group that can be represented by the Lie algebra $\mathfrak{g}_4$. The group consists of infinitesimal vector fields. The corresponding prolongation operators are given by

$$\begin{aligned} \widehat{X}_6 &= \text{pr } X_6 \\ &= x \frac{\partial}{\partial y} + x^+ \frac{\partial}{\partial y^+} + h_+ D_{+h}(x) \frac{\partial}{\partial h_+} \\ &\quad + h_- D_{-h}(x) \frac{\partial}{\partial h_-}, \end{aligned}$$

$$\begin{aligned}
\widehat{X}_7 &= y \frac{\partial}{\partial y} + y^+ \frac{\partial}{\partial y^+} + \left[D(y) - y_1 D(0) \right] \frac{\partial}{\partial y_x} \\
&\quad + h_+ D(0) \frac{\partial}{\partial h_+}, \\
\widehat{X}_8 &= \frac{\partial}{\partial y} + \frac{\partial}{\partial y^+}, \\
\text{pr } \overline{X} &= x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + x^+ \frac{\partial}{\partial x^+} + 2y^+ \frac{\partial}{\partial y^+} \\
&\quad + h_+ D(x) \frac{\partial}{\partial h_+} + \left[D(2y) - y_1 D(x) \right] \frac{\partial}{\partial y_x} \\
&\quad + \left[D\left(\zeta_1\right) - y_2 D(x) \right] \frac{\partial}{\partial y_{x\bar{x}}},
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
x^+ &= S_{+h}(x), \\
y^+ &= y(x^+), \\
\zeta_1 &= D_{+h}(2y) - y_1 D_{+h}, \\
y_1 &= D_{+h}(y), \\
y_2 &= D_{-h+h} D(y).
\end{aligned} \tag{41}$$

There are a few difference invariants of the Lie algebra g_4 :

$$\begin{aligned}
&\frac{h_+}{h_-}, \\
&\frac{y^+}{y}, \\
&\frac{y_1}{h}, \\
&\frac{y_2}{h}, \\
&\frac{x^+}{x}, \\
&\frac{h^+}{x^+}, \\
&\frac{y_2}{y^2}.
\end{aligned} \tag{42}$$

By means of the invariants (42), we can write the following explicit discrete scheme for (26):

$$y_h^2 = \frac{1}{1-2k} \left[k + (1-k) D_{+h}(y) \right]^{(1-2k)/(1-k)}. \tag{43}$$

This scheme is certainly not unique and one can construct another form of the invariant difference equation. In the way

presented in [7], we can further investigate some approximate solutions and stabilities by using the von Neumann condition and the Fourier method. This topic is not further discussed in this paper.

4. New Integrable Dynamical Systems

In the section we discuss another application of the Lie algebra g_4 to generate new integrable dynamical systems, including $(1+1)$ and $(2+1)$ dimensions, according to the Tu and TAH schemes [8, 9]. We have used this approach before to obtain some integrable systems and the corresponding Hamiltonian structures [10–13]. However, we note that the integrable hierarchies derived in this paper possess three potential functions and are different from those in [9–13]. The integrable dynamical systems here consist of an integrable coupling of the standard Burgers equation, a $(2+1)$ -dimensional integrable coupling of the heat conduction equation, and a $(2+1)$ -dimensional integrable coupling of a $(2+1)$ -dimensional hyperbolic equation.

4.1. $(1+1)$ -Dimensional Integrable Systems. We start with a general loop algebra of the Lie algebra g_4 :

$$\overline{g}_4 = \text{span} \{X_6(n), X_7(n), X_8(n), \overline{X}(n)\}, \tag{44}$$

where

$$\begin{aligned}
X_i(n) &= X_i \lambda^n, \quad i = 6, 7, 8; \\
\overline{X}(n) &= \overline{X} \lambda^n, \quad n \in \mathbb{Z}.
\end{aligned} \tag{45}$$

Now we apply $\overline{g}_4$, introducing the isospectral problem

$$\begin{aligned}
\varphi_x &= U\varphi, \quad U = \overline{X}(1) + qX_6(0) + rX_7(0) + sX_8(0), \\
\varphi_t &= V\varphi,
\end{aligned} \tag{46}$$

$$V = \sum_{m \geq 0} (b_m X_6(-m) + c_m X_7(-m) + d_m X_8(-m)).$$

The stationary compatibility condition of (46) leads to

$$\begin{aligned}
c_{m+1} &= -b_{mx} + qc_m - rb_m, \\
d_{m+1} &= -c_{mx}, \\
b_{m+1} &= d_{mx} + rd_m - sc_m.
\end{aligned} \tag{47}$$

Taking three constants $b_0 = \alpha$, $c_0 = \beta$, and $d_0 = \gamma$, we find that (47) is local. For example, we can get

$$\begin{aligned}
b_1 &= \gamma r - \beta s, \\
c_1 &= \beta q - \alpha r, \\
d_1 &= 0, \\
b_2 &= -\beta q s + \alpha r s, \\
c_2 &= -\gamma r_x + \beta s_x + \beta q^2 - \alpha q r - \gamma r^2 + \beta r s, \\
d_2 &= -\beta q_x + \alpha r_x,
\end{aligned} \tag{48}$$

$\vdots$

Note that

$$V_+^{(n)} = \sum_{m=0}^n (b_m X_6(n-m) + c_m X_7(n-m) + d_m X_8(n-m)) = \lambda^n V - V_-^{(n)}. \quad (49)$$

A direct calculation gives

$$-(V_+^{(n)})_x + [U, V_+^{(n)}] = b_{n+1} X_8(0) - c_{n+1} X_6(0) - d_{n+1} X_7(0). \quad (50)$$

According to the Tu scheme, the zero-curvature equation

$$U_t - (V_+^{(n)})_x + [U, V_+^{(n)}] = 0 \quad (51)$$

gives rise to the integrable hierarchy of evolution equations

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_n} = \begin{pmatrix} c_{n+1} \\ d_{n+1} \\ -b_{n+1} \end{pmatrix}. \quad (52)$$

We consider some reduction cases of (52). When we take $n = 1$, we have

$$\begin{aligned} q_{t_1} &= -\gamma r_x + \beta s_x + \beta q^2 - \alpha q r - \gamma r^2 + \beta r s, \\ r_{t_1} &= -\beta q_x + \alpha r_x, \\ s_{t_1} &= -\beta q s + \alpha r s. \end{aligned} \quad (53)$$

Setting $\alpha = \gamma = 0$ and $\beta = 1$, (53) reduces to

$$\begin{aligned} q_t &= s_x + q^2 + r s, \\ r_t &= -q_x, \\ s_t &= -q s. \end{aligned} \quad (54)$$

If we set $\gamma = \beta = 0$ and $\alpha = 1$, then (53) becomes

$$\begin{aligned} r_t &= r_x, \\ q_t &= -q r, \\ s_t &= r s, \end{aligned} \quad (55)$$

which is an integrable coupling of the convective diffusion equation. Equation (55) is solvable.

When one takes $n = 2$, one infers that

$$\begin{aligned} q_{t_2} &= c_3 \\ &= \beta(qs)_x - \alpha(sr)_x - \gamma q r_x + \beta q s_x + \beta q^3 - \alpha q^2 r \\ &\quad - \gamma q r^2 + \beta q r + \beta q r s - \alpha s r^2, \\ r_{t_2} &= d_3 \\ &= \gamma r_{xx} - \beta s_{xx} - 2\beta q q_x + \alpha(qr)_x + 2\gamma r r_x \\ &\quad - \beta(rs)_x, \\ s_{t_2} &= -b_3 \\ &= \beta q_{xx} - \alpha r_{xx} + \beta r q_x - \alpha r r_x - \gamma s r_x + \beta s s_x \\ &\quad + \beta s q^2 - \alpha q r s - \gamma s r^2 + \beta r s^2. \end{aligned} \quad (56)$$

Setting $\alpha = \gamma = 0$ and $\beta = 1$, (56) reduces to

$$\begin{aligned} q_t &= (qs)_x + q s_x + q^3 + q r + q r s, \\ r_t &= -s_{xx} - 2q q_x - (rs)_x, \\ s_t &= q_{xx} + r q_x + s s_x + s q^2 + r s^2. \end{aligned} \quad (57)$$

If we let $\beta = \gamma = 0$ and $\alpha = 1$, (56) becomes

$$\begin{aligned} q_t &= -(rs)_x - q^2 r - r^2 s, \\ r_t &= (qr)_x, \\ s_t &= -r_{xx} - r r_x - q r s. \end{aligned} \quad (58)$$

Setting $\alpha = \beta = 0$ and $\gamma = 1$, (56) gives

$$\begin{aligned} q_t &= -q r_x - q r^2, \\ s_t &= -s r_x - s r^2, \\ r_t &= r_{xx} + 2r r_x, \end{aligned} \quad (59)$$

which is a new $(1+1)$ -dimensional integrable coupling of the standard Burgers equation. It is easy to see that when we take $q = s = 0$, (59) reduces to the Burgers equation. We can also single out other integrable systems in addition to (53)–(59), but this is not discussed further.

4.2. $(2+1)$ -Dimensional Integrable Systems. The Tu scheme is well-known and no review is considered necessary here. This is not the case for the THA scheme, and we briefly discuss how it is used to generate $(2+1)$ -dimensional integrable systems [9]. Let $\mathcal{A}$ be an associative algebra over the field $\mathcal{K} = \mathbb{C}$ or $\mathbb{R}$, and assume that $\partial : \mathcal{A} \rightarrow \mathcal{A}$ satisfies

$$\begin{aligned} \partial(\alpha f + \beta g) &= \alpha \partial f + \beta \partial g, \\ \partial(fg) &= f(\partial g) + (\partial f)g, \end{aligned} \quad (60)$$

where $\alpha, \beta \in \mathcal{K}$ and $f, g \in \mathcal{A}$.

Suppose that $\mathcal{A}[\xi]$ is an associative algebra consisting of the pseudodifferential operators $\sum_{i=-\infty}^N a_i \xi^i$, where $a_i \in \mathcal{A}$ and ξ satisfies

$$\xi f = f \xi + (\partial_y f), \quad f \in \mathcal{A}, \quad (61)$$

from which we have

$$\begin{aligned} \xi^n f &= \sum_{i \geq 0} \binom{n}{i} (\partial^i f) \xi^{n-i}, \\ f \xi^n &= \sum_{i \geq 0} (-1)^i \binom{n}{i} \xi^{n-i} (\partial^i f), \end{aligned} \quad (62)$$

$n \in \mathbf{Z}.$

Then we take operator matrices,

$$\begin{aligned} U &= U(\lambda, \xi, u) \in \mathcal{A}[\xi], \quad u = (u_1, \dots, u_p)^T, \\ V &= \sum V_n \lambda^{-n}, \end{aligned} \quad (63)$$

and seek for solutions of V from the stationary zero-curvature equation

$$V_x = [U, V]. \quad (64)$$

From the expansion

$$\left\langle V, \frac{\partial U}{\partial u_i} \right\rangle = \sum_i g_i^{(n)} \lambda^{-n}, \quad (65)$$

we obtain a recursion relation among $g^{(n)} = (g_1^{(n)}, \dots, g_p^{(n)})^T$, where

$$\begin{aligned} \langle a, b \rangle &= \text{tr } R(ab), \\ R : \mathcal{A}[\xi] &\longrightarrow R^1, \\ \sum a_i \xi^i &\longrightarrow a_{-1}. \end{aligned} \quad (66)$$

Finally, we try to find an operator J from the hierarchy

$$u_{t_n} = J g^{(n)}, \quad (67)$$

whose Hamiltonian structure can be generated from the $(2+1)$ -dimensional trace identity

$$\frac{\delta}{\delta u_i} \langle V, U_\lambda \rangle = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \left\langle V, \frac{\partial U}{\partial u_i} \right\rangle, \quad i = 1, 2, \dots, p. \quad (68)$$

Based on the above steps for generating $(2+1)$ -dimensional integrable hierarchies of evolution equations, we apply the Lie algebra $\mathfrak{g}_4$ to introduce a Lax pair for matrices U and V :

$$\begin{aligned} U &= (\lambda + \xi) \bar{X}(0) + qX_6(0) + rX_7(0) + sX_8(0), \\ V &= bX_6(0) + cX_7(0) + dX_8(0), \end{aligned} \quad (69)$$

where

$$\begin{aligned} b &= \sum_{m \geq 0} b_m \lambda^{-m}, \\ c &= \sum_{m \geq 0} c_m \lambda^{-m}, \\ d &= \sum_{m \geq 0} d_m \lambda^{-m}. \end{aligned} \quad (70)$$

Equation (64) leads to

$$\begin{aligned} b_x &= -(\xi + \lambda)c + qc - rb, \\ c_x &= -(\xi + \lambda)d, \\ d_x &= (\xi + \lambda)b - rd + sc. \end{aligned} \quad (71)$$

Substituting (70) into (71) yields

$$\begin{aligned} c_{m+1} &= -b_{mx} - c_m \xi - c_{my} + qc_m - rb_m, \\ d_{m+1} &= -c_{mx} - d_m \xi - d_{my}, \\ b_{m+1} &= d_{mx} - b_m \xi - b_{my} + rd_m - sc_m. \end{aligned} \quad (72)$$

Assume that $b_0 = \alpha \xi^{-1}$, $c_0 = \beta \xi^{-1}$, $d_0 = \delta \xi^{-1}$, where α, β, δ are constants. We can calculate

$$\begin{aligned} b_1 &= -\alpha + (\delta r - \beta s) \xi^{-1}, \\ c_1 &= -\beta + (\beta q - \alpha r) \xi^{-1}, \\ d_1 &= -\delta, \\ b_2 &= (-\delta r_y + \beta s_y - \beta q s + \alpha s r) \xi^{-1} + \alpha \xi + 2\beta s - 2\delta r, \\ c_2 &= (-\delta r_x + \beta s_x - \beta q_y + \alpha r_y) \xi^{-1} + \beta \xi - \beta q + \alpha r, \\ d_2 &= (\alpha r_x - \beta q_x) \xi^{-1} + \delta \xi, \\ &\vdots \end{aligned} \quad (73)$$

Note that

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^n (b_m X_6(n-m) + c_m X_7(n-m) \\ &\quad + d_m X_8(n-m)) = \lambda^n V - V_-^{(n)}. \end{aligned} \quad (74)$$

Then (64) can be decomposed into

$$-(V_+^{(n)})_x + [U, V_+^{(n)}] = (V_-^{(n)})_x - [U, V_-^{(n)}]. \quad (75)$$

The degree of the left-hand side of (75) is ≥ 0 , while the right-hand side is ≤ 0 . Therefore, the degrees of both sides are zero. Thus, one infers that

$$\begin{aligned} -(V_+^{(n)})_x + [U, V_+^{(n)}] &= R(b_{n+1}) X_8(0) \\ &\quad - R(c_{n+1}) X_6(0) \\ &\quad - R(d_{n+1}) X_7(0). \end{aligned} \quad (76)$$

The zero-curvature equation

$$U_t - (V_+^{(n)})_x + [U, V_+^{(n)}] = 0 \quad (77)$$

admits the $(2 + 1)$ -dimensional integrable hierarchy

$$\begin{pmatrix} q \\ r \\ s \end{pmatrix}_{t_n} = \begin{pmatrix} R(c_{n+1}) \\ R(d_{n+1}) \\ -R(b_{n+1}) \end{pmatrix}. \quad (78)$$

When $n = 1$, (78) reduces to

$$\begin{aligned} q_t &= -\delta r_x + \beta s_x - \beta q_y + \alpha r_y, \\ r_t &= -\beta q_x + \alpha r_x, \\ s_t &= \delta r_y - \beta s_y + \beta q_s - \alpha sr. \end{aligned} \quad (79)$$

When $n = 2$, (78) reduces to

$$\begin{aligned} q_t &= R(c_3) \\ &= 2\delta r_{xy} + \beta(qs)_x - \alpha(sr)_x + \beta q_{yy} - \alpha r_{yy} - \delta q r_x \\ &\quad + \beta q s_x - \beta q q_y + \alpha q r_y + \delta r r_y - \beta r s_y + \beta q r s \\ &\quad - \alpha s r^2, \end{aligned} \quad (80)$$

$$\begin{aligned} r_t &= \delta r_{xx} - \beta s_{xx} + 2\beta q_{xy} - 2\alpha r_{xy}, \\ s_t &= \beta q_{xx} - \alpha r_{xx} \delta r_{yy} + \beta s_{yy} - \beta(qs)_y + \alpha(sr)_y \\ &\quad + \beta r q_x - \alpha r r_x - \delta r_x s + \beta s s_x - \beta s q_y + \alpha s r_y. \end{aligned}$$

When $\alpha = \beta = 0$ and $\delta = 1$, (80) reduces to a $(2 + 1)$ -dimensional integrable coupling of the $(1 + 1)$ -dimensional heat conduction equation

$$\begin{aligned} r_t &= r_{xx}, \\ q_t &= 2r_{xy} - q r_x + r r_y, \\ s_t &= -r_{yy} - r_x s. \end{aligned} \quad (81)$$

When $\beta = \delta = 0$ and $\alpha = 1$, (80) becomes

$$\begin{aligned} r_t &= -2r_{xy}, \\ q_t &= -(sr)_x - r_{yy} + q r_y - s r^2, \\ s_t &= -r_{xx} + (sr)_y - r r_x + s r_y, \end{aligned} \quad (82)$$

which is a $(2 + 1)$ -dimensional integrable coupling of the $(2 + 1)$ -dimensional hyperbolic equation.

When $\alpha = \delta = 0$ and $\beta = 1$, (80) reduces to

$$\begin{aligned} r_t &= -s_{xx} + 2q_{xy}, \\ q_t &= (qs)_x + q_{yy} + q s_x - q q_y - r s_y + q r s, \\ s_t &= q_{xx} + s_{yy} - (qs)_y + r q_x + s s_x - s q_y. \end{aligned} \quad (83)$$

In particular, when we take $r = q = 0$, (83) reduces to

$$s_t = s_{yy} + s s_x, \quad (84)$$

which is a $(2 + 1)$ -dimensional Burgers equation, with (83) being its generalized integrable coupling.

5. Expanding Integrable Models of Integrable Systems (52) and (78)

In the section we want to enlarge the Lie algebra g_4 and deduce the expanding integrable models of the $(1 + 1)$ -dimensional integrable hierarchy (52) and the $(2 + 1)$ -dimensional integrable hierarchy (78). Obviously, the Lie algebra g_5 is the minimum enlarging Lie algebra of g_4 . However, g_5 cannot generate new integrable dynamical systems based on the isospectral problem (46). That is, by applying the enlarging Lie algebra g_5 we are not able to obtain new expanding integrable models compared to the Lie algebra g_4 , except for an arbitrary smooth function with respect to x . Therefore, we have to enlarge the Lie algebra g_5 to the following Lie algebra (not unique):

$$\hat{g} = \text{span}\{X_3, X_4\} + g_5, \quad (85)$$

which is also a Lie subalgebra of the large Lie algebra (15). It is easy to verify that the Lie algebra $\hat{g}$ has the following operation relations:

$$\begin{aligned} [X_3, X_4] &= -X_4, \\ [X_3, X_5] &= -X_5, \\ [X_3, X_6] &= X_6, \\ [X_3, X_7] &= [X_3, X_8] = 0, \\ [X_3, \bar{X}] &= -X_5 - X_6, \\ [X_4, X_5] &= 0, \\ [X_4, X_6] &= X_7 - X_3, \\ [X_4, X_7] &= -X_4, \\ [X_4, X_8] &= -X_5, \\ [X_4, \bar{X}] &= X_3 - X_7, \\ [X_5, X_6] &= X_8, \\ [X_5, X_7] &= [X_5, X_8] = 0, \\ [X_5, \bar{X}] &= X_5 - X_8, \\ [X_6, X_7] &= X_6, \\ [X_6, X_8] &= 0, \\ [X_6, \bar{X}] &= -X_8, \\ [X_7, X_8] &= -X_8, \\ [X_7, \bar{X}] &= X_6, \\ [X_8, \bar{X}] &= X_7. \end{aligned} \quad (86)$$

In what follows, we first introduce an isomorphism between the Lie algebra $\hat{\mathcal{g}}$ and the linear space R^7 . Assume that $a, b \in \hat{\mathcal{g}}$; then a, b can be expressed by

$$\begin{aligned} a &= \sum_{i=1}^6 a_i X_{2+i} + a_7 \bar{X}, \\ b &= \sum_{j=1}^6 b_j X_{2+j} + b_7 \bar{X}. \end{aligned} \quad (87)$$

Suppose that there exists a linear map $f : \hat{\mathcal{g}} \rightarrow R^7$, $\forall a \in \hat{\mathcal{g}} \rightarrow (a_1, \dots, a_7)^T$. It is easy to see that f is an isomorphism between the linear spaces $\hat{\mathcal{g}}$ and R^7 . Based on the commutative relations of the Lie algebra $\hat{\mathcal{g}}$, we define an operation in R^7 by

$$\begin{aligned} [a, b]_R &= (a_4 b_2 - a_2 b_4 + a_2 b_7 - a_7 b_2, a_2 b_1 - a_1 b_2 \\ &+ a_5 b_2 - a_2 b_5, a_3 b_1 - a_1 b_3 + a_3 b_7 - a_7 b_3 + a_6 b_2 \\ &- a_2 b_6 + a_7 b_1 - a_1 b_7, a_1 b_4 - a_4 b_1 + a_4 b_5 - a_5 b_4 \\ &+ a_5 b_7 - a_7 b_5, a_2 b_4 - a_4 b_2 + a_7 b_2 - a_2 b_7 + a_6 b_7 \\ &- a_7 b_6 + a_7 b_1 - a_1 b_7, a_3 b_4 - a_4 b_3 + a_7 b_4 - a_4 b_7 \\ &+ a_6 b_5 - a_5 b_6 + a_7 b_3 - a_3 b_7, 0)^T. \end{aligned} \quad (88)$$

We can prove that the linear space R^7 becomes a Lie algebra if it is equipped with the operation (88).

Next, we deduce the expanding integrable models of the integrable hierarchies based on the Lie algebra R^7 .

5.1. Expanding Integrable Model of the Integrable Hierarchy (52). We introduce a loop algebra

$$\begin{aligned} \widetilde{R^7} &= \{(x_1, \dots, x_6, 0)^T, x_i = \sum x_{im} \lambda^m, x_{im} \\ &= x_{im}(t, x), i = 1, \dots, 6\}. \end{aligned} \quad (89)$$

In terms of $\widetilde{R^7}$, we consider the Lax matrices

$$\begin{aligned} U &= (u_1, 0, u_2, q, r, s, \lambda)^T, \\ V &= (0, a, f, b, c, d, 0)^T, \end{aligned} \quad (90)$$

where λ is a spectral parameter, $a = \sum_{m \geq 0} a_m \lambda^{-m}$, $b = \sum_{m \geq 0} b_m \lambda^{-m}, \dots$ According to the Tu scheme, a solution to the stationary zero-curvature equation

$$V_x = [U, V]_R \quad (91)$$

for V is given by

$$\begin{aligned} c_{m+1} &= -b_{mx} + qc_m + (u_1 - r)b_m, \\ d_{m+1} &= -c_{mx}, \\ b_{m+1} + f_{m+1} &= d_{mx} + qf_m + rd_m - sc_m - u_2 b_m, \\ a_{m+1} &= qa_m, \\ f_{m+1} &= -f_{mx} - u_1 f_m + sa_m, \\ a_{mx} &= (r - u_1)a_m. \end{aligned} \quad (92)$$

Taking

$$\begin{aligned} b_0 &= \alpha, \\ c_0 &= \beta, \\ d_0 &= \gamma, \\ a_0 &= \sigma, \\ f_0 &= \delta, \end{aligned} \quad (93)$$

then (92) admits that

$$\begin{aligned} a_1 &= \sigma q, \\ b_1 &= \delta q + \gamma r - \beta s - \alpha u_2, \\ c_1 &= \beta q + \alpha(u_1 - r), \\ d_1 &= 0, \\ f_1 &= -\delta u_1 + \sigma s, \\ &\vdots \end{aligned} \quad (94)$$

Note that

$$\begin{aligned} V_+^{(n)} &= \sum_{m=0}^n (0, a_m, f_m, b_m, c_m, d_m, 0)^T \lambda^{n-m} \\ &= \lambda^n V - V_-^{(n)}. \end{aligned} \quad (95)$$

A direct calculation yields

$$\begin{aligned} - (V_+^{(n)})_x + [U, V_+^{(n)}]_R &= (-a_{n+1}, 0, -f_{n+1}, \\ &- c_{n+1}, a_{n+1} - d_{n+1}, b_{n+1} + f_{n+1}, 0)^T. \end{aligned} \quad (96)$$

Thus the zero-curvature equation

$$U_{t_n} - (V_+^{(n)})_x + [U, V_+^{(n)}]_R = 0 \quad (97)$$

has an integrable hierarchy

$$\begin{pmatrix} q \\ r \\ s \\ u_1 \\ u_2 \end{pmatrix}_{t_n} = \begin{pmatrix} c_{n+1} \\ d_{n+1} - a_{n+1} \\ -b_{n+1} - f_{n+1} \\ a_{n+1} \\ f_{n+1} \end{pmatrix}. \quad (98)$$

When we take $u_1 = u_2 = 0$, (98) reduces to the (1 + 1)-dimensional integrable hierarchy (52). Therefore, (98) is an expanding integrable model of (52). A simple reduction of (98), when $n = 1$, has the form

$$\begin{aligned}
 q_t &= c_2 \\
 &= -\delta q_x - \gamma r_x + \beta s_x + \alpha u_{2x} + \beta q^2 - \alpha q r + 2\delta q \\
 &\quad + \frac{1}{q} (2\gamma r q_x - 2\beta q_x s - 2\alpha q_x u_2), \\
 r_t &= -\beta q_x - \alpha r_x - \sigma q^2, \\
 s_t &= \delta q u_1 - \sigma q s + \beta q s - \alpha s r + \delta q u_2 + \gamma u_2 r - \beta s u_2 \\
 &\quad - \alpha u_2^2, \\
 u_{1,t} &= \sigma q^2, \\
 u_{2,t} &= \delta u_{1x} + \sigma s_x + \delta u_1^2 - \sigma u_1 s + \sigma q s.
 \end{aligned} \tag{99}$$

5.2. Expanding Integrable Model of the (2 + 1)-Dimensional Integrable Hierarchy (78). We define an associative algebra $R^7[\xi]$ consisting of elements $\sum a_i \xi^i$, $a_i \in R^7$. A linear operator on $R^7[\xi]$ is given by

$$\partial : R^7[\xi] \longrightarrow R^7[\xi], \tag{100}$$

which satisfies

$$\begin{aligned}
 \partial(fg) &= (\partial f)g + f(\partial g), \\
 \partial(\alpha f + \beta g) &= \alpha(\partial f) + \beta(\partial g),
 \end{aligned} \tag{101}$$

where $f, g \in R^7[\xi]$ and α, β are arbitrary constants. In addition, a residue operator is defined by

$$\begin{aligned}
 \mathcal{R} : R^7[\xi] &\longrightarrow R^1, \\
 \mathcal{R}\left(\sum a_i \xi^i\right) &= a_{-1}.
 \end{aligned} \tag{102}$$

Based on the above notations, we introduce two Lax matrices on $R^7[\xi]$:

$$\begin{aligned}
 U &= (u, 0, v, q, r, s, \lambda + \xi)^T, \\
 V &= (0, a, f, b, c, d, 0)^T,
 \end{aligned} \tag{103}$$

where

$$\begin{aligned}
 a &= \sum_{m \geq 0} \lambda^{-m}, \\
 b &= \sum_{m \geq 0} b_m \lambda^{-m}, \\
 &\vdots
 \end{aligned} \tag{104}$$

Similar to previous discussions, there exist relations among a, b, c, d, f such that

$$\begin{aligned}
 b_x &= -(\lambda + \xi)c + qc + (u - r)b, \\
 c_x &= -(\lambda + \xi)d + (\lambda + \xi)a - qa, \\
 d_x &= (\lambda + \xi)b + (\lambda + \xi)f - qf - rd + sc + vb, \\
 a_x &= (r - u)a, \\
 f_x &= -(\lambda + \xi)f + sa - uf, \\
 0 &= -(\lambda + \xi)a + qd.
 \end{aligned} \tag{105}$$

Substituting (104) into (105) gives rise to

$$\begin{aligned}
 c_{m+1} &= -b_{mx} - c_m \xi - c_{my} + qc_m + (u - r)b_m, \\
 d_{m+1} &= -c_{mx} - d_m \xi - d_{my}, \\
 b_{m+1} + f_{m+1} &= d_{mx} - b_m \xi - b_{my} - f_m \xi - f_{my} + qf_m \\
 &\quad + rd_m - sc_m - vb_m, \\
 f_{m+1} &= -f_{mx} - f_m \xi - f_{my} + sa_m - uf_m, \\
 a_{m+1} &= qa_m - a_m \xi - a_{my}, \\
 a_{mx} &= (r - u)a_m.
 \end{aligned} \tag{106}$$

Taking

$$\begin{aligned}
 b_0 &= \alpha \xi^{-1}, \\
 c_0 &= \beta \xi^{-1}, \\
 d_0 &= \delta \xi^{-1}, \\
 a_0 &= \sigma \xi^{-1}, \\
 f_0 &= \gamma \xi^{-1},
 \end{aligned} \tag{107}$$

we have from (106) that

$$\begin{aligned}
 a_1 &= -\sigma + \sigma q \xi^{-1}, \\
 f_1 &= \gamma + (\sigma s - \gamma u) \xi^{-1}, \\
 d_1 &= -\delta, \\
 c_1 &= -\beta + \beta q \xi^{-1}, \\
 b_1 + f_1 &= -\alpha - \gamma + (\gamma q + \delta r - \beta s - \alpha v) \xi^{-1}, \\
 b_1 &= -\alpha - 2\gamma + (\gamma q + \delta r - \beta s - \alpha v - \sigma s + \gamma u) \xi^{-1}, \\
 c_2 &= -\beta q + (\alpha + 2\gamma)(r - u) + [-\gamma q_x - \delta r_x + \beta s_x \\
 &\quad + \alpha v_x + \sigma s_x - \gamma u_x - \beta q_y + \beta q^2 \\
 &\quad + (u - r)(\gamma q + \delta r - \beta s - \alpha v - \sigma s + \gamma u)] \xi^{-1},
 \end{aligned}$$

$$\begin{aligned}
d_2 &= -\delta\xi - \beta q_x \xi^{-1}, \\
f_2 &= -\gamma\xi - 2\sigma s + (-\sigma s_x + \gamma u_x - \sigma s_y + \gamma u_y + \sigma q_s \\
&\quad - \sigma u s + \gamma u^2) \xi^{-1}, \\
b_2 + f_2 &= (\alpha + \gamma) \xi + 2\gamma q - \sigma s + \gamma u + \gamma v + (-\gamma q_y \\
&\quad - \delta r_y + \beta s_y + \alpha v_y - \gamma q u - \beta q s - \gamma q v - \delta r v \\
&\quad + \beta v s + \alpha v^2) \xi^{-1}, \\
a_2 &= -\sigma q + \sigma (q^2 - q_y) \xi^{-1},
\end{aligned} \tag{108}$$

$$\begin{aligned}
q_x &= q(r - u), \\
(q^2 - q_y)_x &= q^2 - q_y.
\end{aligned} \tag{109}$$

According to the TAH scheme, we form a (2+1)-dimensional integrable hierarchy

$$\begin{pmatrix} q \\ r \\ s \\ u \\ v \end{pmatrix}_{t_n} = \begin{pmatrix} \mathcal{R}(c_{n+1}) \\ \mathcal{R}(d_{n+1} - a_{n+1}) \\ -\mathcal{R}(b_{n+1} + f_{n+1}) \\ \mathcal{R}(a_{n+1}) \\ \mathcal{R}(f_{n+1}) \end{pmatrix}. \tag{110}$$

When we set $u = v = 0$, (110) reduces to the (2+1)-dimensional integrable hierarchy (78). Hence, (110) is an expanding integrable model of (78).

A simple reduction of (110), when $n = 1$, has the form

$$\begin{aligned}
q_t &= \mathcal{R}(c_2) \\
&= -\gamma q_x - \delta r_x + \beta s_x + \alpha v_x + \sigma s_x - \gamma u_x - \beta q_y \\
&\quad + \beta q^2 + (u - r)(\gamma q + \delta r - \beta s - \alpha v - \sigma s + \gamma u), \\
r_t &= -\beta q_x + \sigma(q_y - q^2), \\
s_t &= \gamma q_y + \delta r_y - \beta s_y - \alpha v_y + \gamma q u + \beta q s + \gamma q v + \delta r v \\
&\quad - \beta v s - \alpha v^2, \\
u_t &= \sigma(q^2 - q_y), \\
v_t &= -\sigma s_x + \gamma u_x - \sigma s_y + \gamma u_y + \sigma q s - \sigma u s + \gamma u^2,
\end{aligned} \tag{111}$$

which has the constraint condition (109). If we further take $n = 2, 3, \dots$, we obtain different (2+1)-dimensional integrable systems with their corresponding constraint conditions.

6. Conclusions

In the paper we have discussed two applications of the operator Lie algebra (15) and derived some new results. These include a second-order differential equation and its Lagrangians, the quasi-extremal equation and its general solutions,

and the (1+1)-dimensional and (2+1)-dimensional integrable dynamical hierarchies as well as their expanding integrable models. We have adopted the vector Lie algebra R^7 to deduce the expanding integrable models of the integrable hierarchies, so that we can generate Hamiltonian structures of the expanding integrable models by the quadratic-form identity or the variational identity in [14, 15]. It is worth further studying discretizations of the differential equations obtained in this paper. By applying some schemes in [16], we can further investigate the discretizations' exact solutions, symmetries, infinitely many conservation laws, and quasiperiodic solutions by following the ways in [17–32].

Competing Interests

The authors declare that they have no competing interests.

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